Renewing Foundations

past and future of foundations of mathematics

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Introduction: progress of science and renewal of its foundations

A) Philosophy as pre-science

Tracing back the history of a theoretical discipline one often discovers that at certain point in the past this discipline used to be a branch of philosophy. Aristotle's *Physics* is hardly even recognised by today's physicists as belonging to their discipline but it is still widely read at philosophy departments; psychology, political sciences and cosmology separated themselves from philosophy and constituted themselves as independent disciplines only in 20-th century (Note 1). This observation suggests a view on philosophy as an immature science aiming at but never achieving the genuine scientific rigour. Many philosophers would contest this view arguing that common standards of scientific rigour shouldn't be applied for their discipline and that for this reason philosophers don't need to pursue anything like scientific rigour at all; some other would argue that philosophy already got a rigorous scientific method, to which every philosopher should now adhere.

Actually intellectual enterprises of quite different sorts are found presently on the market under the label of philosophy, and I don't feel myself in a position to judge which of them really deserve this name and which don't. These introductory remarks serve another purpose: to explain the reader what kind of enterprise I'm purporting in this book. Unlike aforementioned philosophers I am happy with the notion of philosophy as a pre-scientific theoretic activity closely linked to science itself; by science I mean here all branches of science without any exception. However to make my notion of philosophy more precise I need the following two reservations.

First, I disagree with the view that once a branch of science is well constituted as an autonomous discipline the preparatory activity which made this possible can be stopped or transformed into something else like popularisation of science. This is simply not how science works. Kuhn's picture of development of science, however imprecise, makes this point clear (see Kuhn1962). Kuhn distinguishes between *normal* science which develops sticking to well-established epistemic standards he calls *paradigms* and *scientific revolutions* consisting of drastic changes of the paradigms. The phenomenon of scientific revolutions shows that scientific disciplines are not established once and for all. They need to be re-established from time to time. Moreover, I believe that in case of a mature science this re-establishing activity is permanent and continuous rather than discreet and concentrated only in specific short periods, which Kuhn calls revolutions. The continuity of the re-establishing doesn't preclude science from a cumulative growth as we shall shortly see. Philosophy (or rather the kind of philosophy I'm talking about here) is a technique or art of such re-establishing.

Second, I am certainly *not* happy with the notion of philosophy as a room for all kind of disputable conjectures and revisionary proposals about scientific matters. Such a room cannot be anything like a theoretical discipline, and in spite of my liberal attitude I do count philosophy as a theoretical discipline. This is the reason why I adopt a further restriction, which delimits philosophical competence by *foundations* of science and doesn't allow philosophical arguments in other scientific contexts. To make a sense of it I need now to explain what I mean by foundations, and why taking care about foundations makes philosophy into a pre-science rather than into a particular branch of science.

B) Architectural and biological metaphors of science; educational and conceptual foundations

The term "foundations" makes one to think about construction of a building. To construct a building one should prepare its foundation first. The rest of the construction crucially depends on this first step. The rest can be remade in case: a badly painted wall can be cleaned and painted anew. But a defect in the foundation may cause a crash of the whole building making it unrepairable. Looking for "firm foundations" for science (or any particular branch of science) people usually think about science as such an edifice. But in fact the architectural metaphor of science is very misleading when one looks at science in a wider historical perspective. Foundations of science unlike the foundation of a building don't remain untouched by later developments. At the same time significant changes in foundations don't necessarily cause drastic changes of the rest of a given discipline, as one might probably expect. In particular, significant changes of common views on foundations of mathematics occurring during the long history of this discipline didn't make any harm to the Pythagorean theorem which remains unchallenged since its early discovery. Although these foundational changes modified the precise *sense* of this theorem they didn't brake its identity. In many cases foundational changes in science (or at least efforts aiming at such changes) are even completely disregarded by working scientists as a matter of "philosophical interpretation" irrelevant to their work. Some philosophers may be happy with this situation which allows them to isolate foundations of discipline X from this discipline itself and exercise in their domain whatever they want without any danger of being ridiculed by specialists in X. I am not. The idea to detach foundations from what they are foundations of is, in my opinion, just as wrong as the view suggested by the architectural metaphor of science, according to which philosophy is in a position to disprove any scientific result through a revision of its foundations.

As far as metaphors are concerned science can be better compared with a complex ecological system, which involves populations of different species and develops through continuous reproduction of organisms and underlying evolutionary change. This is slightly more than a mere metaphor because we humans *indeed* form a biological population living in its natural (and partly artificial) environment, and science is our human affair. Any individual knower starts learning science with its basics, that is, with foundations. So the need to "restart" science over and over again follows from the basic biological feature shared by humans with most of other animals, namely the fact that human populations endure through a continuous reproduction of mortal individuals. This educational aspect of foundations is essential and should be never completely forgotten even when one deals with more theoretical aspects of foundations. Thus the *first* notion of foundations I shall deal with in this book is the *educational* foundations by which I mean a first introduction into a given discipline. In this case it is particularly clear why philosophy qualifies as a pre-science: to begin doing science one should learn its foundations first. It is equally clear that educational foundations is a subject of permanent renewal, and that a renewal of educational foundations doesn't necessarily imply a renewal of the whole given discipline. However the educational aspect of foundations is not the only one I would like to take into consideration.

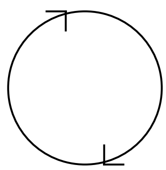
There is actually a sense in which any part of scientific knowledge undergoes repeated renewals also outside of educational contexts. I mean the fact that any theoretical construction, any idea and any established scientific fact endures through series of mental acts of *retention*. Such acts can be attributed both to individual minds and to scientific communities of various scales including the global one. (Note 2) Only in simplest cases such a series of retentions reduces to a mechanical repetition of a given pattern; typically it has the character of progressive *understanding* which involves quite a complicated dynamics. (Note 3) However important for the business of science are written texts, which help fixing identity of various scientific contents, scientific knowledge is not copied in individual minds in anything like the same way in which one may copy texts. The idea that the bulk of existing knowledge can be written into a big book and then transferred to any interested reader stems from archaic practices having little to do with science. Science never canonised its texts in anything like the same way in which religions typically did this. In the individual mind retention is combined with what I shall call *proceeding* just like at higher levels of organisation of science the renewal of foundations is combined with accumulation of knowledge. This can be clearly seen at the example of a mathematical theorem (or any justified proposition in any branch of science). The standard proof theory conceives of a theorem T as a proposition obtained from set of propositions $P_1,\ldots,\ P_n$ called premises as a result of an operation called proof. So we have this picture:

$$P_1,\ldots,\ P_n \quad \text{--------->} \quad T \qquad (1)$$

This analysis takes proceeding into account but leaves the retention out. Notice that unlike what the above picture suggests theorems are usually first announced and only then proved. In particular in Euclid's *Elements* each theorem is stated twice: just before and just after its proof. So Euclid makes explicit what is always involved in the usual practice of theorem-proving: one starts with a meaningful proposition T (Note 4) then performs an appropriate reasoning (which typically involves other propositions $P_1,\ldots,\ P_n$ as premises), and finally concludes (again) with T . This circular process shown at the below diagram preserves T's identity but changes its epistemic status:

$(P_1,\ldots,\ P_n)$

(2)



T

This latter diagram makes retention explicit. But the retention of *what*? Euclid repeats his propositions twice word by word or nearly so. Thus a textual analysis misleads us showing only a trivial retention, i.e. a literal repetition of written symbols. Propositions or thoughts in Frege's sense are eternal things to which the notion of retention doesn't apply. So we cannot count on them either. The notion appropriate for our analysis seems to be that of *concept*. The claim that concepts undergo retention amounts to saying that they are not built once and for all but just like proofs are performed repeatedly in individual minds and in scientific communities of various scales. This recurrent process also typically involves a linear (cumulative) part: higher-order concepts are built on the basis of simpler ones. But since this linear process is supposed to be recurrent it always requires a "point of restart", which I shall call *conceptual foundations*. Conceptual foundations is the *second* notion of foundations, which will be of my concern in this book.

Both notions of foundations just described - educational and conceptual - are equally derived from the observation that science develops in two principal ways: the progress and the renewal. This scheme applies not only to large historical scales where one may point to scientific revolutions as examples of the renewal but also to micro-scales as in the above example of a mathematical theorem where the renewal is called retention and the progress is called proceeding. The general notion of foundations as distinguished from the rest of scientific knowledge helps for making the progress and the renewal mutually compatible: one may think of foundations is the subject of renewal while the rest is the subject of progress. There is a derived sense in which the renewal of foundations implies the renewal of the rest, and in which the progress of the rest implies the progress of foundations. But in order to make the distinction sharper it is helpful to forget about this derived sense and assume that the principal bulk of scientific knowledge is a subject of pure progress (without any renewal) while foundations is a subject of pure renewal (without any progress). Although in the real history the progress and the renewal are always intertwined the suggested model provides a suitable general framework for relationships between philosophy and sciences. Clearly the renewal and the progress require efforts of different sorts. This justifies the notion of philosophy as an art of renewal. This also explains why philosophy cannot avoid to be parasitic on other sciences and why these other sciences cannot perform a sustainable progress without a philosophical ingredient.

C) Systematic foundations; change and identity of scientific theories through time

Having in mind the "recurrent" model of science described above we can now more accurately interpret the traditional static picture, which presents science as a timeless system of disciplines, sub-disciplines and particular theories. This abstract notion of science reveals a feature of foundations which we have already mentioned talking about the architectural metaphor of science: unlike other parts of a given theory (or of a given discipline) foundations have a bearing on the *whole* of it. (A building having a defective foundation is *wholly* defective). This feature of foundations survives even when the metaphor is rejected. What is to be rejected is the view according to which one cannot possibly exchange foundations without destroying the rest. But even if one no longer thinks of science as an edifice one still observes that foundations matter for every part of what they found. One also observes that foundations typically determine how exactly a given theory (discipline) is divided into parts (sub-disciplines), how these parts relate to each other, and what binds them into a whole. This is the sense in which foundations *organise*, *unify* and *identify* what they found. I shall refer to foundations conceived of in this latter sense as *systematic* foundations. This is the *third* notion of foundations, which will be studied in this book.

What the integrating function of systematic foundations has to do with renewal and retention as opposed to progress and proceeding? Before I explain this in general terms let me show that educational and conceptual foundations also perform this integrating role. Take mathematics as an example. All working mathematicians share a bulk of basic mathematical knowledge albeit none of them can today possess the *whole* bulk of mathematical knowledge. This core bulk of knowledge, which can be identified with educational foundations of mathematics, mathematicians learn at early stages of their professional education. This basic knowledge is not only indispensable for making research in any specific domain of mathematics but it also serves for communication between people working in different domains of mathematics. Thus (educational) foundations of mathematics integrate all working mathematicians into a single community and at the same time make mathematics distinguishable from other scientific disciplines. One can identify mathematics as such, i.e. as a whole, by pointing to its foundations. So foundations of mathematics make this discipline into a whole and determine its identity as a discipline. The fact that boundaries between mathematics and some other neighbouring disciplines are often vague and involve further distinctions like that between the pure and the applied mathematics doesn't change the principle point just made. Without a common background shared by all working mathematicians mathematics would split into a variety of practices having no systematic relation to each other and called by the common name only on the basis of a historical kinship and an apparent similarity. Mathematics would be then as disorganised as philosophy (Note 5).

A similar point can be made about conceptual foundations. Retention of mathematical concepts begins with fundamental concepts and then proceeds to more complex and more specific concepts. When people talk about building of concepts they refer to the latter aspect of this process. The fundamental concepts themselves are retended rather than rebuilt. The retention of fundamental concept is the point of restart, from which one can proceed in many different directions. This is, for example, how the concepts of set and structure are used in Bourbaki's Elements. In Bourbaki's setting mathematics is a science of structural sets. Different types of structures are studied by different branches of mathematics. The concepts of set and structure make mathematics into a single whole. So the fundamental concepts of set and structure organise and unify mathematics in a particular way. Fundamental concepts of Euclid's mathematics work similarly but the way in which they organise and unify mathematics is different.

Systematic foundations are most closely related to the traditional architectural image of foundations, so in order to understand them in the context of foundational renewal we need a special effort. This effort concerns some general metaphysical issues, which I shall mention only briefly here postponing a more detailed discussion until the second part of this book. A traditional notion of change stemming from Aristotle assumes that a changing entity has a unchangeable core called its substance. The substance provides the identity of changing entity, which survives through the change. This notion allows one to think of development of science in terms of the architectural metaphor: while science grows and changes its identity is preserved with its unchangeable foundations. Another traditional metaphysical theory explains the notion of change away in terms of timeless momentary states. On this latter account the history of science reduces to a series of images, which are put together for some arbitrary reasons. This view is equally compatible with the architectural metaphor provided that this metaphor is applied to the timeless images of science one-by-one. None of these accounts suits my present purpose since each of them requires a timeless notion of foundations in some form. So my metaphysical choice will be different. First, I shall assume a notion of transformation, which is more general than the aforementioned traditional notion of change since it doesn't imply any preserved identity. Think of a dividing amoeba: it passes away when its two babies get born. This is a transformation but not a mere change. For it doesn't preserve the identity of the old amoeba. Second, I shall consider identity as a specific kind of transformation, which can be informally described as a repetition or renewal. The survival of an amoeba just like its division is a process, it is not given for free by a metaphysical necessity. But it is a process of a different sort, which I shall describe in what follows. On this proposed metaphysical view foundations of science don't change. My account shares this feature with the traditional accounts. But on my account the lack of change – in the precise sense of the term specified above – doesn't imply rigidity. Foundations, on my account, undergo a permanent transformation, which constitutes the identity of science or a particular branch of science. So renewing systematic foundations make (a branch of) science into an enduring whole. A knowledgeable reader can see that the proposed notion of identity as a transformation stems from the Category theory. However I have also an independent epistemological reason to put it forward.

The way in which science constitutes its identity and preserves its through time and through its own evolution appears me fundamental for distinguishing science from various kinds of different science-like activities. As I have already stressed above there is a big difference between reproduction of a ungrounded belief in individual minds through repetition and dissemination of linguistic patterns expressing this belief and retention of mathematical theorem through its competent understanding, which always requires from an individual some critical attitude and an independent effort aiming at proof (or eventual disproof) of the given theorem. But theorems and particular beliefs are very simple epistemic entities which can reproduce but cannot change. (When one changes his or her beliefs this is the identity of the person's mind which is preserved while beliefs themselves don't change but are *exchanged* one for another. Theorems may change their appearance quite dramatically - I shall elaborate on this in the second part of this book - but this phenomenon I qualify as reproduction, not as a change in the precise sense explained above.) On the contrary, scientific theories and disciplines, on the one hand, as well as various non-scientific doctrines, in particular religious doctrines, on the other hand, do change and are usually supposed to do so: they are supposed to be progressively developed. The preserved identity underlying this change (development) is the identity of foundations of a given doctrine or scientific theory (Note 6). Older epistemological accounts took it for granted that proper foundations must be "firm" and rigidly fixed. They identified the lack of change with some form of rigidity, eternity or timelessness. But so they missed the crucial point concerning *how* identities of foundations (and hence identities of doctrines and theories dependent on these foundations) are constituted. What I told above about beliefs and theorems applies to foundations: foundations preserve their identity differently dependently of what they are foundations of. If we are talking about foundations of a religious doctrine then a word-by -word reproduction of chosen sacred texts usually counts as an appropriate way of preserving the identity of this doctrine. But science, on the contrary, preserves its identity through a continuous revision of its foundations. To perform such revision is the principle job of philosophy.

D) Consensus in science and philosophy

When I talk about the revision and renewal of foundations of science I don't assume that this job is done by the global community of thinkers working as a single mind in the universal historical time. Just like the very distinction between the past, the present and the future doesn't require any notion of universal cosmological or historical time but can be best made with the appeal to the individual experience, my talk of past, present and future foundations doesn't refer to the global historical time either. Nevertheless I am not going to take a Cartesian line and describe my proposed revision of foundations as a strictly individual project. For it appears me too obvious that philosophy and science alike are collective affairs. There are however important differences between the two cases. Although science certainly encourages discussions between proponents of conflicting approaches and opinions any controversy about scientific matters is supposed to be finally resolved by achieving a consensus solution. Only consensus solutions gain the status of ready-made scientific knowledge, which is worth to be transmitted through education to further generations. If the scientific community doesn't manage to achieve consensus on some issue this issue can be finally disqualified as non-scientific. This is why the global scientific community can be described as the community of people sharing consensus views about the core scientific knowledge. (Note 7) So at least at the first approximation it is not unreasonable to think about science as a product of a single collective mind.

However this view is obviously inappropriate for philosophy. It is a historical fact that philosophers never achieved consensus during the long history of their discipline; they even hardly ever reached an agreement about what falls under the scope of their discipline and what doesn't. (I'm sure that many philosophers will disagree with my views on this matter.) Were a global philosophical consensus ever occur philosophy would change its usual profile dramatically. But I don't think that this is something to be desired let alone realistically expected. Philosophical communities are unified by shared problems rather than shared solutions. This does not mean that philosophical problems are never solved but this means that they are never solved *finally*, once and for all. Philosophers produce multiple conflicting solutions, which never become the matter of global consensus and always leave a room for revision. Although consensus plays a role in the life of philosophical community too this role is less important for constituting the identity of such community. Although there is a sense in which a philosophical community can be called a collective mind this collective mind is not built through a consensus.

One may argue that a solution of a given problem which is not *final* doesn't deserve to be called a solution at all. I don't think that this severe restriction is justified. One reason to take philosophical problems and their tentative solutions seriously is that such solutions contribute to foundations of science (including pure mathematics). As I have argued the lack of philosophical consensus about foundations of science doesn't prevent scientists from achieving consensus about scientific matters proper. In fact the situation is more involved because foundations concern *both* philosophy and science, and so at this point the two models of intellectual activities - one appropriate for science and the other for philosophy - clash. On the philosophical side the issue of foundations remains as ever open. But on the scientific side certain foundations get widely accepted while some alternative proposals are left out of the mainstream. We can see this at the example of foundations of mathematics in the beginning of 20th century: while Hilbert's *Grundlagen der Geometrie* provided a widely accepted model of how a mathematical theory should look like and what kind of rigour it has to achieve, Brouwer's work on foundations of mathematics survived philosophically but didn't have any comparable effect on the mainstream mathematical research (see Episode 3). In order to treat this clash between science proper and philosophy one could distinguish between scientific and philosophical foundations. Although this distinction suggests itself in most of relevant contexts I don't think that it is really useful. For the differences between science and philosophy, which I stressed above, anyway keep these two kinds of intellectual activities apart while *foundations* of science as a *borderline* between the two domains allows them for fruitful interactions. So I don't see any advantage in dividing the border line between the two parties once again. Like in the case of political borders between countries it is more appropriate to think of foundations as a subject of joint responsibility of philosophy *and* science without trying to ignore further differences between them.

E) Philosophy as art of foundations

Let me resume. Science is a subject of progress, that is, of a permanent growth. The progress requires a transmission of the ready-made scientific knowledge through generations; this is the way in which scientific knowledge endures through time. The transmission of scientific knowledge doesn't reduce to repetition of pre-existing patterns but has a character of permanent reconstruction, revision and renewal. So the talk of ready-made knowledge must be always taken with a pinch of salt. Like any other human institution science either develops or degenerates or both but cannot be simply preserved for long in a frozen condition.

The progress and the renewal are two basic modes of development of any science. Although they are much interwoven the notion of *foundations* helps to keep them distinct: ideally, foundations are a subject of renewal but not of progress while the rest is a subject of progress but not of renewal. Saying that there is no progress in foundations doesn't mean that foundations remain rigidly fixed. A continuous renewal of foundations constitutes their identity; the specific character of this renewal (which can be described as *radical*) determines our notion of science. The renewal of foundations of science is the principle business of a theoretical discipline called philosophy.

The above remarks explain why philosophy doesn't perform any visible progress during its history. Let me now explain why in spite of this fact philosophy contributes to the progress of science. Having no notion of foundations in hands and no special discipline taking care about foundations people would develop the whole of science through a continuous renewal rather than progress: new views would permanently replace older views, innovative trends would struggle with conservative ones like in any other domain of public opinion, but there would be no neutral background for measuring possible progress. But since one introduces the notion of foundations as a "point of renewal" the situation changes. Then the "mainstream science" deliberately leaves philosophical disagreements about its foundations out of its scope and achieves in its proper domain a reasonable consensus allowing one to describe the overall development of science as a continuing progress. The philosophical neutrality of science doesn't mean that scientists shouldn't care about philosophical issues but it means that they should very carefully distinguish between general philosophical and specific scientific arguments in each particular context. Although I promote a strong cooperation between philosophy and science I wholly approve on this requirement. Although this present book contains some mathematical material I definitely qualify it as philosophical, not as mathematical. Nevertheless I hope it can be useful for people studying mathematics, and particularly for those studying mathematical subjects treated in this book.

Foundations present themselves in three well-distinguishable aspects: educational, conceptual and systematic. From the educational viewpoint foundations are basic contents of a given discipline from which one starts its learning. From the point of view of conceptual analysis foundations are primitive blocks out which develop all further theoretical constructions. Finally, from the systematic viewpoint foundations are specific contents binding other relevant contents into a whole (i.e. into a system) and thus providing scientific theories and disciplines with their proper identities. As we have seen these three aspects are closely related, so there is good reason to consider them together rather then apart.

F) Plan of this book

The principle purpose of this book is to present and discuss foundations of mathematics using the notion of recurrent foundations described above. Since recurrent foundations are essentially dynamic and historically-laden I begin with a Brief History of the subject. The first historical part of the book is organised around four mathematical texts, which I use for dividing the history of mathematical foundations into four major Episodes. These texts are the following: Euclid's *Elements* written (or better to say composed) about 300 B.C., *Nouveaux*

Elements de Geometrie by Antoine Arnauld first published in 1667 , *Grundlagen der Geometrie* by David Hilbert first published in 1899, and the first volume of *the Elements of Mathematic* (sic!) by Nicolas Bourbaki first published in 1939. All the four works qualify as Foundations in the sense explained in this Introduction. There is a very simple historical connection between them: the latter three are written after the example of the first and in this sense belong to "Euclidean tradition" broadly conceived. However in all these cases following Euclid's example involves a radical departure from it, which stands in a sharp contrast with various attempts to "improve" or "correct" Euclid, which continued until the end of 19th century. This common feature of Arnauld's, Hilbert's and Bourbaki's works explains why I have chosen them for my story: they best show how foundations undergo and survive radical revision, which according to my view is the very essence of foundations. Since all the three works are purely mathematical while my task is mostly philosophical I shall provide in each case appropriate philosophical contexts. By "appropriate contexts" I mean here *historically* relevant contexts, which make explicit philosophical (i.e. pre-scientific) thinking behind each of the four versions of mathematical foundations. In the case of *Nouveaux Elements* by Arnauld, who was a Cartesian looking for application of Cartesian "method" in sciences, the identification of the wanted philosophical context is obvious; in the other two cases the task is more problematic but also realisable, as we shall see. Even if Euclid, Hilbert and Bourbaki unlike Arnauld didn't have any precise philosophical agenda in their minds their writings were involved in contemporary philosophical discussions, which provide us the needed context. As I have already stressed in the last paragraph of the Introduction my view on history of mathematical foundations is not supposed to grasp the whole subject. Although in addition to the principles sources just mentioned I shall touch upon some other (in particular, talking about Arnauld's *Elements* I cannot avoid considering Descartes' *Geometry* as well) my choice remain limited and guided by my specific purpose, which I reveal in the second part of this book.

While the First Part of this book treats the past the second part treats the future. In this second part I describe and put further forward the ongoing programme of *Categorical* foundations of mathematics, that is, foundations based on Category theory. It seems me evident that Category theory plays in today's mathematics the role similar to that played by Set theory about a century ago and deserves a philosophical treatment similar to that once given to Set theory. In what follows I shall systematically introduce basic concepts of Category theory, discuss their foundational significance and point to some consequences to some other areas of philosophy including philosophy of language and philosophy of science. This discussion involves a significant revision of current views concerning relationships between mathematics and logic, the role of mathematics in natural sciences and some others issues. I conclude with a discussion concerning the role of foundations in mathematical education and arguments in favour of historical approach.

The critical reader might say that this plan, which includes a historical discussion and discussion of prospective future of foundations lacks any systematic discussion of foundations *per se*. What this reader might look for is a systematic *presentation* of foundations of mathematics independent of any historical issues and of any wishful thinking about the future. To respond to this reader let me first make the following metaphysical claim without going to a longer argument: the *present* is *wholly* analysed into (memory about) the past and (anticipation of) the future without any remainder like Aristotle's timeless *now*, which could be identified with the present *per se*. What the present involves over and above the past and the future is a *synthesis* these two things, which is cancelled by the aforementioned analysis. True, only by remembering past foundations and anticipating future foundations one doesn't yet produce any *presentation* of foundations. However a proper synthesis of the two things can make it. To achieve such a synthesis is the principle aim of this book. Let me however stress again that talking about the past and the future I don't mean here the past and the future of the universal history. Although Category theory since its invention in the mid-20th century becomes progressively more important as a common language used in different domains of mathematics and my bet is that Categorical foundations will become more popular in the short historical future, strictly speaking, I talk only about one possible future, which I want to promote and contribute to making it real, but obviously cannot predict what is going to happen.

Anyway when we are talking about Category theory the above metaphysical generalities get a clear pragmatic meaning. What prevents Category theory from entering the mainstream philosophical discussion on foundations of mathematics is apparently not a mere lack of interest but rather the lack of understanding how the new agenda relates to older agendas, including those yet considered by many philosophers as having a potential for further development. As a result the minority of philosophers promoting Category theory often looks in eyes of their colleagues as a team of preachers of a new religion insensitive to arguments of others. Thus to introduce the issue of categorical foundations in a broader historico-philosophical context and establish in this way a true dialogue between categorical foundations and different foundational programmes seems me an important task, which I try to realise in this book. However useful "timeless" accounts on foundations (like Euclid's, Hilbert's or, say, Russell's in his *Principles of Mathematics*) can be, I deliberately construe my account as heavily historically-laden. I believe that it is a historical blindness to bring into a serious philosophical discussion only a relatively recent past - say, only works written in 20th century - disregarding the rest as a barbarian pre-history or leaving it to pure historians who have no ambition to judge about today's state of affairs. Philosophy and mathematics are both old disciplines worth to be conceived in their historical integrity. Their further survival and their future shape crucially depend of how we today projecting this past history into the future. Surely one can simplify the task by cutting the past but the result will be poor: instead of endurable (i.e. revisable) foundations one will get at best just another intellectual fashion, which will parish soon without leaving any significant trace.

Part I. The Past: a brief history of foundations

Unlike physics, biology, psychology and many other sciences mathematics is at least as old as philosophy and arguably much older. It has emerged (as a practice if not as a science) in all known civilisations for obvious practical needs. It is amazing how mathematics managed to preserve its identity across all geographical and historical barriers since a very early period of the history of the humankind. Taking an old text which looks like a philosophical speculation a historian is often embarrassed by hermeneutic problems as to whether or not the given text can be indeed qualified as philosophical in anything like today's sense. But mathematical writings are usually identified as such easily (except only few interesting dubious cases). Whenever people count and calculate this uncontroversially qualifies as a piece of mathematics. For this reason mathematics has a longer history of co-existence and interaction with philosophy than do most of other sciences. The integral impact of this interaction is difficult to overestimate.

The lost history of mathematics written by Aristotle's pupil Eudemus of Rhodes (lived about 370-300 B.C.) and retold us by Proclus (5th century B.C.) in his *Commentary of The First Book of Euclid's Elements* (see Proclus 1873 and Proclus 1970) mentions among other early achievements of Greek mathematics Thales' (624-546 B.C.) proof of the fact that the diameter of a given circle cuts it into two equal halves. We know nearly to nothing about the argument offered by Thales but the very problem is telling. Having only very elementary geometrical notions like circle, square and fraction in hand one may put mathematically non-trivial questions like this one: Which part of a given square takes the circle inscribed in this square? But Thales' problem about the circle is not one of them. It is hard to imagine that many of early mathematicians really doubted that the proposition in question was true. Most likely Thales himself made it problematic and then suggested a proof. He questioned and revised what appeared to be obvious instead of "going further" as many others would do. This was a genuinely philosophical move (in the sense of the notion of philosophy described in the Introduction) albeit we may probably qualify today Thales' proof as mathematical. Actually this is one of earliest mathematical proofs we know about (even if we don't know *it*). It seems reasonable to suggest that the whole idea of proof, which revolutionised mathematics of the time, came out from this kind of philosophical inquiry about mathematics. That proofs also allow for a far-reaching *progress* in mathematics was likely realised only later.

Another famous early Greek proof, that of incommensurability of the diagonal of a square with its side, which the tradition attributes to Pythagoras, is also interesting in the present context. This time the question was mathematically non-trivial. But the tradition tells us that Pythagoras and his school viewed the problem as heavily metaphysically-laden: the related metaphysical problem was, roughly, whether or not "everything is number"; throughout classical philosophical literature (noticeably in Plato) one finds numerous references to the incommensurability of the diagonal with the side of a square used for justification of the idea that mathematics involves two different "first elements" irreducible to each other: number and magnitude. Thus a tricky geometrical problem and its solution contributed to reasoning about foundations.

These and some other similar historical examples suggest that the birth of theoretical mathematics as opposed to earlier practices of calculations and land planning (the ancestor of theoretical geometry) is due to the contact between these older practices and philosophical thought. Obviously neither mathematics nor philosophy existed at the time as established disciplines. This is why it is pointless to ask whether Thales and Pythagoras were philosophers, mathematicians or physicists. However in a historically short period (of about two centuries) Greek philosophy-laden mathematics reached maturity and gained its autonomy from philosophy. Euclid's *Elements* (= *Foundations*), which date back to 4th century BC and comprise principle achievements of earlier Greek mathematics is beyond any doubt *the* most influential mathematical text ever written. It is still read as a sound mathematical (rather than philosophical) text and some parts of its content are still included into standard mathematical textbooks in only a slightly modernised form. This fact looks fairly striking if we compare Euclid's *Elements* with Aristotle's *Physics* , *History of Animals* or any other historical scientific text dating back to about the same epoch. In the following section I shall discuss the Euclid's principle work with some more details.

I.1. Episode One: Euclid's *Elements*

First of all let's make it precise what historical document we are going now to discuss. This is less obvious than one might expect. Even if we leave apart the early history of the *Elements*, about which we anyway know very little, and look only at what has been published in Europe under the title of Euclid's *Elements* since the beginning of book printing (which covers a relatively small period of the 2300 years-long history of the document) we find a surprisingly diverse literature. Most of Euclid's publishers aimed at producing a sound mathematical textbook rather than an accurate reproduction of an older source. So they felt free to correct what they considered to be mistakes and flaws by providing new definitions, new axioms, new theorems and new proofs of old theorems, etc. They didn't even always feel obliged to preserve the principle composition of the source. This is moreover noticeable since many of Euclid's publishers like I. Barrow (edition of 1733) and C.L. Dodgson (also known under the name of Lewis Carrol, edition of 1875) took a conservative stand against new trends and produced their editions of the *Elements* as a secure alternative to new contemporary mathematical textbooks. So when they urged to return "back to Euclid" they meant the Euclid's spirit, not the Euclid's letter. The situation is even more involved since many old geometry textbooks, which don't mention Euclid's name in their titles, are still based on Euclid's *Elements*; this makes it apparently impossible to distinguish clearly between a modified version of the *Elements* and an original textbook based on the *Elements*. Comparing, for example, once popular *Elements of Geometry* first published by A. Tacquet in 1654 and the edition of Euclid's *Elements* (the first eight books thereof) published by M. Dechales 6 years later in 1660 it is difficult to say why the later work has Euclid's name in its title while the former doesn't. The difference between the two titles seems to be unrelated to the content of the two books although it might point to different intentions of their authors. When Tacquet's book was republished in 1725 (long after the authors death) it actually got Euclid's name on its cover! Thus stressing the long life of the *Elements* as a standard textbook one shouldn't forget that the identity of this ancient text until quite recently used to be understood very liberally.

Even if some earlier editors of Euclid's *Elements* like J. Keill (edition of 1754) sincerely tried to restore the old text rather than to improve upon it, the view on the *Elements* as a historical

document, which should be judged from a historical rather than a purely mathematical viewpoint and which for this reason requires a careful interpretation rather than remedies, is relatively recent. A great philological work aiming at fixing the *urtext* of the *Elements* through comparing available manuscripts and tracing their history back to lost earlier sources has been made in the end of 19th - beginning of 20th century by J.L. Heiberg and his assistant H. Menge. Noticeably these people were not mathematicians but historians and philologists. Their edition of the original text of the *Elements* (1883-1885) and its English translation by Th. Heath (1908), which follows Heiberg's urtext as closely as possible, until today remain standard references. All the later translations of the *Elements* into English and other modern languages equally rely on Heiberg's urtext and apply a similar translating standard.
Modifications introduced into the *Elements* during the long history of this document shed a light on the history of foundations. But here I mention them for a different purpose, namely for comparing different treatments of the classical text. From a historian's point of view the way in which the *Elements* were usually treated until 20th century looks barbarian, since earlier editors of this document didn't properly distinguish between the original source from later amendments; often they didn't have this distinction in their minds at all. However in eyes of a mathematician having little or no interest to history this cavalry's attitude may look quite justified. Moreover he or she may argue that the historian's approach simply misses the genuine mathematical content of the *Elements* and instead pays too much attention to superficial details. A typical revisor of Euclid's *Elements* would say he doesn't *change* the mathematical content of this text but tries to express it in a better manner and correct Euclid's errors. It is easy to resolve this controversy by saying that historians and mathematicians have different research interests. In my view, however, it deserves a different solution.
The notion of *recurrent foundations* described in the above Introduction partly justifies the traditional cavalry's approach of mathematicians to older sources and at the same time meets principle concerns of historians. For this notion takes the *revision* of foundations to be their essence. It removes foundations from a hypothetical timeless realm and brings them into the timely realm of human intellectual history; at the same time it wholly justifies the usual mathematicians' eagerness to modernisation and encourages one to think more seriously about the future than about the past. The permanent revision of Euclid's *Elements* during last few centuries shows that the notion of recurrent foundations has more traditional features than one might probably expect. However this example also highlights its non-traditional side. A naive Platonic view on mathematics suggests that improvements made on foundations bring them closer to a hypothetical perfect form. It seems that at least a part of Euclid's revisors had

something like this idea in their minds. But whether this notion of perfect foundations is conceived as a realistic goal or only as a regulative ideal it is obviously in odds with the notion of recurrent foundations explained above. So *this* aspect of the traditional view the notion of recurrent foundations doesn't approve. Actually I can see no *historical* reason to approve on it either. For in the real history foundations of mathematics never converged to a stable form. After all the attempts to find the best possible formulation of the mathematical content of Euclid's *Elements* this content is no longer seen today as satisfactory foundations (Note 8). The post-Euclidean history of foundations doesn't demonstrate any global convergence either, as we shall shortly see.

Having the notion of recurrent foundations in mind I shall treat Euclid's *Element* as follows. I shall use Heiberg's urtext of the *Elements* and try to avoid its anachronistic interpretations as the common standard of historical research requires it. However I shall not try to reconstruct the outdated mathematics of the *Elements* on its own rights but choose for my analysis only those of its features which seem me relevant to foundations of mathematics today. It might sound paradoxical but such relevant features are mostly archaic features swept away by later modifications. In fact there is nothing paradoxical here. To extract from the *Elements* its purely mathematical (as distinguished from historical) content means basically to translate the theory of the *Elements* into the language of today's mathematics. As a result of this translation one gets a sound piece of elementary mathematics, which however no longer qualifies as foundations. At the same time the basic architecture of the *Elements* and some philosophical ideas *behind* the mathematics of the *Elements* may be still considered for a new use (in a properly modified form). As I have already mention the analysis of older mathematical foundations, which I am going to suggest, requires some historico-philosophical reconstructions. In the next two paragraphs I provide basics of philosophy of mathematics of Plato and Aristotle, which will provide us with two different views on the *Elements*. In addition to texts of the two classical philosophers just mentioned I shall extensively use the *Commentary on the First Book of Euclid's Elements* (hereafter referred as *Commentary*) written by Proclus in 5th century A.D. from a Neo-Platonic perspective. I opt for Plato and Aristotle for three obvious reasons. The first reason is their historical relevance: what mathematics discussed by these two philosophers can be identified with Euclid's mathematics with a degree of historical precision sufficient for our purpose. (Remind that the *Elements* is not a fully original work but a systematic presentation of earlier results.) The second reason is the impressive amount of available original writings, which allows for a sound reconstruction of views of these two authors.. The third reason is related to the second: the two authors greatly influenced the following tradition. My view on Proclus, who lived about eight centuries later than Plato and Aristotle is different. I don't consider his *Commentary* as an authentic source of philosophy of mathematics of Euclid's epoch but rather see him as a colleague doing a job similar to my own. From a historical viewpoint Proclus' *Commentary* is very valuable because working on it this author had access to some important sources, which are no longer available. The limits of this book don't allow me to bring into this discussion further philosophical sources relevant to Euclid's mathematics. It also goes without saying the short historico-philosophical summaries found in this book shouldn't be considered as systematic presentations of philosophical views of the authors they mention.

Section 1.1. Plato's philosophy of mathematics

A) Basics

According to Plato the "highest" and actually the only "true" science is *dialectics* (Plato's name for philosophy), which deals with eternal *ideas.* It is this eternal and immutable nature of *ideas*, which makes science possible and guaranties the eternity and the immutability of scientific truths. At the lowest epistemic level Plato places various kinds of practical knowledge and technical skills dealing with the ever-changing material world. This latter kind of knowledge is doomed to be ever-changing itself because such is its subject-matter. Epistemic capacities related to the two kinds of knowledge Plato calls, correspondingly, *reason* (dianoia) and *opinion* (doxa). From the ontological viewpoint the domains dealt with by dialectic and by the practical knowledge are distinguished as *Being* and *Becoming* . While the former domain is accessible only by the pure reason the latter is accessible through the sensual experience. The two domains are not independent: "becoming" material stuff *partakes* and *mimics* their corresponding ever existing ideas. Think about a potter who tries to achieve the best possible match between his material production and the ideal pattern he has in mind beforehand. Similarly the Demiurg of Plato's *Timaeos* makes up the material world looking at its pre-existing ideal prototype. This asymmetric relation between Being and Becoming provides a sense in which the former determines the later and, correspondingly, in which a pure theoretical reasoning about what there *is* (i.e. about ideas) supersedes any practical argument. (Note 9)

B) Intermediate status of mathematics

Today's popular term *Mathematical Platonism* suggests that Plato viewed mathematics as a part of dialectics and took mathematical objects to be eternal ideas. However this is plainly wrong. In fact today's Mathematical Platonism has very little to do with Plato's own philosophy of mathematics and with the historical Platonism, i.e. philosophy developed in Platonic schools of late Antiquity (Note 10). According to Plato mathematics doesn't enter into either of the two ontological domains (Being and Becoming) and for this reason cannot be straightforwardly qualified either as a science or as a practical skill. It appears to be *intermediate* between the two worlds. The fact that mathematics turns to be a problematic case doesn't make it less important for Plato but on the contrary makes it central for his thinking. Consider the following two passages.

(Socrates talks to Gaucon)

"[Socrates:] - Next proceed to consider the manner in which the sphere of the intellectual is to be divided.

- In what manner?

- Thus: --There are two subdivisions, in the lower or which the soul uses the figures given by the former division as images; the enquiry can only be hypothetical, and instead of going upwards to a principle descends to the other end; in the higher of the two, the soul passes out of hypotheses, and goes up to a principle which is above hypotheses, making no use of images as in the former case, but proceeding only in and through the ideas themselves.

- I do not quite understand your meaning, he said.

- Then I will try again; you will understand me better when I have made some preliminary remarks. You are aware that students of geometry, arithmetic, and the kindred sciences assume the odd and the even and the figures and three kinds of angles and the like in their several branches of science; these are their hypotheses, which they and everybody are supposed to know, and therefore they do not deign to give any account of them either to themselves or others; but they begin with them, and go on until they arrive at last, and in a consistent manner, at their conclusion?

- Yes, he said, I know.

- And do you not know also that although they make use of the visible forms and reason about them, they are thinking not of these, but of the ideals which they resemble; not of the figures which they draw, but of the absolute square and the absolute diameter, and so on --the forms which they draw or make, and which have shadows and reflections in water of their own, are converted by them into images, but they are really seeking to behold the things themselves, which can only be seen with the eye of the mind?

- That is true.

- And of this kind I spoke as the intelligible, although in the search after it the soul is compelled to use hypotheses; not ascending to a first principle, because she is unable to rise above the region of hypothesis, but employing the objects of which the shadows below are resemblances in their turn as images, they having in relation to the shadows and reflections of them a greater distinctness, and therefore a higher value.

- I understand, he said, that you are speaking of the province of geometry and the sister arts.

- And when I speak of the other division of the intelligible, you will understand me to speak of that other sort of knowledge which reason herself attains by the power of dialectic, using the hypotheses not as first principles, but only as hypotheses --that is to say, as steps and points of departure into a world which is above hypotheses, in order that she may soar beyond them to the first principle of the whole; and clinging to this and then to that which depends on this, by successive steps she descends again without the aid of any sensible object, from ideas, through ideas, and in ideas she ends.

- I understand you, he replied; not perfectly, for you seem to me to be describing a task which is really tremendous; but, at any rate, I understand you to say that knowledge and being, which the science of dialectic contemplates, are clearer than the notions of the arts, as they are termed, which proceed from hypotheses only: these are also contemplated by the understanding, and not by the senses: yet, because they start from hypotheses and do not ascend to a principle, those who contemplate them appear to you not to exercise the higher

reason upon them, although when a first principle is added to them they are cognizable by the higher reason. And the habit which is concerned with geometry and the cognate sciences I suppose that you would term understanding and not reason, as being intermediate between opinion and reason."
(*Rep.*, 510b-511d, B. Jowett's translation)

"[Socrates:] - <...> as to the mathematical sciences which, as we were saying, have some apprehension of true being --geometry and the like --they only dream about being, but never can they behold the waking reality so long as they leave the hypotheses which they use unexamined, and are unable to give an account of them. For when a man knows not his own first principle, and when the conclusion and intermediate steps are also constructed out of he knows not what, how can he imagine that such a fabric of convention can ever become science?

- Impossible, he said.

- Then dialectic, and dialectic alone, goes directly to the first principle and is the only science which does away with hypotheses in order to make her ground secure; the eye of the soul, which is literally buried in an outlandish slough, is by her gentle aid lifted upwards; and she uses as handmaids and helpers in the work of conversion, the sciences which we have been discussing. Custom terms them sciences, but they ought to have some other name, implying greater clearness than opinion and less clearness than science: and this, in our previous sketch, was called understanding. But why should we dispute about names when we have realities of such importance to consider?"
(*Rep.*, 533b-d, B. Jowett's translation)

There is a number of points to be made about these passages:

1) The principle difference between mathematics, on the one hand, and dialectics, on the other hand, concerns two different treatments of hypotheses. While mathematics takes appropriate hypotheses for granted and then "descends" to certain conclusions dialectics moves in the opposite direction and purports to "ascend" from given hypotheses to some absolute non-hypothetical principles and so to "do away" with the hypotheses.

As examples of hypotheses Plato mentions in the first quoted passage "the odd and the even and the figures and three kinds of angles and the like". This and other contexts found in Plato suggest that one shouldn't understand *hypotheses* mentioned by Plato as (or at least only as) certain *propositions* , i.e. some basic *truths*, taken for granted. Comparing Plato's examples with the content of Euclid's *Elements* one finds that Plato's *hypotheses* correspond to Euclid's Definition*s* rather than Axioms or Postulates. This shows that Plato's *hypotheses* can be better understood as basic concepts rather than basic propositions.

2) Plato distinguishes mathematics both from practical "arts" and from dialectics and relates mathematics to a special epistemic capacity of *understanding* (dianoia), which intermediates between dialectical *reason* and practical *opinion* (Note 11). Remarkably this general scheme doesn't reserve any special room for natural science. See D) for a further discussion.

3) Dialectical reasoning is expressed in dialogues, and first of all in oral discussion rather written texts (Note 12). This shows that Plato's eternal ideas are not supposed to be matched by fixed linguistic patterns. Thus "two times two equals four" wouldn't be for Plato an example of eternal truth about eternal entities as one might expect. Plato would qualify the "operational" aspect of "two times two equal four", that is, the notion of "producing" the number four by multiplication, as related to Becoming, and he would insist that this operational aspect is essential for mathematical thinking.

4) Notice Plato's critique of geometrical reasoning (or more precisely geometrical *understanding*) aimed against using *images*. This critique follows from a more general notion, according to which *opinion* relies entirely on senses, *reason* operates with pure ideas without any help of sensual representations while mathematical understanding in general and geometrical understanding in particular do something in between. Geometry demonstrates this double nature of mathematics in the most explicit form. Thus Plato's critique amounts to pushing mathematical understanding from the domain of opinion toward a dialectical pure reasoning.
It is amazing how this Platonic thinking about mathematical matters could survive through centuries in spite of the fact that Platonic philosophy for long lost its direct appeal. As we shall see a major concern of people working on foundations of mathematics in the end of 19th - beginning of 20th century was to get rid with any epistemic reliance on imagery and intuition. Although these people justified this move differently the resemblance is just too

striking to be explained by a mere coincidence. This is just one reason why a serious historical analysis of the impact of Platonism (by which I mean the long-term intellectual tradition originated from historical Plato but not a branch of philosophy of mathematics developed in Analytic tradition of 20th century) on mathematics is important.

5) As we have seen Plato's epistemological notions have precise ontological counterparts: *reason* corresponds to the domain of *ideas*, otherwise referred to as Being while *opinion* corresponds to material *Becoming*. Although one doesn't find in Plato's dialogues a systematic ontological account corresponding to mathematical *understanding*, that is, a systematic ontology of mathematical objects, it has been later developed in Plato's circle. Such ontology is referred to and severely criticised in Aristotle's *Metaphysics.* In particular, Aristotle describes the notion of "intellectual matter" which is a counterpart of sensual matter relevant to mathematical objects (see *Met.* 1031a).
Particularly interesting is the notion according to which mathematical objects unlike their ideal prototypes exist in an indefinite number of copies (*Met.* 987b). According to this account there is an indefinite number of copies of mathematical number 2 (i.e. an indefinitely many of such numbers) all of which correspond to the same *ideal* number 2. A related difference between ideal and usual mathematical numbers is that the former unlike the latter cannot be a subject of arithmetical operations; this in particular implies that ideal numbers unlike mathematical ones cannot be thought of as sums of units and so are "indivisible" (*Met.* 1081a-1082b). This shows that the intermediate nature of mathematical objects is indeed essential: if one follows Plato's advise and "ascends" from mathematical objects to their ideal prototypes one certainly stops doing mathematics!
Plato's view, according to which material and mathematical objects are "imperfect copies" of their ideas has interesting implications concerning the issue of identity of mathematical objects. There are multiple passages in Plato where he speaks of "*X* itself", "*X* (thought of) through itself" (kath'autoo) and "Idea of *X*" interchangeably or explains the latter through the former. For example in *Symposium* 210-211 Plato does this with the notion of Beauty, and in *Phedon* 96-103 with number 2 (Note 13). I interpret these passages in the sense, which seems me straightforward: the notion of "being identical to itself" applies to ideas but not to material things, nor to mathematical objects. Material objects don't have identities at all (except of identities of their ideas): their mutual transformations form a Heraclitean flow where nothing ever remains the same (Note 14). Now, taking into consideration the aforementioned hints from Aristotle's *Metaphysics*, it seems reasonable to treat the intermediate case of

mathematical objects as follows: although these things don't allow for the strict identity they allow for its weaker version which is *equality*. In other words (on the Platonic account) mathematical objects are determined *up to equality* but not up to strict identity. On this account judgements of the form *A is B* , which describe how things *are*, are available only in dialectics, while mathematics provides judgements (if this name is still appropriate) of the form *A equals to B* .

C) Quadrivium

I conclude this brief sketch of Plato's philosophy of mathematics with Plato's classification of mathematical disciplines, which he provides in *Republic*, ch.7. Plato distinguishes four mathematical disciplines: arithmetic, geometry, music (by which he means a mathematical theory of musical harmony otherwise called *harmonics*) and astronomy. This quadruple (given the name of *Quadrivium* in the Scholastic tradition) is structured in the following way. Since *arithmetic* doesn't rely on images (let alone sensual perception) it is the closest to dialectics and hence the "highest" and the "purest" among the mathematical disciplines. *Geometry* is the next in the Platonic epistemic hierarchy. One way to justify this lower epistemic status of geometry with respect to arithmetic is to refer to its reliance to imagery. A different account of apparently the same issue is found in Aristotle's *Posterior Analytics* (Proclus in his *Commentary* attributes it to Pythagoreans): while the Unit is *the* basic concept (=foundation) of arithmetic *the* basic concept of geometry is Point, which can be described as a *positioned* (thetos) Unit (*An. Post.* 87a36). *Position* is conceived of here as a "degree of freedom", which makes mathematical reasoning (=understanding) less precise; it seems appropriate to describe this notion of *position* in the modern language as *spatial intuition*. Harmonics and astronomy, which Plato's calls "sisters", share in Quadrivium the same lower epistemic status. In modern terms one may describe them as two areas of *applied* mathematics, namely harmonics as applied arithmetic and astronomy as applied geometry. Plato is, of course, aware of the fact that material things, which are less noble than musical harmonies and celestial motions, also allow for a mathematical treatment. In *Republic* he stresses a practical utility of mathematics for affairs of the state. He grants the status of *sciences* only to harmonics and to astronomy (leaving now aside his reservation concerning dialectics) but not to any other "material application" of mathematics because these other applications aim at practical purposes and so leave one no chance to ascend to dialectics. Notice that geometry has in this Platonic hierarchy of mathematical sciences an intermediate position between arithmetic, on the one hand, and harmonics and astronomy (placed at the same epistemic level), on the other hand. Given the intermediate status of mathematics itself (with respect to dialectics and practical skill) the intermediate position of geometry makes it into the most *representative* (albeit not the *purest*) mathematical discipline. This is why to refer to mathematics in general Plato often says "geometry and the like".

D) Mathematical physics

One may wonder if Plato finds a place in his epistemolgy to anything like natural science. An answer can be found in Plato's dialogue *Timaeos*. In this dialogue Timaeos who is described as "the most of an astronomer" among the participants presents them a piece of mathematical physics and cosmology. Using various mathematical considerations he tells a story of creation of the material universe on the basis of its ideal prototype. In particular Timaeos associates regular polyhedra with physical "first elements" (Fire, Air, Water and Earth) and explains, to give a funny example, the usual sensual effect of fire by the fact that the associated tetrahedron is more picky than other polyhedra. Timaeos' story once again shows that Plato doesn't identify mathematical objects with eternal ideas; in this dialogue he rather blurs the boundary between mathematics and physical world. Noticeably before telling his story Timaeos makes a strong reservation warning his listeners that what he is going to deliver is nothing but a "probable tale". He explains that this low esteem of the delivered mathematical speculation about physical matters has nothing to do with his personal modesty but has a more profound reason: the chosen subject matter in principle doesn't allow for a rigorous account (Note 15). Although Plato's apparently enjoys Timaeos' mathematical tale his official view doesn't allow him to take it seriously.

Section1.2. Aristotle's philosophy of mathematics

A) Nature of things and their form

Aristotle is the first philosopher in the Western tradition who made a systematic effort to built an intellectual history and find his own place in it. In the historical account provided by Aristotle in the beginning of his *Metaphysics* the author traces the development of philosophical thought from the legendary times of the "seven wisemen" up to Plato. Much of Aristotle's own views are presented as a critical reply to this earlier tradition. I shall mention here only few of lines of Aristotle's thinking, which diverge from Plato and Platonism. Leaving aside Plato's dialectic as a philosophical business, which has little to do with science in anything like today's sense, one can reasonably identify Plato's notion of science with his

Quadrivium. On this account science reduces to pure and applied mathematics. Aristotle believes that this reduction is fundamentally erroneous because it misses what he calls (after some older thinkers) the *nature* (physis) of things. The following curious story told by Aristotle's *Physics* helps to understand his intuition.

"Antiphon points out that if you planted a bed and the rotting wood acquired the power of sending up a shoot, it would not be a bed that would come up, but wood - which shows that the arrangement in accordance with the rules of the art is merely an incidental attribute, whereas the real nature is the other, which, further, persists continuously through the process of making." (*Phys.* 193a12-17, translated by R.P. Hardie and R.K. Gaye)

Indeed, in spite of its aristocratic flavour one finds behind Plato's philosophy the intuition of a craftsman who has a definite idea of what he wants to make but never manages to realise his task perfectly. The visible world on Plato's account is a craftsman's work: remind the Demiurg from *Timaeos*. Aristotle, on the contrary, stresses the differences between natural and artificial things and argues that the former are far more powerful than the latter. The geometrical form of bed artificially given to a piece of wood unlike the proper *nature* of the wood has no reproductive power. So the funny observation due to Antiphon in Aristotle's eyes is an evidence against Platonic way of thinking, which would treat the wood as a passive matter and the geometrical form of the bed as a reflection of eternal ideas. It turns out that nature survives the form through the perpetual reproduction.
This difference between Plato's and Aristotle's views has far-reaching epistemological consequences, as we shall now see. In particular, it leads Aristotle to a very different understanding of mathematics.

B) Mathematical abstraction

To counter Plato's argument according to which one cannot possibly *reason* about sensible physical things Aristotle says the following:

"[C]learly it is possible that there should also be both propositions and demonstrations about sensible magnitudes, not however *qua* sensible but *qua* possessed of certain definite qualities. For as there are many propositions about things merely considered as in motion, apart from what each such thing is and from their accidents, and as it is not therefore necessary that there should be either a mobile separate from sensibles, or a distinct mobile entity in the sensibles,

so too in the case of mobiles there will be propositions and sciences, which treat them however not *qua* mobile but only *qua* bodies, or again only *qua* planes, or only *qua* lines, or *qua* divisibles, or *qua* indivisibles having position, or only *qua* indivisibles. Thus since it is true to say without qualification that not only things which are separable but also things which are inseparable exist (for instance, that mobiles exist), it is true also to say without qualification that the objects of mathematics exist, and with the character ascribed to them by mathematicians. And as it is true to say of the other sciences too, without qualification, that they deal with such and such a subject - not with what is accidental to it (e.g. not with the pale, if the healthy thing is pale, and the science has the healthy as its subject), but with that which is the subject of each science - with the healthy if it treats its object *qua* healthy, with man if *qua* man: - so too is it with geometry; if its subjects happen to be sensible, though it does not treat them *qua* sensible, the mathematical sciences will not for that reason be sciences of sensibles - nor, on the other hand, of other things separate from sensibles.
Many properties attach to things in virtue of their own nature as possessed of each such character; e.g. there are attributes peculiar to the animal *qua* female or *qua* male (yet there is no 'female' nor 'male' separate from animals); so that there are also attributes which belong to things merely as lengths or as planes. And in proportion as we are dealing with things which are prior in reason and simpler, our knowledge has more accuracy, i.e. simplicity. Therefore a science which abstracts from spatial magnitude is more precise than one which takes it into account; and a science is most precise if it abstracts from movement, but if it takes account of movement, it is most precise if it deals with the primary movement, for this is the simplest; and of this again uniform movement is the simplest form. <...>
Each question will be best investigated in this way - by setting up by an act of separation what is not separate, as the arithmetician and the geometer do. For a man *qua* man is one indivisible thing; and the arithmetician supposed one indivisible thing, and then considered whether any attribute belongs to a man *qua* indivisible. But the geometer treats him neither *qua* man nor *qua* indivisible, but as a solid. For evidently the properties which would have belonged to him even if perchance he had not been indivisible, can belong to him even apart from these attributes. Thus, then, geometers speak correctly; they talk about existing things, and their subjects do exist." (*Met*. 1077b16 - 1078a30, Ross' translation, corrected, bold mine)

There are several points to be made about this passage.

1) Unlike Plato Aristotle doesn't assume that epistemic boundaries between different branches of knowledge (and between different types of knowledge) reflect ontological distinctions between different kinds of entities. He doesn't assume that the subject-matter of any given branch of knowledge constitutes a "separate" ontological domain (Note 16). The mere fact that mathematicians successfully reason about numbers and magnitudes doesn't mean for Aristotle that numbers and magnitudes should be taken ontologically seriously. At the same it is not Aristotle's aim to disqualify the mathematicians' talk as merely fictious. Aristotle doesn't try to refute the realism of mathematicians about numbers and magnitudes but tries to make it less naive and more sound. He doesn't say that claims to the effect that numbers and magnitudes *exist* are plainly false but says that these claims are true *only* provided the meaning of "exist" is properly specified in the given context.

The idea that the verb "exist" has multiple context-dependent meanings is a cornerstone of Aristotle's philosophy. In the Platonic vein Aristotle distinguishes between the *prior* sense of "exist" (namely, exist as a *substance*) and a plethora of *secondary* senses of this verb. This theory is systematically presented by Aristotle in his *Categories*. As we shall shortly see the kind of existence, which Aristotle grants to mathematical objects is *not* prior. So there is a sense in which mathematical objects exist and there is a different sense in which they don't exist. Everything depends on the sense of "exist" taken into the account.

2) In various contexts (but not in the above passage) Aristotle distinguishes between things prior *for us* (hemin proteron) and things prior *by nature* (physei proteron). Here is an example. *For us* the notion of solid body is prior - in the sense that we acquire it in the early childhood through the uneducated sensual experience - while the notions of surface, line and point are acquired later through mathematical education. But *by nature* the ("real") order of things is the opposite: moving points generate lines, moving lines generate surfaces and moving surfaces generate solids. This is why in mathematical theories, which are supposed to represent the natural order of things adequately, points and lines are introduced before solids (like in the *Elements*). The order in which we *learn* about these things, i.e. the epistemological order, doesn't correspond to the ontological order in which these things naturally emerge.

The above example shows that the opposition *for us* versus *by nature* remains well compatible with Platonism. So in this case Aristotle uses new words to tell us the old Platonic story rather than proposes something really new. But Aristotle's opposition *prior in reason* versus *prior in essence* (proteron too logoo / proteron the ousia) referred to in the above passage (in the locution "... as we are dealing with things which are prior in reason and simple ...") definitely

goes beyond Platonism. Like the former opposition this latter opposition differentiates between what we *know* about things (or how we *reason* about them) and how things really *are* (i.e. their essence) (Note 17). This new way to distinguish between epistemological and ontological priority brakes with fundamental principles of Plato's philosophy. For it challenges the fundamental Platonic assumption according to which conceptual beauty, simplicity, coherence, precision and related qualities of *reasoning* are the best available evidences of what there is. As Aristotle puts it boldly in one place "not all things which are prior in reason are also prior in essence" (all'ou panta hosa too logoo protera kai the ousia protera: *Met.* 1077b1-2, my translation). Aristotle's message to Plato, Platonics and other funs of pure mathematics seems to be here the following: However clear, precise and beautiful your beloved mathematical theories might be they may fail to account for anything real; this is why pure mathematics provides no privileged access to what there is.
In the analysis of concrete examples the two pairs of oppositions (*for us* versus *by nature* and *in reasoning* versus *in essence*) lead to different results. In particular, it turns out that the order of generation of geometrical concepts (point --> line --> surface --> solid) corresponds to the order of "natural reasoning" but not to the *essential* order of things, i.e. not to the relevant ontological order. This essential order is reversed; in fact it coincides with the naive epistemological order *for us*: solid bodies are ontologically prior while surfaces, lines and points are ontologically dependent (i.e. their existence depends on that of bodies). For "we have no experience of anything that can be put together out of lines ot planes or points" (*Met.* 1077a34-35, Ross' translation). Aristotle's motivation behind this argument seems to be again physical (or rather biological): only bodies can be *animated* , i.e. can be living creatures (Note 18). Thus from this new perspective the theoretical order of concepts found in mathematics is no longer seen as an adequate presentation of the natural order of real things.

3) Let us now see what kind of existence Aristotle grants to mathematical objects. It is an *abstract* existence. Aristotle's notion of abstraction has two principle components. The first component is Aristotle's notion according to which any thing *A* , which has a certain feature *B*, can be conceived of *qua B* disregarding all of its other features. This epistemic operation is widely used by Aristotle in very different contexts. For example in *Metaphysics* Aristotle describes the subject-matter of his *first philosophy* (which today we call *ontology*) as *being qua being* (on he on). This means that the subject-matter of this science includes everything that there is *but* conceived only in the aspect of its existence disregarding anything else. This *qua*-operation, for which Aristotle has no special name, should not be confused with

abstraction. *First philosophy* in Aristotle's understanding, is *not* an abstract science, i.e. its subject-matter is not abstract. (In fact this is the *only* science, which is not abstract in this sense as we shall now see.) For Aristotle's notion abstraction involves a further step: after taking *A qua B* one "separates" *B* and conceives of *B* on its own rights as if *B* were a self-standing entity. Aristotle describes this step somewhat paradoxically as "setting up by an act of separation what is not separate". This further step can be also described as *hypostatisation*, which amounts to thinking of specific features of things distinguished with the *qua*-operation *as if* they were full-fledged self-standing entities. Aristotle's view, if I understand him correctly, is that this kind of thinking is excusable for a mathematician (and any scientist dealing with abstractions) but not for a philosopher, who pursues real things (Note 19). So, in Aristotle's view, it is up to philosopher rather than mathematician to explain what mathematics is *really* about.

4) *Mathematical* abstraction amounts to (i) taking things *qua* numbers (in arithmetic) or *qua* magnitudes (in geometry), *qua* moving points (in astronomy), etc, and (ii) hypostatisation of these features. There is a remarkable trade between precision and abstraction, stressed by Aristotle in the above passage. The more abstract is a given subject matter (i.e. the less is the number of features simultaneously taken into consideration) the more precise is the corresponding theory. This explains, in particular, why (and in which sense) arithmetic is more precise than geometry. However on Aristotle's account the more abstract implies the less real. Thus unlike Plato Aristotle doesn't think of theoretical precision as a direct evidence of truth about what there is. He rather thinks of it as one specific epistemic criterion competing with other epistemic criteria, which are equally important. It is easy to built a very precise theory about a very abstract subject-matter (i.e. about a subject matter, which comprises only very few features of real things) but such a theory will be of a little epistemic value because of its exceeding abstractness.

Aristotle's epistemological views on mathematics imply a new account of relationships between different mathematical disciplines, which turns Platonic Quadrivium upside down. Remind that in the Quadrivium the science of *astronomy* is given the lowest possible grade, which it shares with the science of *harmonics*. Aristotle, on the contrary, sees astronomy as a science, which achieves the best balance between mathematical precision and physical substantiality. This makes astronomy, by Aristotle's word "most akin to philosophy". As explains Aristotle "this science speculates about substance, which is perceptible but eternal, while the other mathematical sciences, i.e. arithmetic and geometry, treat of no substance"

(*Met*. 1073b5-7, Ross' translation). This is, of course, a polemic exaggeration. For, on the one hand, Aristotle's theory of abstraction still grants a limited substantiality to numbers and geometrical magnitudes (i.e. to subject-matters of arithmetic and geometry). And, on the other hand, it provides a sense in which the subject-matter of astronomy also qualifies as abstract. So the difference stressed by Aristotle in the above quote is rather a matter of degree. Nevertheless Aristotle's message seems to be clear: it is wrong to evaluate the epistemic value of a given theory taking into account only its precision, its substantiality must be also taken into the account. Judged by this double criterion astronomy gets a higher rank than geometry and arithmetic. Celestial bodies turn to be "more real" than numbers and geometrical magnitudes (we shall shortly see more precisely why).

5) It must be stressed that Aristotle's notion of abstraction is *not* specific for mathematics but provides a universal epistemic mechanism of constitution of the subject-matter of any special science. Notice, in particular, the example of *the healthy* in the above passage. Aristotle goes as far as claiming that abstraction (i.e. "setting up by an act of separation what is not separate") is *the* way in which "each question will be best investigated" - and this obviously concerns *not* only mathematical questions. This is why the subject-matter of astronomy is also abstract: although celestial bodies are perceivable astronomy studies them not *qua* perceivable but *qua* moving and *qua* geometrical. Aristotle doesn't make it what distinguishes *mathematical* abstraction from different kinds of abstraction except saying that arithmetic treats things *qua* numbers, while geometry treats them *qua* magnitudes (and more specifically - *qua* lines and *qua* planes).

6) Although Aristotle approves on abstraction as an epistemic operation, which facilitates a scientific investigation, he warns us about its possible misuse. Consider Aristotle's example of *male* and *female* mentioned in the above passage. It may be developed in the following way. One can reasonably constitute a research area like *female studies* provided that it will study only *human* females or only female individuals of some other particular species. But the notion of *general* female studies, which is a science about females of all biological species, is absurd. This is in spite of the fact that the general notion of female makes a perfect sense and applies across the species. Such generality doesn't allow one to abstract the property of being a female from the underlying species and make it into a subject matter of a special research. Studying females *in general* one should still properly distinguish between their *genera* (in this case - biological species) and avoid any uncontrolled switching between the genera. Such

erroneous switching prohibited by Aristotle is known after him under the name of *metabasis*. I shall say more about Aristotle's notion of generality in 1.4E.

Even if Aristotle tries to justify what mathematicians are doing in their domain rather than dismiss the mathematical way of thinking he apparently has a suspicion that mathematics works like the general female studies just described. Why indeed one cannot abstract the property of being female and study it on its own rights but can abstract and study the property of being in such-and-such number or have such-and-such magnitude? What makes the difference? Aristotle doesn't give any definite answer to this question. His strategy is not to distinguish between appropriate and inappropriate abstraction on some general methodological grounds but rather always control abstraction through ontological considerations. As far as one properly distinguishes between what exists in the *primary* sense and what exists only in a *secondary* sense as a hypostasised property abstraction remains well-controlled. Mathematics may be just as unsound as the general female studies but it is harmless unless one takes numbers and magnitudes ontologically too seriously. What really approves on mathematics is astronomy, harmonics and similar disciplines (we may call them all by the modern name of *mathematical physics*), which take into their account not only mathematical properties of things but also these very things (substances) from which the mathematical properties are abstracted. This doesn't mean that mathematical physics unlike pure mathematics has a direct access to things but this means that mathematical physics has a mechanism of ontological control, which the pure mathematics lacks. While a pure mathematician typically doesn't care about sensible things behind his beloved mathematical abstractions an astronomer always has these sensible objects in his sight. While a pure mathematician can forget about abstraction and study mathematical objects *as if* they were real for an astronomer the mathematical (and physical) abstraction is a part of his job. In *this* sense astronomy is closer to philosophy than geometry and arithmetic as Aristotle states this. (Remind that on Plato's view it s arithmetic, which is the closest.)

The Aristotelian project of mathematical physics just outlined sounds very modern. In Aristotle's own works it remained underdeveloped: the above reconstruction is based almost exclusively on what Aristotle says about astronomy in his *Metaphysics* and *Posterior Analytics*. Aristotle principle means to control abstraction is different: in his view the most effective ontological control is provided not by the mathematical physics but by the *first philosophy* (i.e. ontology and metaphysics).

C) Mathematics, physics and logic: The Classical Model of Science

Although the notion of mathematical physics described above seems to be perfectly adequate to Aristotle's epistemological assumptions he apparently couldn't point to any other science except astronomy as a concrete realisation of this idea (Note 20). His principle epistemological project is different: to invent and justify a general notion of science such that physics and mathematics could be seen as its two special cases. Since mathematics in Aristotle's time was already well-developed while the contemporary physics could be hardly qualified as a theory in anything like the same sense (remind that Plato viewed it as a mixture of mythological tales and practical techniques) this Aristotle's project can be also described as an attempt to establish physics as a science on its own rights. To realise this ambitious project Aristotle invents and develops two new disciplines (in addition to physics). The first is his *first philosophy*, which can be described as a science of everything provided that the *everything* is viewed *only* in the aspect of its existence. This science distinguishes between abstract and *primary* entities, between substances and their essential (and accidental) properties, etc. The second discipline Aristotle calls *analytics*; it has two different parts, which roughly correspond to the content of the *Prior Analytics* and the content of the *Posterior Analytics*. The first part is known today under the name of *logic* (first used by Stoics) while the second can be identified as *epistemology*, i.e. a general theory of science. *First philosophy* and *analytics* are closely related: a part of their content, which today we qualify as logical (for example, the law of non-contradiction) Aristotle qualifies as ontological, i.e. as a description of how things generally *are*. Aristotle's ontology, logic and epistemology found in his *Metaphysics* and the two *Analytics* provide a general framework for doing physics, mathematics and any other science. It can be roughly described as follows:

a) Every particular science accounts for a certain domain of being and claims certain truths (true propositions) about entities belonging to this domain of being.

b) Scientific truths are divided into two classes: first (fundamental) truths taken for granted and secondary (derived) truths.

c) Derived truths are obtained from fundamental truths through logical inference. Fundamental truths are taken as premises of inferences and derived truths are obtained as conclusions of inferences.

d) Logical inferences are governed by laws of logic, which reflect general ontological principles and are universal for all sciences.

This model of science called in (Jong, R., Betti, A., forthcoming) *Classical* had a great influence throughout the history. It remains quite influential until today (in spite of the fact that today people don't usually understand by logic the old-fashioned Aristotelian syllogistics). It is moreover important to see that the above model of science is not unique. We have already seen that Platonism offers a very different view on science in general and on mathematics in particular. Curiously the philosophical view usually called today *Mathematical Platonism* , according to which mathematical objects exist beforehand rather than are created by mathematical activities, derives from the Aristotelian views on mathematics rather than from the Platonic views. For the Aristotelian views unlike the Platonic views leave no room for the idea that mathematical objects generate and develop; on the Aristotelian view they (in an appropriate sense) always exist. True, according to Aristotle, mathematical objects exist only *qua* abstract entities and this is not what today's Mathematical Platonists usually have in mind. However this part of the story has little or no impact on the mathematics itself: the Aristotelian model of science is universal and doesn't depend on the ontological status of entities studied by this or that science. So mathematicians, on the Aristotle's account, have no other choice but to take the existence of mathematical objects for granted and leave it to philosophers to specify the ontological status of mathematical objects. The Platonic notion of generation has, on the contrary, precise mathematical counterparts, as we shall now see.

Section1.3. Euclid via Plato

Plato's philosophy of mathematics outlined in I.1.1 turns to be impressively coherent with the content of Euclid's *Elements*. I don't think that this fact means that Euclid, about whom we know near to nothing, was a convinced Platonic and tried to implement Plato's philosophical doctrine in his mathematical work. It seems me more likely that it was rather Plato who developed a good deal of his philosophy through a critical reflection upon his contemporary mathematical tradition, which Euclid later inherited. In any event my aim here is not to develop a historical speculation about mutual influences of philosophy and mathematics in Antiquity but to understand the *Elements* properly. For obvious reason I cannot (and need not) analyse here the content of this work in full; I shall not even try to provide here a survey of this content. I shall discuss instead only some architectonic principles of the *Elements* , which in my view are relevant to the today's discussion on foundations of mathematics.

The *Elements* are foundations of mathematics but they also have foundations of their own, which I shall call *fundamentals* (Note 21). I mean Definitions, Postulates and Axioms, which have a bearing onto all the following Propositions and play a major role in organising the content of the *Elements* into a coherent whole. In what follows I shall analyse these three kinds of fundamentals one after another and then discuss some general issues concerning Propositions.

A) Definitions

The *Elements* consist of 13 Books, which contain in total 128 Definitions. Each Book starts with a relevant list of Definitions except Books 8, 9, 12 and 13, which use only Definitions from the preceding Books. Book 10 has two additional lists of Definitions introduced in the middle. Here are basic Euclid's geometrical (Book 1) and arithmetical (Book 7) Definitions (hereafter I use the recent translation of Euclid's *Elements* by Richard Fitzpatrick):

Definitions of Book 1:

D1.1. A point is that of which there is no part.

D1.2. And a line is a length without breadth.

D1.3. And the extremities of a line are points.

D1.4. A straight-line is whatever lies evenly with points upon itself.

D1.5. And a surface is that which has length and breadth alone.

D1.6. And the extremities of a surface are lines.

D1.7. A plane surface is whatever lies evenly with straight-lines upon itself.

D1.8. And a plane angle is the inclination of the lines, when two lines in a plane meet one another, and are not laid down straight-on with respect to one another.

D1.9. And when the lines containing the angle are straight then the angle is called rectilinear.

D1.10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called perpendicular to that upon which it stands.

D1.11. An obtuse angle is greater than a right-angle.

D1.12. And an acute angle is less than a right-angle.

D1.13. A boundary is that which is the extremity of something.

D1.14. A figure is that which is contained by some boundary or boundaries.

D1.15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from a single point lying inside the figure are equal to one another.
D1.16. And the point is called the center of the circle.
D1.17. And a diameter of the circle is any straight-line being drawn through the center, which is brought to an end in each direction by the circumference of the circle. And any such (straight-line) cuts the circle in half.
D1.18. And a semi-circle is the figure contained by the diameter and the circumference it cuts off. And the center of the semi-circle is the same (point) as (the center of) the circle.
D1.19. Rectilinear figures are those figures contained by straight-lines: trilateral figures being contained by three straight-lines, quadrilateral by four, and multilateral by more than four.
D1.20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.
D1.21. And further of the trilateral figures: (a) a right-angled triangle is that having a right-angle, (b) an obtuse-angled (triangle) that having an obtuse angle, and (c) an acute-angled (triangle) that having three acute angles.
D1.22. And of the quadrilateral figures: (a) a square is that which is right-angled and equilateral, (b) a rectangle that which is right-angled but not equilateral, (c) a rhombus that which is equilateral but not right-angled, and (d) a rhomboid that having opposite sides and angles equal to one an other which is neither right-angled nor equilateral. And (e) let quadrilateral figures besides these be called trapezia.
D1.23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).

Definitions of Book 7:
D7.1. A unit (or monad) is (that) according to which each existing (thing) is said (to be) one.
D7.2. And a number (is) a multitude composed of units (monads).
D7.3. A number is part of a(nother) number, the lesser of the greater, when it measures the greater.
D7.4. But (the lesser is) parts (of the greater) when it does not measure it.
D7.5. And the greater (number is) a multiple of the lesser when it is measured by the lesser.
D7.6. An even number is one (which can be) divided in half.

D7.7. And an odd number is one (which can) not (be) divided in half, or which differs from an even number by a unit.
D7.8. An even-times-even number is one (which is) measured by an even number according to an even number.
D7.9. And an even-times-odd number is one (which is) measured by an even number according to an odd number.
D7.10. And an odd-times-odd number is one (which is) measured by an odd number according to an odd number.
D7.11. A prime number is one (which is) measured by a unit alone.
D7.12. Numbers prime to one another are those (which are) measured by a unit alone as a common measure.
D7.13. A composite number is one (which is) measured by some number.
D7.14. And numbers composite to one another are those (which are) measured by some number as a common measure.
D7.15. A number is said to multiply a(nother) number when the (number being) multiplied is added (to itself) as many times as there are units in the former (number), and (thereby) some (other number) is produced.
D7.16. And when two numbers multiplying one another make some (other number) then the (number so) created is called plane, and its sides (are) the numbers which multiply one another.
D7.17. And when three numbers multiplying one another make some (other number) then the (number so) created is (called) solid, and its sides (are) the numbers which multiply one another.
D7.18. A square number is an equal times an equal, or (a plane number) contained by two equal numbers.
D7.19. And a cube (number) is an equal times an equal times an equal, or (a solid number) contained by three equal numbers.
D7.20. Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third (is) of the fourth.
D7.21. Similar plane and solid numbers are those having proportional sides.
D7.22. A perfect number is that which is equal to its own parts [= equal to the sum of its factors].

These Definitions can be classified into the following three types.

(i) "Technical" Definitions like D1.11, D1.23 and D7.22. A technical Definition specifies the intended meaning of a special mathematical term. These Definitions look like today's mathematical definitions and for this reason seem unproblematic.

(ii) "Philosophical" Definitions like D1.1 and D7.1. Definitions of this kind unlike Definitions of the former kind cannot be "used" in following mathematical proofs in anything like the usual sense.

(iii) Definitions, which look like technical but are not used in the following proofs. This, in particular, is the case of D1.22: terms *rhombus* and *rhomboid* nowhere appear in the *Elements* except this definition itself. Instead Euclid uses the notion of *parallelogrammic area* (first time in Proposition P1.34) or *parallelogram* without trying to define it (so apparently he takes this name as self-explaining).

I leave it to the reader to sort out the above list into the three suggested categories (minding dubious cases like D1.13 - D1.14) and discuss here from a Platonic perspective only a more general issue concerning the role of definitions in the *Elements*.

The question "what is *X*", to which a definition of *X* provides an answer, is a basic question asked in Plato's dialectics. For example, in *Symposium* Plato looks for "true" definition of Eros, in *Protagoras* - of virility, etc. According to Aristotle (*Met.* 1078b) Plato was actually the inventor of the method of definition employing appropriate genus and species. Taking this into consideration one may look at Euclid's mathematical definitions as a way to meet concerns about mathematical "hypotheses", which Plato expresses in his *Republic* (see the above quote), by applying the dialectical method of definition within mathematics. As I showed elsewhere (Rodin 2003) definitions of Book 1 (with an exception of D1.23) indeed form a precise Platonic hierarchy, which well explains the order of the definitions and the choice of their definienda. Here I shall demonstrate this only at the particular example of D1.22 in which Euclid classifies triangles.

This classification looks usual but in fact it slightly differs from the one we normally use today: on Euclid's account an equilateral (regular) triangle is *not* a special case of isosceles triangle and an isosceles triangle is *not* a special case of scalene triangle; in Euclid the three classes of triangles are disjoint. Euclid's three kinds of triangles are ordered by the "degree of regularity", not by their generality. For a suggestive analogy think about a potter classifying his pottery into the three categories: perfectly good, slightly distorted and heavily distorted. The today's usual convention according to which a regular triangle *is* isosceles would likely sound for Euclid as a sheer contradiction. D1.22 provides an example of Platonic hierarchical ordering, where further elements are conceived of as distorted images of preceding elements:

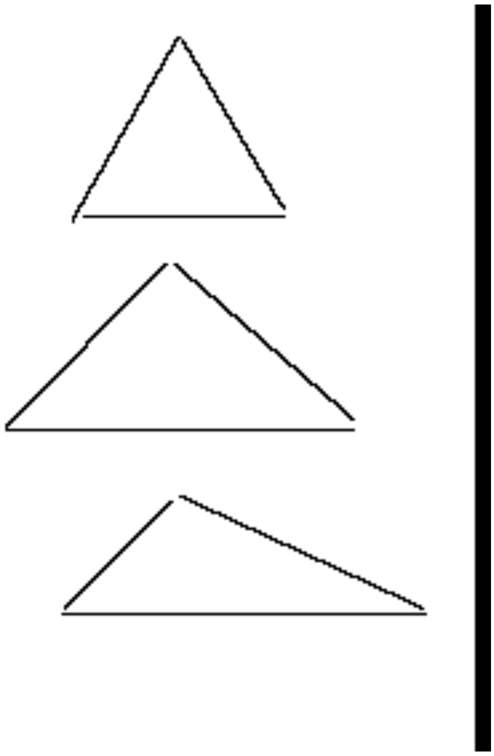


Fig. 1

The regular triangle in D1.22 is a *generic figure*, which generates the other two kinds of triangles through a progressive distortion. This specific generic principle underlies most of Euclid's definitions. The basic *geometrical* generator in the theory of the *Elements* is *point* (D1.1) while the basic *arithmetical* generator is *monad* or *unit* (D7.1). (Beware that by D7.1 and D7.2 the unit is *not* itself a number.) Given the aforementioned Platonic notion of point as a "positioned monad" one may consider the monad as a generator of the whole mathematical universe.

As we have seen the Platonic view on definitions doesn't reduce them to auxiliary devices serving for conjecturing and proof of propositions like we usually do this today. On the Platonic view mathematical definitions – and systems of definitions - are thought of as self-sustained pieces of mathematical knowledge. And actually this older view much better squares with Euclid's Definitions than the modern view. In particular, it explains why many of Euclid's definitions are left in the *Elements* without any further treatment (Note 22). As we shall see in 1.4A Aristotle's view on the role of definition is quite different.

B) Postulates and Axioms

Unlike the case of Definitions, which are spread over the 13 Books of the *Elements*, this treatise contains a unique list of Postulates and a unique list of Axioms; both lists are placed one after the other in the beginning of the Book 1. The main purpose of this section is to make clear the difference between Postulates and Axioms, which has been largely forgotten in modern mathematics but is quite important in Euclid. In the second part of this book I shall argue that this forgotten distinction is relevant to the today's discussion on foundations.
Here are the Postulates:

P1. Let it have been postulated to draw a straight-line from any point to any point.
P2. And to produce a finite straight-line continuously in a straight-line.
P3. And to draw a circle with any center and radius.
P4. And that all right-angles are equal to one another.
P5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angle (and do not meet on the other side).

Postulates P1-P3 describe the well-known rules of "construction by ruler and compass" (Note 23). The point I want to emphasise is that *prima facie* these Postulates are not propositions. They are not *truths* taken here for granted. They are not things to be evaluated as "true" or "false". If these three Postulates can be qualified as assertions at all they assert the following: it is *postulated*, (literally: *asked for*) to perform such-and-such *operation*. P1 requires to draw a (segment of) straight line between two given points, P2 requires to extend a given straight segment indefinitely beyond any of its two endpoints and P3 requires to draw a circle using a given straight segment. What these Postulates allow one to take for granted are these very operations described by verbs in the infinitive form. Clearly an operation is not something that can be evaluated as true or false.
The two key points allowing one to make a Platonic sense of P1-P3 are (i) Platonic fundamental ontological distinction between Being and Becoming and (ii) Platonic doctrine about the intermediate status of mathematics. As Proclus explains it in length in his *Commentary* mathematics in general and geometry in particular has this double, almost paradoxical nature: it partly belongs to the domain of Becoming (or Generation) and partly to the domain of Being. Although these two aspects of geometry cannot be separated from each

other one can nevertheless distinguish between Postulates as principles of "geometrical Becoming" (geometrical generation) and Axioms as principles of "geometrical Being". Any assertion that something *is* the case makes sense *only* if it concerns Being but not Becoming. Only such assertions can be evaluated as true or false. Postulates 1-3 describe what and how geometrical objects *generate* but not what and how they *are*. Thus Platonic philosophy of mathematics allows for making sense of P1-P3 without paraphrasing them into propositions. The reader may notice that P4 and P5 have a different character. P4 says that all right angles (see D1.10) *are* equal. The case of P5 is dubious. In principle P5 can be interpreted like P1-P3 as a description of an operation, namely of the construction of the intersection point of the two straight lines in question. However it reads more naturally (and more literally) as a description of a property of a ready-made construction rather than description of the operation bringing this construction about (Note 24).

Proclus resolves the difficulty of Platonic interpretation of Euclid's Postulates by claiming that P4 and P5 don't really qualify as Postulates and must be proven as theorems. He offers the reader a proof of P4 borrowed from "other commentators". This obvious proof is based on superposition and relies on Axioms 4,5; arguably it meets Euclid's usual standard of rigor. For P5 Proclus proposes his own proof, which is obviously mistaken (Note 25). It may be argued that Proclus simply cheats when he denies to qualify P4 and to P5 as postulates. But I don't think that this argument is justified. In the given case Proclus acts as a critic but not only as a commentator. He doesn't cheat against his source but make a suggestion of how to improve on it.

P1-P3 make more mathematically precise basic Platonic generic principles, which we already found in Definitions. Think of *point* as *the* fundamental generator of our geometrical universe. This unique Platonic "ideal point" is indistinguishable from a pure ideal unit. Then put some "geometrical matter" in it. This geometrical matter allows for distinguishing between *different* points. In other words (more appropriate to the Platonic way of thinking) - it allows points to be many. This multiplication of points can be conceived of in two different (mutually dependent) ways. First, one may boldly introduce (after Pythagoreans) the notion of "position" and claim that points differ by their positions. Second, one may think of change of point's position in terms of continuous motion. This gives one the notion of line as a trajectory of point moving from one fixed position to another (cf. D1.2, D1.3). To get P1 one should now choose among all possible trajectories the straight one. The fundamental Platonic assumption according to which any variety of things belonging to the same genus always contains a generator turns to be helpful again: think of a curvilinear motion as a disturbed rectilinear motion. Even if the principle of inertia just stated is anachronistic we can see that it perfectly squares with the Platonic way of concept-building (Note 26).

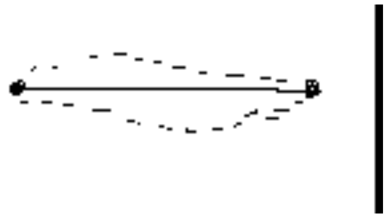


Fig.2

P2 follows from the idea to continue the rectilinear motion of a mobile point inertially instead of stopping at the previously decided position:

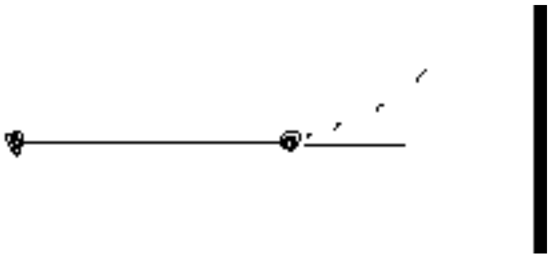


Fig.3

Finally P3 performs an alternative scenario which unlike P2 doesn't lead to infinite construction. Instead of continuing the rectilinear motion one turns the mobile point around the fix point and so gets a circle.

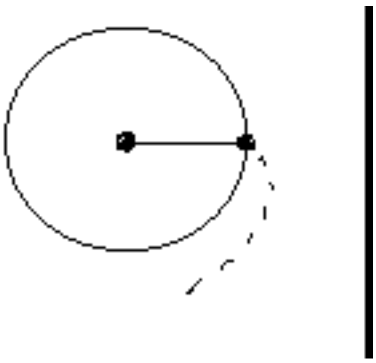


Fig.4

One finds in Proclus' *Commentary* a version of the same dialectics (Note 27). In the case of circle (P3 and D1.15, 16) it involves fundamental Neo-Platonic categories of Unity (monhe) represented here by the generic fix point, Progress (proodos) represented by the radius and

Return (epistrofhe) represented by the circumference. It is not my aim here to get deep into it. Instead I would like to stress that my interest to this issue is not purely historical. Notions of generator and genericity are abundant in today's mathematics. They can be precisely defined in many different settings including a category-theoretic one. In the second part of this book I shall show that the notion of genericity plays in new category-theoretic foundations a more important role than in standard set-theoretic foundations.
Let's now consider Axioms of the *Elements*. Euclid calls them *Common notions* (koinai ennoiai) but their identification as Axioms (axiomata) is clearly documented by Aristotle (see 1.4C below) and so is uncontroversial. Here they are:

A1. Things equal to the same thing are also equal to one another.
A2. And if equal things are added to equal things then the wholes are equal.
A3. And if equal things are subtracted from equal things then the remainders are equal.
A4. And things coinciding with one another are equal to one another.
A5. And the whole [is] greater than the part.

As we see unlike Postulates (or more precisely - unlike P1-P3) Axioms are propositions. From a Platonic viewpoint this is related to the fact that unlike Postulates Axioms describe mathematical Being (rather than Becoming). Syntactically the relevance of Axioms to Being is expressed by the fact that each Axiom contain the word *are* or *is* (Note 28). However Euclid's Axioms account for the "intermediate" *mathematical* Being, not for Being *simpliciter*. This is why A1-A4 involve not the basic non-qualified copula *is* but the specific "mathematical copula" *is equal to*. (See I.1.1B5 above where we discuss a Platonic interpretation of the notion of *equality*.) Taking into consideration that A5 immediately implies that *the whole is* not *equal to the part* we can claim that (under Platonic interpretation) the main purpose of Euclid's Axioms is to specify the notion of equality, which is a mathematical substitute of the notion of identity. Since equality is the basic distinguishing feature of mathematical Being as distinguished from Being *simpliciter* the Axioms tell us in general terms what the whole science of mathematics is about. In fact they say even more than that since they also tell us something about how mathematical Being relates to Being *simpliciter*.
A1 is the only axiom in the list, which involves the notion of identity in an explicit form: notice the word “same” used in this axiom. The interplay between identity and equality made explicit by A1 is worth a special analysis. Observe that identity is stronger than equality in the

following obvious sense: identical things are equal but equal things are generally not identical. (This is why by exchanging identity and equality in A1 one gets a false statement: things identical to equal things are identical.) Having this in mind one may think of A1 as a description of what happens when identity is weakened up to equality. Consider things a, b, c, ... equal to self-identical thing I : $a = I$, $b = I$, $c = I$, ... In Platonic terms this construction can be described as splitting of self-identical idea I into an indefinite number of mathematical "copies". A1 states that this splitting is uniform in the following sense: all the obtained copies turn to be mutually equal. This allows to think of the copies as defined "up to equality" rather than up to identity (Note 29).

Now observe that A2-A4 involve such notions as addition, subtraction and "putting into coincidence", which obviously belong to mathematical Becoming. Moreover taken at their face value these notions may also belong to the domain of sensual material experience (beware that addition and subtraction mentioned in A2 and A3 don't necessarily mean addition and subtraction of numbers). That is why they are so telling for people doing mathematics at all levels and in all ages. But A2-A4 put precise conditions under which these operations qualify as mathematical; each of these conditions involves the notion of equality. In this sense we can say that A2-A4 delimit the boundary between the domain of pure mathematics and the domain of sensual experience. Remarkably A2-A4 don't involve the notion of identity. Euclid doesn't say instead of A2

If equal things are added to the same thing then the wholes are equal

or

If the same thing is added to equal things then the wholes are equal

or

If the same thing is added to the same thing the whole is the same.

For the operation of addition (or "putting together") makes an obvious sense only in the domain of copies. Indeed we have no problems with the meaning of $a + b = c$ when a, b, c exist in an indefinite number of copies. In this case the addition of a and b doesn't affect a and b themselves: one gets a new entity c and still keeps a stock of copies of a and b, which can be eventually used for making up new copies of c or for some other purpose. It is less clear what happens when a, b, c are unique. Do a and b survive the addition and continue to exist as parts of c or they get destroyed by this operation? I shall not explore here this controversial metaphysical question but only mention that the standard Platonic answer to it is in negative:

self-identical entities, i.e. *ideas*, are indivisible; moreover they cannot be a subject of any operation like addition, subtraction and the like. Thus from a Platonic viewpoint the fact that Euclid avoid the notion of identity talking about operations is quite justified.

Let me now provide more concrete details, which show that the above Platonic reading of Euclid's Axioms is indeed well-grounded. A remarkable feature of Euclid's Axioms is that these Axioms don't mention explicitly any specific mathematical object but talk only about abstract "things". An analogy with Hilbert's *Foundations of Geometry* of 1899, where this latter author similarly refers to "systems of things" (see I.3 below), is tempting but rather misleading. While Hilbert's axiomatic theory of geometry is supposed to be *modelled* in a certain theoretic domain (or domains) Euclid's Axioms are supposed to be universal truths about mathematical objects. Remind that mathematical objects considered by Euclid are of two basic kinds: arithmetical objects (numbers) and geometrical objects. This provides one of two traditional explanations of Euclid's term *Common Notions*: arithmetic and geometry share Axioms in common. (The other traditional explanation, which in my view is less plausible, refers to the fact that the Axioms are allegedly supposed to be commonly known and commonly granted.) Let's now consider how A1-A5 work in arithmetic and geometry. As I have already mentioned the crucial notion involved into Euclid's Axioms is that of equality. In case of numbers we can take it (provisionally) for granted. Similarly we can take for granted elementary arithmetical operations like addition and subtraction. So at least at the first approximation the arithmetical meaning of A1-A3 is unproblematic. The question about an arithmetical meaning of A4 (if any) is, on the contrary, the most difficult. Another word for the notion of *coincidence* involved into this axiom is *congruence* (this latter term stems from the standard Latin translation of Euclid's "ta farmozonta ep allhla" by "quae inter se congruunt"). Today this term has a geometrical meaning, which is roughly the same as Euclid's (see below) and a number of more specific mathematical meanings (including arithmetical), none of which is suitable for interpreting A4. Thus in order to claim that A4 applies to arithmetic one needs to suggest an appropriate notion of congruence (coincidence) of numbers. D7.2 turns to be helpful for this end: this definition shows that Euclid's notion of number is actually not quite the same as today's. The definiendum of D7.2 can be translated into the modern terms as *finite set* of units (provided, of course, that we use the most naive notion of set possible) (Note 30). By congruence of numbers so conceived Euclid and his contemporaries might mean one-to-one correspondence between their units (elements of finite sets). This is moreover likely given the quasi-geometrical way in which Euclid treats numbers in arithmetical Books of his *Elements*: he systematically represents units by segments of

straight line and numbers by bigger segments composed of unit segments. Under this representation the arithmetical operation of subtraction requires the mutual application of straight lines (Note 31). Although the case of coincidence (congruence) is nowhere revoked by Euclid in this context it remains an obvious possibility. Proclus definitely considers A4 as universal (i.e. applicable both in geometry and arithmetic) on equal footing with other Euclid's axiom although he doesn't explain its arithmetical meaning (Note 32). Apparently he takes the arithmetical meaning of A4 to be obvious. The suggested interpretation of the definiendum of D7.2 as a finite set also sheds a light on A5. The part/whole relation doesn't seem to be immediately applicable to the modern notion of number. But it does so when after Euclid one thinks of numbers as sets of units. Beware that talking about parts Euclid always means *proper* parts. Notice also D7.3, which explicitly defines the notion of *part* in the case of numbers. This definition provides the term *part* with a more restrictive meaning than one could expect. What Euclid calls a *part* of a given number is sometimes called today an *aliquote* part. However this restriction doesn't make A5 false in the domain of numbers.
Let me now clarify the geometrical meaning of Euclid's axioms. The notion of congruence in this case can be taken for granted. But beware that on Euclid's account the case of congruent geometrical objects is nothing but a special case of *equal* geometrical objects. In order to see what Euclid means by equality of geometrical objects one needs to consider how A2-A3 are used in his geometrical proofs. In the case of straight segments the meaning of addition and subtraction is obvious but Euclid also applies these operations to other kinds of geometrical objects, noticeably to rectilinear figures, i.e. polygons. For the sum $A+B$ of two given polygons A, B Euclid takes a polygon resulted from application of A, B side-by-side; subtraction is taken to be the reverse operation. Notice that results of both operations are not defined uniquely up to congruence: there are many ways in which one of the two given polygons can be applied to the other. However according to A2 all polygons $A+B$ are *equal*. Thus Euclid's *equality* of geometrical objects doesn't imply their *congruence* although according to A4 the converse is (obviously) the case. In modern terms Euclid's *equality* best translates as *equicomposability*: A and B are said to be equicomposable when they both can be cut into the same (up to congruence) disjoint parts or in other words - when B can be obtained from A trough a re-arrangement of its parts. Since polygons are equicomposable if and only if they have the same *area* Euclid's theory of polygons found in Book 1 of the *Elements* is often interpreted as a theory of areas (Note 33). This modern reading, however, assumes a very different setting in which geometrical objects are put into correspondence with real numbers

called their measures while Euclid's notion of equality applies to geometrical objects directly (Note 34).
We see that although the five Axioms of the *Elements* apply both to geometry and arithmetic meanings of basic terms involved into the Axioms depend on a given domain of application. In geometry and arithmetic (as these disciplines are presented in the *Elements*) *equality*, *addition*, *subtraction*, *congruence*, *whole* and *part* are understood differently, so one may reasonably wonder if there is any sense at all in which these notions can be grasped in abstraction from this or that specific subject-matter. This question opens an important issue in the early philosophy of mathematics, which concerns so-called *universal mathematics* (also known under its Latin name of *mathesis universalis*). I shall postpone a discussion on universal mathematics until the next section and conclude this section with few general remarks about Euclid's Postulates and Axioms.

1) We have seen that the Axioms provide a very general framework for doing mathematics, which can be compared with a *logical* framework (see 1.4C below). Although this framework doesn't apply outside of mathematics, the Platonic notion according to which mathematics is distinguished by its epistemic status rather than by its specific subject-matter, allows one to call this framework universal in a strong sense. However in the *Elements* this general framework doesn't have a form of a mathematical theory on its own rights. To develop any branch of arithmetic or geometry Euclid needs additional principles introduced through Definitions and Postulates; such additional principles provide a *subject-matter* for any particular branch of mathematics. Unlike Axioms these additional principles are *not* basic *truths* from which one may obtain further truths; they are rather primitive constructions with the help of which one may perform further constructions.

2) P4 and P5 look dubious not only because of their *content* but also because of their epistemic *form*. P4 is arguably provable from other principles (without P5) - and hence is superfluous. But P5 makes a real problem. The project of developing geometry on the double foundations, which comprise five very general Axioms, on the one hand, and three generic operations described by P1-P3, on the other hand, doesn't work as it should work from a Platonic viewpoint. It works only up to certain point which delimits what (after Bolyai) is still sometimes called *Absolute geometry* , i.e. the part of Euclid's theory independent of P5. Attempts to prove P5 as a theorem on the basis of the rest of Axioms and Postulates (and perhaps some additional fundamental principles) was a major driving force of the long-term

history of geometry until the second half of 19th century when geometry and the whole of mathematics changed its shape dramatically (see Episode 3 below).

3) Leaving now the problem of P4-P5 apart we can specify a more general sense in which Euclid's foundations prove insufficient from a Platonic viewpoint. P1-P3 are supposed to generate all of the available geometrical content. However the idea to identify the geometrical content with the geometrical universe generated by P1-P3 (i.e. constructed by ruler and compass) doesn't go through. One obvious problem is that basic geometrical definitions of Book 1 cover much more than P1-P3 generate. Consider, for example, D1.2 (general definition of line which covers all possible curves) and D1.14 (plane figure). So the geometrical universe RC (for ruler and compass) generated by P1-P3 is only a minor part of the larger geometrical universe G described by Definitions. In order to tackle this problem a Platonic could stress the special status of RC and claim, in particular, that only "perfect" lines like straight lines and circumferences allow for a more precise mathematical treatment while other kinds of curves don't (Note 35). However this strategy in fact doesn't fully work either. For similar problems arise within RC itself. Euclid's fundamentals easily allow one to construct a square and then double it. They equally allow for building a cube but not for doubling a cube. This latter fact has been firmly established only in 19th century. Ancient geometers were unaware of it and tried hard to solve the problem (known as *Delian* problem) by ruler and compass. They obtained a number of interesting solutions, which required different instruments, but didn't achieve what they aimed at. Plato rejected such "mechanical" solution on philosophical grounds (Note 36). However one didn't need to be a convinced Platonic to opt for the rejection. Since P1-P3 belonged to the core of foundations of geometry of the time no Greek geometer could possibly consider mechanical solutions of Delian problem as satisfactory without a radical revision of foundations of his science. But as a matter of historical fact the needed revision didn't take place until 17th century B.C. (see Episode 2 below).

This example shows two important things: (i) that a mathematical problem, which seems to be quite specific or even purely technical, can undermine foundations and (ii) that a revision of foundations can significantly contribute to mathematical progress even if this progress is measured merely in terms of problem-solving. The question of whether or not Delian and other similar problems undermine Plato's philosophy of mathematics requires a more nuanced answer. It is clear that this philosophy cannot provide a sound explanation of why a square can be easily doubled by ruler and compass but a cube, on the contrary, cannot. In this sense the impossibility of the desired solution makes for Platonism a real problem. However one doesn't necessarily need to dismantle the whole of Platonism to fix it. One may argue instead that circle and straight line turn to be wrong generators and look for better ones without changing basic philosophical principles behind mathematical theories.

4) Noticeably there is no *arithmetical* Postulates in the *Elements* although one can easily conceive of them. Consider, for example, these:

To compose a number from its given units (compare D7.2)

To compose a given number with a given unit
(alternatively: Given a number to construct the following number)

Given two numbers to construct their sum

The notion of arithmetical Postulate seems me moreover reasonable (I mean reasonable by Euclid's own standard) since Euclid's arithmetical Propositions just like his geometrical Propositions have a part, which can be described as *construction* (in a precise sense of the term, which I shall explain in the next section). Euclid certainly uses these and some other basic operations in his arithmetic but he doesn't stipulate them as Postulates. I shall suggest a possible explanation of this puzzle in 1.3C3 below.

C) Propositions: Problems and Theorems

Editors of the *Elements* traditionally give the name of *Propositions* to numbered blocks of text, which constitute the principle content of the *Elements*. Unlike the case of Definitions, Postulates and Axioms the title "Propositions" doesn't appear in the original text. This relatively late terminological invention looks particularly controversial given the today's standard meaning of the term "proposition" stemming from Frege and Russell as a sentence having a definite truth-value. For we shall shortly see that Euclid's Propositions generally don't qualify as propositions in this later sense. Following the established tradition I shall use the term "Proposition" referring to the *Elements* in the usual way but I warn the reader that this term describes here only a textual feature and says nothing about the content. In what follows I shall denote by "E*m*.*n*" *n*-th Propositions of *m*-th Book of the *Elements.*

A plausible reason why Euclid and Greek mathematics avoid to use any general name in this case could be this: Euclid's Propositions are of two quite different kinds: some of them are Problems while some other are Theorems. Although these latter terms don't appear in the *Elements* either they were widely used already in the Plato's circle. Proclus describes the distinction between Problems and Theorems several times in his *Commentary*. Below is one such passages. It begins with a more general epistemological point, which I include in the quote for clarity:

"Science as a whole has two parts: in one it occupies itself with immediate premises, while in the other it treats systematically the things that can be demonstrated or constructed from these first principles, or in general are consequences of them. Again this second part, in geometry, is devided into the working out of problems and the discovery of theorems. It calls "problems" propositions whose aim is to produce, bring into view, or construct what in a sense doesn't exist, and "theorems" those whose purpose is to see, identify, and demonstrate the existence and non-existence of an attribute. Problems require us to construct a figure, or set it at a place, or apply it to another, or fit it upon or bring it into contact with another, and the like; Theorems endeavour to grasp firmly and bind fast by demonstration the attributes and inherent properties belonging to the objects that are the subject-matter of geometry." (*Commentary* 200.20-201.14, Morrow's translation)

I should warn the reader that Morrow's English translation of Proclus' *Commentary*, which I use here, is strongly modernising: in particular, there is no counterpart in the Proclus' text for the word "proposition" used by the translator; Proclus doesn't say in fact that Problems "require" us to do something. I deliberately use here this modernised translation without trying to correct it to begin with because I think that the proposed modernisation is rather advantageous for the first reading. After doing some necessary hermeneutic work I shall propose a more literal translation, which will provide a closer grasp on Proclus' thinking.
The difference between the two kinds of Propositions is made quite explicit in the *Elements*: Problems always end up with the words "[what] it was required to do" ("oper edei poihesai" or in Latin translations "quod erat faciendum") while Theorems always end up with the words "what it was required to show" ("oper edei deixai" or in Latin "quod erat demonstrandum") (Note 37). In what follows I shall point to another textual criterion of a similar type. Even if the word "problem" in this context has a meaning, which is more specific than the colloquial one, the distinction between Problems and Theorems in the *Elements* appears to be clear:

Problems realise geometrical constructions with certain desired properties while Theorems establish some non-trivial properties of given geometrical constructions. One can remember this from the school: pupils are asked either to construct something, i.e. to solve a Problem, or to prove something, i.e. prove a Theorem. What remains unclear is how Euclid combines the two kinds of Propositions into a single theory. One might expect that Euclid's Theorems establish facts concerning geometrical constructions, which are realised in the preceding Problems. One could then describe this situation as follows: Problems grant the *existence* of geometrical construction while Theorems treat their further properties (see Backer 1957). However the order of Propositions in the *Elements* doesn't meet this expectation. The first Proposition of the *Elements*, namely Problem E1.1, shows how to construct an equilateral triangle by the ruler and the compass but one doesn't find in the *Elements* any Theorem establishing a property of the equilateral triangle. The first Theorem of the *Elements* E1.4 establishes a property of general triangles, which are not built beforehand (except the special case treated by E1.1). The following Theorem E1.5 treats an isosceles triangle, which is not built beforehand either. The reader can easily check that these immediate examples are anything but exceptional.

Let me now show that Proclus' Platonic analysis allows for a better understanding of the role of Problems and Theorems in Euclid's mathematics. Before I shall discuss differences between the two kinds of Propositions I shall stress their similarities. From the modern viewpoint the fact that there are profound similarities between Problems and Theorems is perhaps more surprising than the fact that there are differences. For a (solution of a) Problem looks like a particular *method* of geometrical construction while a Theorem is a proved mathematical *truth*. So it is not immediately clear how such different things can be taken on equal footing and put into one and the same category of Propositions. However Euclid's Problems and Theorems indeed share a common structure, which represents a fundamental pattern of Euclid's reasoning. Says Proclus:

"Every Problem and every Theorem that is furnished with all its parts should contain the following elements: [i] an *enunciation*, [ii] an *exposition*, [iii] a *specification*, [iv] a *construction*, [v] a *proof*, and [vi] a *conclusion*. Of these *enunciation* states what is given and what is being sought from it, a perfect *enunciation* consists of both these parts. The *exposition* takes separately what is given and prepares it in advance for use in the investigation. The *specification* takes separately the thing that is sought and makes clear precisely what it is. The *construction* adds what is lacking in the given for finding what is sought. The *proof* draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The *conclusion* reverts to the *enunciation*, confirming what has been proved." (*Commentary*, 203.1-15, Morrow's translation) (Note 38)

Before I shall try to clarify what Proclus tells us here let's consider two examples: Problem E1.1 (Proclus' own example) and Theorem E1.5. This is how the six aforementioned parts are identified in these two cases:

Proposition E1.1 (Problem):

“[*enunciation*]

To construct an equilateral triangle on a given finite straight-line.

[*exposition*]

Let AB be the given finite straight-line.

[*specification*]

So it is required to construct an equilateral triangle on the straight-line AB.

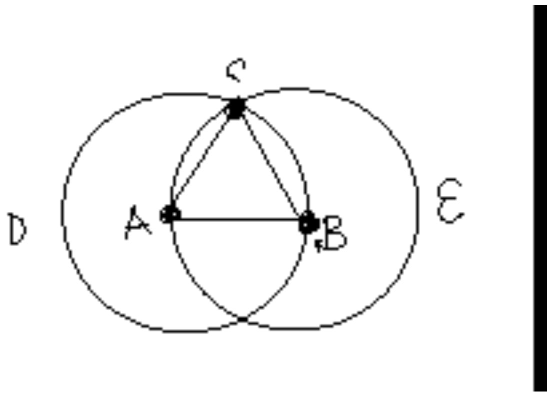


Fig.5

[*construction*]

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center radius BA have been drawn. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another to the points A and B (respectively).

[*proof*]
And since the point A is the center of the circle CDB, AC is equal to AB . Again, since the point B is the center of the circle CAE, BC is equal to BA. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another . Thus, CA is also equal to CB. Thus, the three (straight lines) CA, AB, and BC are equal to one another.

[*conclusion*]
Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do."

Proposition E1.5 (Theorem):
“[*enunciation*]
For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another.

[*exposition*]
Let ABC be an isosceles triangle having the side AB equal to the side AC, and let the straight-lines BD and CE have been produced in a straight-line with AB and AC (respectively).

[*specification*]
I say that the angle ABC is equal to ACB, and (angle) CBD to BCE.

[*construction*]
For let the point F have been taken somewhere on BD, and let AG have been cut off from the greater AE equal to the lesser AF . Also, let the straight lines FC and GB have been joined.

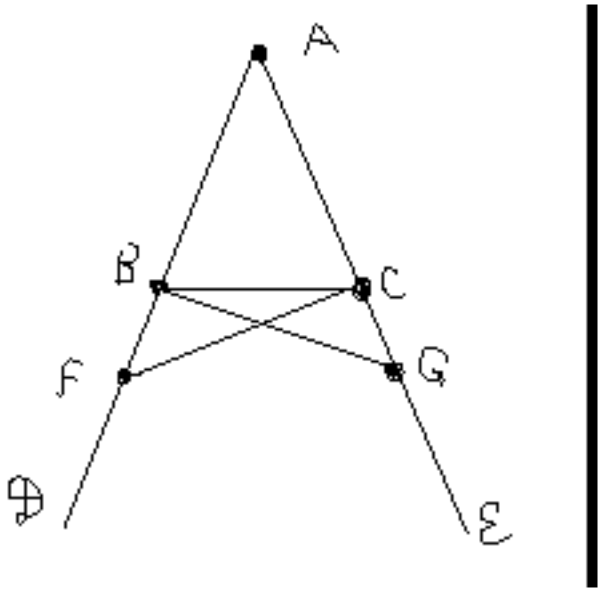


Fig.6

[*proof*]

In fact, since AF is equal to AG, and AB to AC, the two (straight-lines) FA, AC are equal to the two (straight lines) GA, AB, respectively. They also encompass a common angle FAG. Thus, the base FC is equal to the base GB, and the triangle AFC will be equal to the triangle AGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. (That is) ACF to ABG, and AFC to AGB. And since the whole of AF is equal to the whole of AG, within which AB is equal to AC, the remainder BF is thus equal to the remainder CG. But FC was also shown (to be) equal to GB. So the two (straight lines) BF, FC are equal to the two (straight lines) CG, GB, respectively, and the angle BFC (is) equal to the angle CGB, and the base BC is common to them. Thus the triangle BFC will be equal to the triangle CGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. Thus, FBC is equal to GCB, and BCF to CBG. Therefore, since the whole angle ABG was shown (to be) equal to the whole angle ACF, within which CBG is equal to BCF, the remainder ABC is thus equal to the remainder ACB . And they are at the base of triangle ABC. And FBC was also shown (to be) equal to GCB. And they are under the base.

[*conclusion*]

Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show."

As we see this Proclus' analysis of a Proposition (a Problem and a Theorem) into six parts applies to Euclid's text immediately without paraphrasing. The above partitioning of E1.1 is made by Proclus himself (*Commentary*, 208.1-210.16). The partitioning of E1.5 is mine but there is hardly any possible controversy about it. The reader can check it independently that this six-part structure is present and easily identifiable in most of Euclid's Propositions including arithmetical Propositions (Note 39).
I shall now generalise upon the two above examples considering them more precisely one after the other.

C1) Problem

Enunciation of a given geometrical problem is usually read today as a requirement to make a construction with certain desired properties. But the original text makes this interpretation unconvincing since in Greek the infinitive form of a verb (*to construct* and the like) normally doesn't imply any deontic modality (Note 40). More significant is the fact that *enunciations* of Problems have the same grammatical form as Postulates P1-P3 . This fact shows that the usual reading of Postualtes as stipulated truths (see 1.3C below) and the usual reading of *enunciations* of Problems as requirements are hardly compatible with each other: the original text strongly suggests that these locutions should be interpreted uniformly. The principle difference between a Postulate and a Problem is just this: a Postulate (at least one of the first three Postulates) grants some basic construction while a Problem treats some more complicated construction. That is why a Problem doesn't reduce to its *enunciation* just like a Theorem doesn't reduce to its *enunciation*. Euclid explains us in E1.1 how to construct an equilateral triangle but doesn't explain in P1 how to produce a straight line: the latter construction unlike the former is taken for granted. However this important difference doesn't concern the meaning of a given *enunciation* itself. The *enunciation* of E1.5 "For isosceles triangles, the angles at the base are equal to one another" comes with an epistemic requirement according to which it should be proved but not simply taken for granted. But this *enunciation* is not itself a requirement of any sort. I claim that *enunciations* of Euclid's Problems should be understood similarly. "To construct an equilateral triangle on a given finite straight-line" is not by itself a requirement but a description of an operation. This description is sufficient for understanding of *what* this operation does but not sufficient for understanding *how* it does it. Similarly the *enunciation* of E1.5 is sufficient for understanding *what* is claimed here to be the case but not sufficient for understanding *why* this claim is justified. *How* and *why* are indeed required but these epistemic requirements shouldn't be confused with *what* the two *enunciations* tell us. People hardly ever make this confusion in the case of a Theorem but in the case of a Problem the confusion is common. By correcting it we gain the desired uniform reading of Problems and Postulates. *Enunciations* of Problems just like Postulates (at least P1-P3) describe constructive operations. The only difference between them is that the latter are fundamentals while the former are not. A brief look at the Proclus' quote concerning the distinction between Problems and Theorems in its original form fully confirms this interpretation: unlike the translator (Morrow) the author (Proclus) doesn't describe Problems and Theorems in terms of aims and purposes. Here is my modified version of this translation:

"Science as a whole has two parts: in one it occupies itself with immediate enunciations, while in the other it treats systematically the things that can be demonstrated or constructed from these first principles, or in general are consequences of them. In the geometrical reasoning this second part is again divided into solving problems and finding theorems. The name "problem" is appropriate where what in a sense doesn't exist is produced, set, brought into view and arranged, while the name "theorem" is appropriate where something that is attributed or not attributed is seen, known and proved. The former [have to do with] generation, setting, application, ascription, inscription, insertion, touching and the like; the latter [have to do with] properties and essential attributes of geometrical objects, which are grasped and firmly bound by demonstration." (*Commentary*, 200.20-201.14, Morrow's translation, corrected) (Note 41)
Notice that in the first sentence of this passage Proclus uses the same word "protasis" (*enunciation*) (Note 42), which he uses for the first element of a Problem or a Theorem. By *immediate enunciations* he clearly mean here both Axioms and Theorems - notice that in the second half of the sentence he mentions both proofs and constructions. This justifies the reading of Postulates as self-standing *enunciations* of the same type as the *enunciations* of Problems. For a better understanding of the above quote it is helpful to remind some Platonic generalities. When Proclus says that geometrical constructions "in a sense don't exist" this should hardly be understood in the sense that these things are first wholly absent and then brought into the existence by a constructive procedure. One should rather think about Platonic basic ontological distinction between Being and Becoming (Generation). Because of its "intermediate" ontological status any geometrical object has both these aspects: it can be considered both as generated and as eternally existent. I shall call these two aspects *generic*

and *ontic* correspondingly. To put it roughly, Problems provide the former and Theorems the latter view onto the same geometrical subject-matter. When one considers a geometrical object as generated then this object "in a sense doesn't exist". But in a different sense this object always exists.

The *enunciation* of a given Theorem is a proposition in the usual logical sense and we distinguish it from one's belief that this proposition is true. Propositions and propositional beliefs are different things. Platonic ontology allows for a similar distinction in the case of a Problem. Here one can distinguish between a constructive operation objectively conceived and a particular individual action realising this operation. The *enunciation* of a given Problem describes an operation in the latter rather than in the former sense. However the modern notion of operation, which I have used earlier, once again doesn't exactly fit the Platonic way of thinking. For this modern notion is related to notions of general *method* or *rule*, while the Platonic ontology suggests to conceive of it rather as an *event* or *process* related to the generic aspect of a given construction. This ontology allows one to conceive of such constructive events objectively in the same sense in which, more generally, it allows one to conceive of mathematical objects objectively. For the Platonic doctrine of the intermediate ontological status of mathematical objects implies precisely this: mathematical objects are not just eternal things but also events and processes; more precisely they are neither of these but somehow combine features of both. Importantly this doctrine doesn't suggest a reduction of the generic aspect of mathematics to its subjective aspect, i.e. to the issue of how *we*, people, calculate and make geometrical constructions. Unlike the modern notions of method and rule Plato's Becoming (in general) and Mathematical Becoming (in particular) are ontological but not epistemological notions. It is a common knowledge that Platonism grants to mathematical objects an "independent existence". It is often forgotten that it equally grants to them an "independent becoming".

In order to understand the role of the *exposition* and the *specification* of a given problem observe first of all that any construction described in *Elements* usually assumes some of its elements as previously given. This is equally true for primitive constructions described in Postulates and for further constructions described in Problems. For example, P1 describes the construction of straight line by its given endpoints and E1.1 describes the construction of equilateral triangle by its given side. This notion of the given is tricky. Apparently it is not informative. For example, saying that a straight line is given doesn't provide any new information about it: it can be any straight line whatsoever. But in fact this is already a crucial piece of information! To see this consider a modified version of E1.1, which proposes one

E'1.1: to construct an equilateral triangle

without mentioning any data as previously given. This modification looks innocent but actually it is not. To solve the modified problem one starts with a choice of a straight line and then proceeds just like in E1.1. But now the whole construction depends on the choice made at the first step. It is easy to see that in this particular example the choice doesn't matter but this is not a general rule. For another example consider E1.2, which proposes one

E1.2: "To place a straight line equal to a given straight line at a given point"

(Note 43) and drop the requirement concerning the given point:

E'1.2: To produce a straight line equal to a given straight line

Now the problem becomes trivial: given a straight line *AB* one can straightforwardly construct another straight line *AC* equal to *AB* using P3 and D1.15. or even offer as a solution the straight line *A'B'* congruent to *AB*. If we further drop the requirement concerning the given straight line:

E"1.2: to produce a straight line

the Problem in its usual sense disappears (Note 44). We can see that saying that an object is *given* is tantamount to saying that it is chosen *arbitrarily* and not by some special purpose. And this condition is essential because it makes the solution of a given Problem into a *general method* valid for every possible choice of the initial data. E1.1 and E'1.1 equally allow one to construct an equilateral triangle. But E1.1 is general in a sense in which E'1.1 is not. For there is a sense in which E1.1 produces *all* equilateral triangles - each particular triangle corresponding to each particular choice of the given straight line - while E'1.1 produces only one particular triangle determined by the choice made at the first step. Keeping this notion of given in mind let's now consider the *exposition* and the *specification* of E1.1.

The *exposition* of the Problem provides the given data (viz. a straight line) with a proper name (viz. *AB*). *Specification* restates the *enunciation* referring to what is given by its name (or names). Since details of the given data are already explicitly mentioned in the *exposition* they can be omitted in the *specification* (like in E1.9, for example). Thus the *exposition* and the *specification* taken together make more explicit the distinction between what is given and what is sought. This analysis of the *enunciation* into the two components is obviously important. However it could be equally done without naming. Consider this modification of the *exposition* and the *specification* of E1.1:

A finite straight line is given. It is required to construct an equilateral triangle on it.

It is rather evident that after this modification the *exposition* and *specification* cannot play the same role but it is not immediately clear why. What is the epistemological impact (if any) of giving the name "*AB* " to the straight line mentioned in the *enunciation* of E1.1? Notice that this naming doesn't restrict the generality : the given line can be still any straight line whatsoever. So *the given straight line* and *the given straight line AB* is exactly the same thing in E1.1.

One reason why Euclid nevertheless introduces the name "*AB* " in the *exposition* of E1.1 is that this is advantageous from a notational viewpoint. Let's see what precisely this advantage consists of.

An obvious remark is that the name "*AB* " is shorter than the expression “the given straight line”, so using the former instead of the latter is economical (Note 45). However as we shall now see this is not the only advantage of Euclid's mathematical notation. This system of notation based on the principle of identification of geometrical objects by their distinguished points is in fact very smart and perfectly adapted to Euclid's geometrical reasoning. Giving the name "*AB* " to a straight line Euclid also provides names to its endpoints, which can be now referred to as *A* and *B* without a special notice (provided the reader knows how this system of notation works). So when Euclid mentions point *A* in the *construction* of E1.1 he doesn't need to explain that *A* is one of the two endpoints of the straight line described as *given* in the *enunciation*. To show that the gained economy is indeed significant I shall reformulate the beginning of the *construction* of E1.1 in English without using any special notation:

Draw a circle with the centre at one of the two endpoints of the given finite straight line and the radius equal to this given straight line. Draw another circle with the centre at the other endpoint of the given finite straight line and the radius again equal to this given straight line. Then produce a straight line from one of the two endpoints of the given straight line to the point where the two drawn circles cut each other. Produce also a straight line from the other endpoint of the given straight line to the point where the two drawn circles cut each other.

In spite of the fact that the geometrical construction described here is very simple it is difficult to grasp it through the above description. In case of more complicated constructions (and associated proofs) the purely verbal description becomes impossible or at least practically useless. One may wonder why Euclid doesn't explain the reader the basic syntax of his notation, in particular the fact that the name "*AB* " of a straight line is composed of the names "*A* " and "*B* " of the two endpoints of this line. An obvious reason for it is the following: Euclid's notation is not a system of shortcuts to verbal descriptions but an interface between the verbal discourse and diagrams. Letters involved into Euclid's mathematical notation appear both in the written text side-by-side with words of the natural language and on diagrams side-by-side with drawn figures. Thus Euclid's notation serves not only for the economy of thought and/or the economy of writing but also for linking the verbal reasoning with the diagrammatic reasoning. To see more precisely how it works consider the following two scriptures:

(i) *A*______________*B*

(ii) straight line *AB*

(i) is a diagram supplied with a letter notation and (ii) is a linguistic expression supplied with a letter notation. In (i) the letters *A*, *B* play the role of hieroglyphics: what matters in this case is only the ability of their users to identify and distinguish their shapes correctly. In (ii) the same symbols are used as letters of a phonetic alphabet along with all the other letters used for writing down sentences of the natural language. This allows for spelling out (ii) without difficulties. So the linguistic expression (ii) can be described as *voicing* of the diagram (i). Clearly the voicing of mathematical diagrams doesn't reduce to a simple exchange of one material means of representation for another; it is a cognitive procedure laying at the core of Euclid's geometrical thinking.

Now I would like to stress another aspect of *exposition*, which is related to the issue of notation but doesn't reduce to it. Observe that in the *enunciation* of E1.1 the expression "straight line" stands with the indefinite article *a* while in the following *exposition* the same

expression stands with the definite article *the*. This is a mere artefact of translation since Greek has no indefinite article. Nevertheless the translator quite correctly expresses here by means of modern English an important difference between the meaning of the expression "straight line" in the *enunciation* and in the *exposition*: the *enunciation* tells us about *some* given straight line while the *exposition* points to a particular straight line *AB*. The expression "straight line" in the *enunciation* of E1.1 refers to a general concept rather than a concrete mathematical object. The *exposition*, on the contrary, refers to a particular object, which is a concrete instance of the general concept described in the *enunciation*. At the stage of *exposition* one can forget for a while (namely, until the *conclusion*) about the general aspect of *enunciation* and work with a particular object. I shall call such switching from general concepts to particular instances of these concepts *instantiation*. The instantiation doesn't lead to any loss of generality because instances of general concepts are supposed to be arbitrarily chosen. As I have already stressed, the *straight line AB* referred to in the *exposition*, *specification*, *construction* and *proof* of E1.1, on the one hand, and *a given straight line* mentioned in the *enunciation* of E1.1, on the other hand, is one and the same thing. This is what the *exposition* of E1.1 explicitly tells us. Or more precisely this is what the *exposition* stipulates. And this stipulation doesn't reduce to introduction of a notational convention but involves instantiation of a given concept (the concept of straight line in our example).
Let me now show that this instantiation concerns not only mathematical objects but also mathematical *subjects*. This latter issue is no longer about *what* is involved into a given mathematical reasoning but about *who* reasons mathematically in a given case. Observe that the name "*AB* ", which Euclid uses in E1.1, is conventional in a sense in which the general name "straight line" is not. One may repeat E1.1 using letters *L*, *M*, *N* instead of letters *A*, *B*, *C* used by Euclid without changing anything essential. One cannot do the same with the term "straight line" or any other general name. As we shall see in the Episode 3 Hilbert in his *Foundations* made general names of mathematical concepts exchangeable in the same way but this modern approach is, of course, very different from Euclid's. In the traditional Euclid's geometry an individual mathematician is allowed to choose letters for denoting points but not allowed to choose general terms in anything like the same way. One cannot, for example, to call straight lines "points" and to call points "straight lines". For these general terms belong to a mathematical community but not to an individual mathematician. This different use of proper names (letter notation) and general names reveals something deeper: while the *enunciation* is attributed to a collective thinking subject (or, if one prefers, to a universal mathematical mind) the *exposition* and all the following parts of a given Problem until the

conclusion have a direct appeal to an individual thinker. This can be also seen through a more precise analysis of Euclid's wording. In the *exposition* of E1.1 he uses the imperative form of the verb "einai" (to be), which makes it clear that the name "*AB* " is given here to a straight line by an individual act of naming. Formally speaking this is a *performative* but not a descriptive sentence: it realises this act here and now (i.e. in any particular reading) rather than describes some objective state of affairs. This is Euclid who speaks to the reader here, not the universal mathematical mind. (We shall shortly see that in the case of a Theorem this feature is expressed even more explicitly.) In the *specification* (but not in the *enunciation*!) there first appears the word "dei" translated into English by the expression "is required", which in the given context only makes sense as an appeal to an individual. While the *enunciation* merely describes an operation the *specification* urges the reader to work it out. An individual act of reasoning instantiates here an universal (or collective) reasoning in a sense similar to which an individual object like straight line *AB* instantiates a general mathematical concept .

The switch from general concepts to their instances achieved through the *exposition* and the *specification* can be also described as *imagining*. When in the *enunciation* of E1.1 one reads about a straight line it is appropriate to consult definition D1.4, which defines this very notion. The following *exposition* pushes the reader into a new direction, namely to imagining the straight line mentioned in the *enunciation*. At this stage one normally starts to draw a diagram. The following *specification* restates the *enunciation* in terms of the straight line imagined by the reader. This is a prerequisite for the next step.

Construction is the main part of a Problem. It realises the operation described in the *enunciation* through combination and reiteration of primitive constructions described in P1-P3 and of further constructions realised in preceding Problems (if any). This is what one usually means by construction by ruler and compass. Arguably Euclid doesn't make explicit some important details here. In particular, many modern commentators point to the fact that in E1.1 Euclid refers to the point *C* of intersection of the two circles without having any appropriate principle allowing him to identify such a point. I would like however to stress a different issue, namely that a pure *construction* by itself doesn't constitute a Problem or a solution of a Problem. First of all a Problem should be stated in general terms. This is what the *enunciation* serves for. Second, one's imagination should be switched on. This is done through the *exposition* and the *specification* as I have already explained. Finally and crucially, one should *prove* that the proposed *construction* indeed produces what is required in the *specification* . This is why a *proof* is an indispensable element of any Problem. This shows that the generic aspect of geometry cannot be autonomous. A pure combination and reiteration of primitive constructions granted by P1-P3 leads to nowhere. This generic process needs to be tightly controlled by ontic means, i.e. by *proofs* based on Axioms. I leave a more detailed analysis of the *construction* of E1.1 to the reader and postpone further generalities concerning *construction* until the next paragraph.

The *proof* of E1.1 relies on D1.15 and A1: D1.15 implies that *AB*=*AC* and *AB*=*BC*; then A1 implies that *AC*=*BC* . Hence the *conclusion*: triangle *ABC* is equilateral. *Enunciation* and *conclusion* in the case of a Problem look differently: while *enunciation* refers to an operation ("to construct") *conclusion* refers to its result ("triangle ... has been constructed"). Moreover the resulting triangle is referred in E1.1 by its proper name "*ABC*". The straight line given in the *enunciation* is also referred in the *conclusion* by its proper name "*AB* ". However in this case Euclid also repeats its complete description as the "given finite straight line". This allows for a return to the general point of view expressed in the *enunciation* : the *conclusion* achieves exactly what the above *enunciation* merely describes. (Note 46)

C2) Theorem

Enunciation of a given Theorem is a proposition in the familiar Fregean sense: it is a truth about a specific mathematical subject-matter. *Enunciations* of Theorems always refer to general concepts like that of isosceles triangle but not to particular mathematical objects like triangle *ABC*. In the case of a Theorem this contrast between general concepts and their instances is even more obvious than in the case of a Problem discussed above. The general character of the *enunciation* of Theorem E1.5 can be described by saying that this Theorem holds for *all* isosceles triangles or equivalently - that it holds for *any* isosceles triangle. However this modern interpretation, in my view, is not quite appropriate or at least it doesn't precisely fit the Platonic viewpoint to which I stick here. For it assumes that all the individual isosceles triangles are in some sense previously given and form a domain on which one can quantify. This view might square well with what is often called Mathematical Platonism today but it doesn't square with Platonism in the historical sense of the term. On Plato's account the existence should be attributed to the concept itself rather than to its instances. From this point of view the instances belong to the generic but not to the ontic aspect of mathematics; they *become* rather than exist, and they don't *become* all at once. I shall come back to this point shortly.

The *exposition* and the *specification* work in the case of a Theorem in a way very similar to which they work in the case of a Problem. The *exposition* of E1.5 amounts to picking up a particular isosceles triangle *ABC* (instantiation) while the *specification* amounts to restatement of the *enunciation* applied to the case of this particular triangle. The *exposition* and the *specification* together make explicit the distinction between what is assumed in this Theorem and what is supposed to be further shown. (In E1.5 it is assumed that the given triangle is isosceles and it is to be shown that its angles at and under its base are equal.) Just like in the case of a Problem the instantiation is realised through naming. And once again one observes that this instantiation has a subjective aspect. While the *enunciation* claims an universal truth the *exposition* involves choices of notation left to an individual mathematician. The *specification* of any Euclid's Theorem including E1.5 makes this subjective aspect even more explicit: this part of the Theorem is formulated as a personal claim of the form "I say that..." This expression can be used as a formal criterion for distinguishing between Problems and Theorems along with the other criterion mentioned above (the *specification* of a Problem always begins with the words "It is required...").

The instantiation doesn't restrict the generality of the Theorem because the instance *ABC* is supposed to be picked up arbitrarily. There is a sense in which this triangle *ABC* - in spite of the fact that it is a particular triangle of the given kind, just one among many of its likes - represents *all* possible triangles of its kind and hence fully represents the corresponding general concept. The arbitrariness involved into the constitution of *ABC* turns to be equivalent to generality. For similar reasons the instantiation doesn't restrict the generality on the subjective side. When Euclid says in the *specification* of E1.5 " I say that the angle ABC is equal to ACB, and (angle) CBD to BCE." he doesn't simply express his personal opinion. Anybody at his place - noticeably, the reader - would be obliged to tell the same (possibly using a different letter notation) if he or she is a rational being and accepts the same fundamentals. The individual choice concerns here only the letter notation and insignificant features of wording. The author instantiates a universal mathematical mind like isosceles triangle *ABC* instantiates the general concept of isosceles triangle. The Theorem works equally for any other competent thinker and for any other isosceles triangle. I shall provide more details concerning this fundamental trick of Euclid's mathematics shortly when we shall discuss the *conclusion* of E1.5. (Note 47)

The presence of *construction* in E1.5 may appear surprising since the *enunciation* of this Theorem unlike the *enunciation* of Problem E1.1 doesn't say anything about constructing. However anyone having a basic experience in elementary geometry knows that proofs of geometrical theorems typically require constructions, which are called today *auxiliary*. Moreover one knows that in spite of this modest name these constructions are usually crucial elements of proofs. An appropriate auxiliary construction can make a non-obvious geometrical property obvious reducing the rest of the proof to mere formalities. Even if this is not exactly the case of E1.5 where the *construction* is followed by a relatively long *proof* this following *proof* essentially relies on this auxiliary construction (Note 48). Thus constructions generally play a very important role in Euclid's Theorems even if *enunciations* of these Theorems don't make it immediately clear why.

Before continuing let me make an important terminological remark. Today we usually analyse a mathematical theorem into two parts: a proposition and its proof. Even if in Euclid's mathematics *enunciations* of Theorems also have a distinguished status, which allows one to use an *enunciation* as a shortcut to the corresponding Theorem, the modern bipartition of theorems hardly applies here. The notion of mathematical *proof* mentioned by Proclus is by far more specific than the corresponding modern notion (I am talking now about the current informal notion of mathematical proof, not about a proof in a refined logical sense). Notice that on Proclus' account the *proof* of a Theorem doesn't include the preceding *construction*, which we usually call today *auxiliary*. Talking above about auxiliary constructions as elements of geometrical proofs I meant today's notion of proof, not Proclus'. I think that this historical change of terminology is not without a reason, and in the next section I shall point to a possible reason for it. But it is worth mentioning already now that a part of the problem concerns translations from Greek to modern languages. There are two verbs in (the scientific) Greek, which are often translated by the same English verb to "prove": "deiknumi" and "apodeiknumi". Correspondingly, there are two Greek nouns, which are often translated by the word "proof": "deixis" and "apodeixis". But in fact they don't have quite the same meaning. This can be clearly see, in particular, in Aristotle's logical writings. Proclus uses the term "apodeixis" for denoting the part of a Problem or a Theorem, which we call here "*proof* ". Euclid in his turn finishes every Theorem by the standard expression "oper edei deixai", which is translated by Fitzpatrick as "which is the very thing it was required to show". The translator quite correctly distinguishes here between the verbs "deiknumi" and "apodeiknumi" and translates the former verb by the English verb to "show" but not by the verb "to prove". Thus Proclus' terminology agrees with Euclid's: as a whole a given Theorem *shows* (deiknusi) something while its *proof* proves (apodeiknusi) something; the latter procedure is a part of the former. However it was common already in early Latin translations of the *Elements* to pay no attention to this subtlety and translate Euclid's expression "oper edei deixai" by the Latin expression "quod erat demonstrandum". Since "demonstratio" is an established Latin translation of Greek "apodexis" widely used by Aristotle in his logical writings the difference between "deiknumi" (to show) and "apodeknumi" has been lost.

As one can see at the example of E1.5 the *construction* of a Theorem looks just like *construction* of a Problem: it is a combination of elementary operations granted by P1-P3. The only difference between the two cases concerns the purpose of a given *construction*. In the case of a Problem the purpose is evident: the *construction* realises a complex operation described in general terms in the corresponding *enunciation* through repeated application of the three Postulates; the following *proof* makes it sure that the *construction* fits the *enunciation* correctly. Thus the *construction* achieves the principle purpose of a given Problem while the proof plays an auxiliary role in it. In the case of a Theorem this relation of purposes is reversed. The main purpose of the whole reasoning in this case is to show that the *enunciation* of a given Theorem is true. This is achieved in the *proof* of this Theorem and the following *conclusion*. The purpose of the *construction* is in this case auxiliary: to make the *proof* possible via introduction of new elements into the initially given construction. However as I have already stressed the "auxiliary construction" involved into a Theorem is in fact crucially important. *Mutatis mutandi* the same can be said about the "auxiliary proof" involved into a Problem (Note 49). These observations suggest that the subordination of purposes shouldn't be taken too seriously in either case, and that, more generally, to describe Problems and Theorems in terms of aims and purposes is not a particularly good idea.

The role of *conclusion* of a given Theorem is not purely rhetorical in spite of the fact that it repeats the corresponding *enunciation* almost literally. Notice that each Euclid's *proof* results into a claim concerning particular geometrical objects like triangle *ABC* while the corresponding *enunciation* always tells us something about *any* object of a certain kind. In particular the *proof* of E1.5 results into the claim that in the isosceles triangle *ABC* angles *ABC* and *ACB* at its base *AB* are equal while the preceding *enunciation* says that angles at the base of any isosceles triangle are equal. The *conclusion*, which follows the *proof*, repeats the *enunciation* and thus marks a logically important step from the claim about the isosceles triangle *ABC* to the general claim about any isosceles triangle. We have already noticed a similar return to a general viewpoint in the *conclusion* of a Problem. In the case of a Theorem this return presents itself in a sharper form. While the *conclusion* of a Problem still involves the letter notation the *conclusion* of a Theorem doesn't: it repeats the *enunciation* literally. From a logical point of view the situation looks puzzling: the *proof* proves only a proposition concerning a particular mathematical object (the proposition specified in the *specification*) but in the following *conclusion* this proved proposition is claimed to hold for any object of the given type. One can reasonably ask for justification of this step.

One way to justified it I have already mentioned several times throughout this analysis: since the mathematical object in question is *arbitrarily* chosen among its likes everything that is proved about the chosen object (*ABC* in our example) equally applies to any other object of corresponding type (in our example - to any isosceles triangle). To assure this one should only avoid to take into account any specific feature that the chosen object might have but some differently chosen object of the same type might not have (Note 50). But as I have already said this argument hardly fits the Platonic way of thinking. From a Platonic viewpoint the arbitrariness of instantiation should be thought of in terms of distortion of *ideas* rather than in terms of one's free choice. This distortion amounts to the following: mathematical *ideas* belonging to the domain of Mathematical Being have multiple *copies* with arbitrary accidental properties in the domain of Mathematical Becoming. This is why in E1.5 the general concept of isosceles triangle splits into particular triangles like *ABC*. The task of mathematical reasoning is to distinguish between essential and accidental properties of particular objects and disregard the latter in favour of the former. As far as its accidental properties are systematically ignored a distorted *copy* is just as good as its ideal prototype.

One may argue that from a logical point of view there is no difference between Plato's story about *ideas* and their distorted *copies* and the modern story about arbitrarily chosen instances: both can be viewed as metaphors for the universal quantification. One's preferred metaphysics arguably is not essential for mathematical reasoning in this case. I don't think this argument is correct. The modern idea of logical semantics according to which quantifiers range over classes of individuals certainly makes part of modern logic. The talk of *all* or equivalently *any* isosceles triangle requires in this framework a notion of an infinite class comprising *all* isosceles triangles (i.e. of the extension of the concept of isosceles triangle). But such notion hardly makes sense for a Platonic. For elements of a given class are supposed to be full-fledged individuals with definite identity conditions. But this is not how Plato thinks about distorted *copies*. On his account only *ideas* have proper identities and can be distinguished clearly. On Plato's account the introduction of letter notation shouldn't be thought of as identification: the name "*ABC* " doesn't pick up in E1.5 one isosceles triangle among a bunch of others but points to the fact that we are dealing with an imaginary *copy* of the *idea* of isosceles triangle rather than with this *idea* directly. The principle epistemic trick of Platonic science amounts to "seeing the *idea* through its *copy*" without being misled by accidental features of this *copy*.

From a logical point of view this still means that one should consider only those features of the *copy*, which are explicitly stated in the *enunciation*, and ignore any other feature, which could be later introduced by imagination or by drawing. But this logical puzzle - How a proof concerning one particular object can be valid for other objects of the same type? - simply doesn't arise in the Platonic setting. Or at least it doesn't arise in the same form. One can rather ask here the following: How it is possible to learn anything about *ideas* by looking at their material and imaginary *copies*? This is a central epistemological question of Platonic philosophy, to which Plato and his followed provided various elaborated answers. I cannot consider these answers here but want to stress that the question in the Platonic context sounds very radical. For the possibility of "ascending" from the sensual and imaginary experience to a purely intellectual conception of *ideas* is equivalent for Plato and Platonics to the very possibility of knowledge. It seems that mathematics - and more specifically the kind of mathematics one finds in Euclid's *Elements* - served for Plato and his followers as a strong evidence that this procedure actually works.

A Platonic may also wonder why after putting forward a general *enunciation* of a Theorem, which has a direct appeal to geometrical *ideas*, and before coming to its equally general *conclusion* Euclid needs to "descend" to the domain of imaginary "copies" of these *ideas*? Wouldn't it be better to stick to the general *ideas* without helping oneself with their distorted *copies*? From a Platonic viewpoint this might appear desirable: even if mathematics unlike dialectics is incapable to grasp *ideas* in their purity it would always work in this case at the upper level of its epistemic capacity. Expressed in a more modern language this suggestion amounts to the following: imagination should be barred from any serious mathematical reasoning. However as we shall see in the Episode 3 such a strategy of reforming mathematics became quite influential in 20th century,. But actually Platonic philosophy allows one to justify not only this proposed reform but also the more traditional Euclid's mathematical practice. Here is a Platonic argument in its defence.

The fact that the *enunciation* of a given Theorem is expressed in general terms doesn't provide any guarantee that the reader can immediately grasp appropriate *ideas* behind these general words. This might work for a divine universal mind but not for a human being. By a mere reading of an *enunciation* one (a human being) doesn't get any *knowledge*. That is why the use of bare *enunciations* is limited by the case of fundamentals (Definitions, Postulates and Axioms). A typical Theorem, on the contrary, requires a progressive "ascending" from particular imaginary objects to corresponding general *ideas*. This is what the four intermediate elements of a given Theorem (*exposition*, *specification*, *construction* and *proof*) serve for. It is naive to think that mathematical *ideas* can be immediately grasped by a miracle. In fact it is a technical procedure, which is obligatory for every particular geometrical *enunciation* unless it expresses a fundamental (Note 51). By Euclid's legendary word there is no royal road to geometry.

C3) Problems and Theorems in Arithmetic

Proclus says that the distinction between Problems and Theorems applies only in geometry. But all Euclid's arithmetical Propositions except special cases mentioned in Note 40 have the same six-part structure. How this structure is realised in the arithmetical case can be seen at the example of E1.7 quoted in the Note 32. This example shows, in particular, that the notion of *construction* is not specifically geometrical as one might easily think.

Using the first of two aforementioned formal criteria allowing for distinguishing between Problems and Theorems one can see that all Euclid's arithmetical Propositions without exceptions qualify as Theorems: all of them finish with "(what) it was required to show" but not with "(what) it was required to do". This observation squares well both with Proclus' remark that the Problem/Theorem distinction applies only in geometry, and with the fact that in the *Elements* there is no arithmetical Postulates, which I have earlier stressed. This also squares well with (albeit, in my understanding, is not implied by) the Platonic view according to which arithmetic stands in the Platonic hierarchy higher than geometry: on this ground a Platonic can argue that imagination and other things related to the generic aspect of mathematics are appropriate in geometry but not in arithmetic. What remains puzzling is that quite a few of Euclid's arithmetical Propositions nevertheless look very much like Problems rather than Theorems. Consider, for example, *enunciation* of E7.2

"To find the greatest common measure of two given numbers (which are) not prime to one another."

Moreover this and some other Euclid's arithmetical Propositions qualify as Problems by the second criterion: their *specifications* begin with words "It is required..." but not with "I say that....". I don't have a definite solution of this puzzle but one thing seems to be clear. From a modern viewpoint the major difference between Problems and Theorems concerns the logical form of their *enunciations*: while *enunciations* of Theorems are propositions (i.e. have a truth-value) *enunciations* of Problems are not. But for Euclid this difference seems to be not very significant and so he can render E7.2 into a Theorem on different grounds disregarding the distinctive logical form of its *enunciation*. What these other grounds could be is suggested by Euclid's wording of E7.2. He says "to find" the greatest common measure, not to produce or construct it. Numbers are not produced in a mathematical reasoning in anything like the same sense in which triangles and circles are produced. My guess is that pointing to this Platonic intuition provides at least a partial answer.

C4) Conclusion on Problems and Theorems

Proclus relies on the distinction between Problems and Theorems for clarifying a less obvious distinction between Postulates and Axioms, which we have discussed earlier. This could give one a wrong impressions that the generic and the ontic aspects of mathematics can be perfectly separated one from the other, so that Euclid's mathematics would be split into two parts with different fundamentals. However as we have seen the two aspects of Euclidean mathematics are in fact tightly intertwined: every Problem involves a *proof* and hence depends on the Axioms while a typical Theorem involves a *construction* and hence depends on the Postulates. Proclus reports that already in ancient times some people tried to reformulate every Problem as a Theorem while some other people tried to reformulate every Theorem as a Problem. Among "radical Platonists" trying to banish Problems Proclus mentions Speusippus (Plato's nephew and his official successor as the Head of the Academy) and certain Amphinomus (about whom nothing else is known) (*Commentary* 77-78). According to Proclus these people argued that the notion of *generation* involved in Problems is irrelevant to mathematics because mathematical objects are eternal. Among "radical constructivists" trying, on the contrary, to banish Theorems in favour of Problems Proclus mentions Menaechmus (a pupil of Plato and of Eudoxus) and his pupils. I would like to stress that Proclus describes this controversy as internal for Platonic thinking; I called the first aforementioned position the "extreme Platonism" only in order to point to an obvious analogy with debates on foundations of mathematics taking place in 20th century (see Episode 3), where the name of Platonism was largely devoid of its historical content. This analogy provides a certain ground to Whitehead's view on European Philosophical tradition as "a series of footnotes to Plato" (Whitehead 1979, p. 39). But even if philosophy is indeed doomed to the endless repetition of positions and arguments it doesn't make it trivial and easy, as I have already explained in the Introduction.

Section 1.4. Euclid via Aristotle

We have seen that the historical Platonism allows one to clarify many features of the *Elements*, which otherwise look obscure like the distinction between the Postulates and the Axioms or plainly unsound like the presence of definitions, which are not used in the following theory. As we shall now see Aristotle's philosophy doesn't provide anything like the same effect in spite of the fact that this philosopher repeatedly refers to mathematical examples for illustration of his logical and epistemological points (Note 52). In I.1.2 I have already mentioned one reason for it: while Plato basically identifies mathematics with science (as distinguished from *opinion*, one the one hand, and from *dialectics*, on the other hand) Aristotle aims at a more general notion of science supposed to include mathematics as a special case along with physics (i.e. natural science including biology - see section 2 above). Aristotle puts forward his mathematical examples along with examples coming from physics and everyday life, and so it is hardly surprising that his general scheme supposed to embrace that much doesn't provide a precise grasp on mathematical reasoning.

Another reason why Aristotle's philosophy seems to be less clarifying for interpreting the *Elements* is more prosaic: many of Aristotle's polemic points later became common places and are often taken by modern readers as a matter of course. This makes Aristotelian features of the *Elements* in the eyes of the modern reader clear to begin with, so one doesn't need any longer Aristotle for clarifying them. However this apparent clarity can be quite misleading as we shall shortly see. In any event the significance of Aristotle's philosophy and of Aristotelian philosophical tradition for later developments in the foundations of mathematics provides a strong reason to look at this author carefully. Let's now see how Aristotle's Classical Model of Science described in section 2 applies to Euclid's mathematics.

A) Definitions

In his *Posterior Analytics* (Part 2, chapters 3-12) Aristotle systematically compares definitions and proofs as two different means of acquiring knowledge, and argues in favour of the latter at the expense of the former. So Aristotle shares the common today's view according to which definitions alone cannot provide knowledge. (Remind that Plato's philosophy allowed us to argue for the opposite; this helped us to make sense of some problematic Euclid's definitions.) But in spite of this important common point modern and Aristotle's views on definition are quite different. Aristotle considers but rejects the view according to which definitions explain nothing but meanings of defined terms; he still believes that they can say something essential about defined objects (definienda). His worry is that what definitions say about their definienda is never *proved* even if it is true. For example, if one defines a man as a wingless two-footed mortal animal, it is quite appropriate, in Aristotle's view, to ask questions like Why a man is wingless? and insist that the corresponding claim must be proved rather than merely postulated (*An. Post.* 92a1ff). Aristotle uses quite complicated arguments, which I leave here aside, for showing that definitions by themselves cannot provide answers to such questions (Note 53). Aristotle doesn't say that *every* proposition implied by a given definition (like *men are wingless*) must be proved. Some of them can and should be taken for granted as *immediate premises*. This is why Aristotle doesn't dismiss definitions but considers them as a particular kind of fundamentals. His point is that people who try to use definitions for acquiring knowledge usually don't care whether the propositions implied by these definitions are to be proved or taken for granted as fundamentals. In order to put this issue under control Aristotle reserves for definitions the role of fundamentals and combines them with a deductive reasoning.

Let's now see how this Aristotle's account applies to Euclid's Definitions. Euclid's Definitions are listed in the *Elements* among fundamentals of other types (Postulates and Axioms); some (but not all) of them are used in the following proofs. For example, Euclid repeatedly uses the fact that radii of a given circle are equal, which is implied by D1.15 (definition of circle); see again E1.1. Since it is not unreasonable to take this premise as immediate rather then try to find a proof for it D1.15 satisfies Aristotle's epistemic requirements. Even if today we tend to consider the proposition *radii of a given circle are equal* as an immediate consequence of a terminological convention concerning the term "circle" rather than as a basic mathematical truth like 2x2=4 Aristotle's views on definition square well in this case not only with Euclid's mathematics but also with the today's common attitude. Talking about those of Euclid's Definitions, which are not used in the following proofs we should distinguish between two cases. I have already identified these cases in the previous section as "philosophical" and "technical" Definitions. Aristotle's views on definition don't really imply that "philosophical" Definitions like D1.1 or D7.1 are redundant as one might expect. Aristotle might well ask Euclid to clarify the impact of the immediate premise "a point has no parts" (cf. D1.1) in his theory. But he would hardly agree to drop this Definition out even if Euclid would reply him that this premise actually plays no role. Aristotle would rather try either to redefine the notion of point or show that Euclid somewhere uses D1.1 tacitly (Note 54). The reason why Aristotle cannot leave the notion of point without a definition is not that he purports to "define everything" by explaining away every notion in terms of some other notions. He systematically avoids an infinite regress in deduction and there is no reason to suggest that he would embrace it in definition. But on Aristotle's account, the two cases are not parallel: the regress in deduction is stopped by fundamentals (immediate premises) while definitions (or at least good definitions), in Aristotle's view, are themselves fundamentals. The role of definition is not to explain the definiendum away but to introduce it as a primitive (Note 55). Primitive notions, on Aristotle's account, cannot be introduced only by their names; they should be introduced through a number of primitive propositions, i.e. of immediate premises. Hilbert's notion of "definition by axioms" (see Episode 3) is, in my view, not so alien to Aristotle's thinking as it might seem (but beware that Aristotle uses the term "axiom" in a different sense, which I explain later). In my understanding, Aristotle wouldn't see the difference between sentences "Point is that of which there is no part" and "Points have no parts" as logically and epistemically significant. What Aristotle's logic and epistemology really doesn't allow is the idea to define, for example, both points and straight lines by postulating propositions like "Two different straight lines share at most one point". This is because for Aristotle the notion of relation is secondary: in his view one should first make it clear what things are by themselves and only then describe how they relate to each other. This is indeed a major difference between Aristotle's and modern logic. In this sense Aristotle's logic and epistemology remain "essentialist" in spite of the requirement according to which definitions should be always followed by proofs. We shall shortly see that this feature of Aristotle's logic makes its applicability for analysis of Euclid's mathematical reasoning very limited.

For example of a "technical" definition not used by Euclid in his proofs consider the Definition of *romboid* D1.22d. Properties of a *romboid* postulated in D1.22d are later proved in E1.32 where the same thing is called by a different name of "parallelogrammic figure". Thus D1.22d is a clear example of the kind of definitions disapproved by Aristotle in the *Posterior Analytics*: they merely postulate what can and hence must be proved. The fact that Euclid uses here two different terms for the same geometrical notion, which seems particularly inappropriate from a modern viewpoint, would be for Aristotle less significant.

B) Postulates

Aristotle doesn't use systematically the term "postulate" (aithema) in his logic but in *Posterior Analytics* he provides the following distinction between a postulate and an hypothesis:

[A]nything that the teacher assumes, though it is a matter of proof, without proving it himself, is an hypothesis if the thing assumed is believed by the learner [...] but, if the same thing is assumed when the learner either has no opinion on the subject or is of a contrary opinion, it is a postulate. This is the difference between an hypothesis and a postulate; for a postulate is that which is rather contrary than otherwise to the opinion of the learner, or whatever is assumed and used without being proved, although matter for demonstration". (*An.Post.* 76b26-34, Heath's translation)

The above passage tells us that a postulate is a provable hypothesis, which is non-obvious, so one is not inclined to belief it to begin with. There is little doubt that Aristotle reports to us here correctly a popular meaning of the word. However it is not so clear what this description has to do with the five Postulates found in the *Elements*. Although Euclid indeed begins his list of Postulates with the expression "let it have been postulated..." (literally "...asked for" or "demanded"), which squares well with the Aristotle's description, it is unclear what kind of proof of P1-P3 one could think of and why P1-P3 are non-obvious. Interestingly P4-P5, which in Proclus' view are not genuine Postulates at all, are in fact the only Postulates fitting Aristotle's description! However this is apparently not a matter of disagreement between Proclus and Aristotle about philosophical or mathematical principles but rather a terminological matter: with the accordance with Aristotle’s notion of postulate Proclus believes that both P4-P5 can and should be proved as Theorems (Note 56). It is however extremely unlikely - or even plainly impossible - that Euclid considered all his Postulates including P1-P3 in this way - as provisionary hypotheses to be eliminated in an improved version of the same theory. For even today it is fairly impossible to imagine what the required improvement might consist of. For this reason I am fully agree with Barnes who says commenting on *An. Post.* 76b that Euclid and Aristotle use the term "postulate" in quite different senses (see Aristotle 1993, p. 141). It is not however without a reason that Aristotle points to this meaning of the term rather than to the other. For Aristotle's model of science unlike Platonic model of science doesn't have any place for such specific non-propositional fundamentals as P1-P3. So the only way to make sense of Euclid's Postulates from an Aristotelian viewpoint is to render (i.e. paraphrase) them into propositions (Note 57). This is actually the way in which Euclid's Postulates are usually read today. There are two principle ways in which P1-P3 are commonly paraphrased into propositions; I'll show them at the example of P1. Consider first this paraphrase:

P1M: Given two different points it is always *possible* to draw a straight line between them.

P1M asserts the feasibility of the operation described in P1. I shall call this paraphrase *modal* because it involves the notion of possibility. Apparently it squares very well with the Euclid's intended meaning. The principle problem concerning this paraphrase is that it remains logically sterile. It would be an interesting project to translate Euclid's geometry in terms of modal logic taking the modal interpretation of Postulates seriously but for the best of my knowledge it has been never done so far, and I'm not going to pursue this project in this book either. But usually the modal paraphrase serves a different purpose, namely a mere blurring of the Euclid's distinction between Postulates and Axioms; this can allow one to claim that every Euclid's Proposition follows from a number of unproved basic propositions (usually called "axioms"). The term "proposition" as a common name for Euclid's Problems and Theorems is introduced as a part of the same modification, which aims at rebuilding of Euclid's geometry after the Classical Model of Science. But unless these efforts are followed by much deeper changes of Euclid's setting they remain purely rhetorical. When one tries to make it precise what does it mean that Euclid's Propositions "follow from" assumed first principles, one immediately observes that modal notions play no role in it.
The idea of modal paraphrasing of Euclid's Postulates apparently stems from Aristotle's own attempts to use modal notions for resolving certain mathematical and mathematico-philosophical questions, in particular for making sense of mathematical infinity. (A different Aristotle's attempt to use modal notions for an analysis of a Theorem will be discussed below in the paragraph D of this section.) In Aristotle's view, what has been called in the later tradition *potential* infinity is the only sound notion of infinity. A given straight segment, in Aristotle's view, doesn't actually contain an infinite number of points but only provides a possibility to mark as many points on it as one wishes. This sounds appealing but rises this question: what is a precise difference between a conceived possibility to mark a point and marking it? What we can do about it on a diagram cannot provide an exhaustive explanation until one identifies the drawn straight line with the drawing itself - and Aristotle certainly wouldn't go that far. One can only imagine eating a cake and really eat it: in this case the modal distinction between possible and actual things seems to be clear. But to imagine a geometrical point and actually "get" it is arguably one and the same thing. If the distinction between actual and potential (i.e. merely possible) mathematical objects can make any sense at all it should be described more precisely but not only by an analogy with the common practice. In spite of the continuing work on formal mathematical theories of modal logic

philosophy and mathematics of 20th century hardly clarified this question significantly. In fact the mainstream developments in foundations of mathematics in 20th century rather followed the idea to lift the traditional Aristotelian ban of *actual* infinity and so get rid with modal distinctions in mathematics (Note 58). This brought about another method of paraphrasing Euclid's Postulates into propositions, which I shall call *existential*. Consider the following paraphrase of P1:

P1E: Given two different points *there exist* a straight segment bounded by these points.

While P1M says that the straight line can be produced P1E says that it is already there, it eternally (or timelessly) exists. In 20th century this approach has been misleadingly called Platonic although, as we have seen, it has little to do with the historical Platonism. While the formalisation of P1M requires a modal logic P1E can be easily formalised with the usual modern First Order logic. There were few recent attempts to formalise Euclid's geometrical reasoning and more generally the traditional geometrical reasoning in terms of First Order logic (see Avigad et al. 2008). Such interpretations can demonstrate the power of modern methods and also clarify the mathematical content of ancient mathematical texts from a modern viewpoint but they can hardly clarify specific ancient foundations of mathematics, which are of our concern here.

C) Axioms

A comparison of Euclid's Axioms with the notion of axiom found in Aristotle brings a richer outcome. In *An. Post.* ch.10 and several other places Aristotle discusses the distinction between those first principles, which are specific for a given science, and those, which are shared in common by certain sciences. In this context Aristotle gives the following mathematical examples:

"Instances of first principles peculiar to a science are the assumptions that a line is of such-and-such a character, and similarly for the straight line; whereas it is a common principle, for instance, that if equals be subtracted from equals, the remainders are equal." (*An. Post*. 76a38-43, Heath's translation)

Euclid's A3 is recognised in this quote immediately and uncontroversially. Aristotle's terminology agrees here with Euclid's: both writers describe A3 as *common* (common notion or common principle). "Axiom" is Aristotle's alternative term for "common opinion" or "common principle" (see, for example, *Met.* 997a12); although one doesn't find in Aristotle's corpus a place where any of Euclid's Axioms would be called by this name (axioma) the tradition to use this name for Euclid's *common notions* also perfectly complies with Aristotle's terminology. Let's now look at non-mathematical examples of axioms given by Aristotle; they turn to be more relevant to our topic than one might expect. Says Aristotle:

"By first principles of proof [as distinguished from first principles in general] I mean the common opinions on which all men base their demonstrations, e.g. that one of two contradictories must be true, that it is impossible for the same thing both be and not to be, and all other propositions of this kind." (*Met.* 996b27-32, Heath's translation, corrected)

We see that by "common opinions" (=axioms) Aristotle calls here basic logical laws. This Aristotle's terminological peculiarity should be taken into consideration in any discussion of Aristotle's logic. This author doesn't call by the name "axiom" any assumed immediate premise like we do this today. Axioms for Aristotle are always "common" while immediate premises are usually specific. As first principles of demonstration the axioms (in Aristotle's sense) are common for all sciences, which involve demonstrations. This makes a difference between axioms and first principles, which are specific for geometry, astronomy, physics, etc. One may wonder how this later notion of axiom (universal logical law) relates to the notion of mathematical axiom. Although both kinds of axioms are "common" for many sciences they are not common for the same sciences: logical axioms are common for all demonstrative sciences without any exception while mathematical axioms are common only for mathematical sciences (in particular for geometry and arithmetic). The problem of relationships between mathematical and logical axioms is touched upon by Aristotle in the following two passages:

"We have now to say whether it is up to the same science or to different sciences to inquire into what in mathematics is called *axioms* and into [the general issue of] essence. Clearly the inquiry into these things is up to the same science, namely, to the science of the philosopher. For axioms hold of everything that [there] is but not of some particular genus apart from others. Everyone makes use of them because they concern being *qua* being, and each genus is. But men use them just so far as is sufficient for their purpose, that is, within the limits of the genus relevant to their proofs. Since axioms clearly hold for all things *qua* being (for being is what all things share in common) one who studies being *qua* being also inquires into the axioms. This is why one who observes things partly [=who inquires into a special domain] like a geometer or a arithmetician never tries to say whether the axioms are true or false" (*Met.* 1005a19-28, my translation)

"Since the mathematician too uses common [axioms] only on the case-by-case basis, it must be the business of the first philosophy to investigate their fundamentals. For that, when equals are subtracted from equals, the remainders are equal is common to all quantities, but mathematics singles out and investigates some portion of its proper matter, as e.g. lines or angles or numbers, or some other sort of quantity, not however *qua* being, but as [...] continuous." (*Met.* 1061b, my translation)

In order to see that the above passages indeed say something about logical laws one should take into consideration an important feature of Aristotle's thought, which I have briefly mentioned in I.1.2: Aristotle conceives of logical laws (like "one of two contradictories must be true" or "it is impossible for the same thing both be and not to be") as fundamental features of Being ("being *qua* being") but not (only) as rules of thought. This is why the title of "logical axiom" is in fact more appropriate in this case than that of "logical law". By the "science of philosopher" and "first philosophy" Aristotle means a "science of Being" called today *ontology*. Having in mind the traditional title of the collection of Aristotle's texts presenting this "science of philosopher", it is also appropriate to call this science *metaphysics*. (It is not now my aim to distinguish these philosophical disciplines precisely.) Thus when Aristotle says that the inquiry into the axioms makes part of the "science of the philosopher" which treats "being *qua* being" there is no doubt that by "axioms" he means here [what we call today] logical laws.

Mathematical axioms are mentioned in the beginning of the second quote. Since Aristotle repeatedly points in his writings at Euclid's A3 as a standard example of mathematical axiom the notion of mathematical axiom referred to in the above passage can be safely identified with that found in Euclid's *Elements*. But immediately after mentioning mathematical axioms Aristotle describes a notion of axiom, which is quite different: he obviously talks here about his notion of logical axiom. The problem is that Aristotle doesn't say it explicitely that he uses the word "axiom" in two different senses, so the reader might think that sentences like "one of two contradictories must be true" could be found under the title of axiom in some mathematical writings known to Aristotle.

But this impression is superficial and most certainly wrong. In my understanding the following is going on here. Talking about mathematical axioms Aristotle refers to an established concept, about which he has learnt from his contemporary mathematics. But then Aristotle suggests an original and far-reaching generalisation of this notion supposed to cover not only mathematical disciplines but also other sciences including natural sciences. This is how the notion of logical axiom (logical law) came about: Aristotle didn't borrow it in mathematics but invented. Mathematical axioms like those used by Euclid in his *Elements* suggested to Aristotle this great invention.

Thus the unnoticed change of the meaning of the term "axiom" can be intentional. Aristotle perfectly knows what mathematicians call by the name "axiom" but he is not satisfied with this current notion; so he explains us what thinking people - including mathematicians - *should* understand by this word. Aristotle couldn't see a mathematical treatise where explicitly stated logical laws would take the place of traditional mathematical axioms as we know them after Euclid's *Elements.* But Aristotle might dream that once such a treatise will be written. We shall shortly see why neither Aristotle nor any of his followers didn't make - and couldn't possibly make - any significant progress in this direction. But as we shall also see in Episode 3 this project in a different form was realised in 20th century: *Elements* of Bourbaki begin with a presentation of a system of logic (Bourbaki 1954).

The thesis according to which Aristotle's logic is of a mathematical origin is well-known and hardly controversial (see Smith 1978 and further references thereof). For a popular illustration let me show to the obvious analogy between Euclid's A1 and Aristotle's logical axiom of *perfect syllogism* (PS). In order to make this analogy more apparent I present here both A1 and PS in a modernised form:

A1M: If A=B and C=B then A=C

PSM: If all A are B and all B are C then all A are C

The fact that Euclid is younger than Aristotle shouldn't confuse the reader: the core content of Euclid's *Elements* is older than Aristotle's logic; while the latter is a genuine invention of its author the former is not. So historically speaking it is more likely that Aristotle invented PS after the model of A1 rather than the other way round.

It can be also mentioned that the core of Aristotle's logical terminology stems from the earlier established mathematical terminology. We have seen this already at the example of the term "axiom" but there are also other examples. A telling one is the expression "figure of syllogism" having an obvious geometrical origin. In what follows I shall also point to Aristotle's letter notation and show that it likely derives from the mathematical letter notation, which we discussed above. My principle aim here, however, is not to justify the thesis about mathematical origins of Aristotle's logic, which I take to be already established, but rather try to explain why and how Aristotle used his mathematical sources. One might suggest that Aristotle managed to grasp a logical form of his contemporary mathematics, and this allowed him to formulate his logic. But this is definitely wrong. As we shall shortly see Aristotle's logic in fact doesn't fit the kind of mathematical reasoning he could be aware of, namely one, which is present in Euclid's *Elements*. A plausible way in which Aristotle could proceed is different.

Notice that Euclid's Axioms are used only in *proofs* - now I'm talking about *proofs* in the technical sense of the term explained in 1.3C above - not in *constructions* or some other parts of Propositions. These *proofs* are all based on the same mathematical Axioms whether corresponding Propositions are geometrical or arithmetical. Aristotle's notion of proof generalises upon mathematical *proofs* in this technical sense - not upon mathematical reasoning generally. Aristotle's notion of logical axiom (law of logic) generalises upon the notion of mathematical axiom, as I have already explained. The main purpose of this generalisation was, in my view, the extension of the notion of science (episteme) beyond mathematics into the domain of natural sciences. The new notion of proof based on universal logical axioms was supposed to be applicable in all of these disciplines. Understandably Aristotle was more concerned about consequences of his project for physics (i.e. natural sciences) rather than for mathematics, which was already well-established. But as I have already mentioned in a longer run this project greatly influenced mathematics too.

Let's now return to the two Aristotle's passages quoted above. The first passage begins with the question of whether axioms should be treated by the "science of essence", i.e. by ontology and metaphysics, or by some other science. What other science Aristotle could have here in mind? One possibility is that he conceives here of a notion of logic as distinguished from ontology, then asks whether or not it is appropriate to merge the two disciplines into one, and finally answers in positive. However from a historical viewpoint it seems more likely that Aristotle points here to the Platonic notion of *universal mathematics*, which provides an alternative approach to the same issue. I shall discuss this notion in paragraph E) below.

In the second passage Aristotle mentions that mathematical axioms are common for all quantities (poson). Nevertheless he avoids here to discuss a possibility of treating quantities in general. The alternative he puts forward is this: on the one hand, there is mathematics, which applies axioms separately to particular genus, and on the other hand, there is ontology, which relates axioms to Being. But axioms relevant to this latter approach are logical axioms, not mathematical. If I'm right that in the first passage Aristotle tacitly rejects the *universal mathematics* this explains why he doesn't consider a possibility of general science of quantity in the second passage. For the *universal mathematics* can be described as such a science, as we shall see. In the next Episode we shall also see how this missed possibility was eventually realised in 17th century.

The specific relationships between mathematical and logical axioms just described make it difficult to analyse the former in terms of the later. When one tries to describe Euclid's Axioms in terms of Aristotle's logic and epistemology Aristotle's own notion of axiom strikes the eye first. But then one immediately sees that the two authors use the term in different senses. The reason is that unlike Plato's philosophy Aristotle's philosophy suggests a radical revision of his contemporary mathematics rather then accounts for it in ontological and epistemological terms. For this reason Aristotle's philosophy is less helpful for understanding Euclid. However it is more helpful for understanding what happened in foundations of mathematics afterwards. Consequences of Aristotle's philosophical reform for mathematics are described more precisely in the next paragraph.

D) Propositions

As it has been noticed by other scholars Aristotle's syllogistic is "hopelessly inadequate" for grasping Euclid's reasoning (Smith 1978 and Mueller 1974). (Note 59). This is why instead of making another hopeless attempt I prefer to analyse a mathematical example, which Aristotle provides himself:

"Let *A* be two right angles, *B* triangle, *C* isosceles. Then *A* is an attribute of *C* because of *B*, but it is not an attribute of *B* because of any other middle term; for a triangle has [its angles equal to] two right angles by itself, so that there will be no middle term between *A* and *B*, though *AB* is matter for demonstration." (*An. Pr.* 48a33-37, Heath's translation, corrected)

Aristotle discusses here a syllogism, which amounts to application of the theorem about the sum of internal angles of a triangle to a special case of isosceles triangle. In a modernised form this syllogism can be written as follows:

All triangles have [...] two right angles (Premise 1: *AB*) (Note 60)
All isosceles triangles are triangles (Premise 2: *AC*)

All isosceles triangles have [...] two right angles (Conclusion: *BC*)

The principle point made by Aristotle in the above passage is this: Premise 1 is *immediate* , i.e. it cannot be proved by introducing another middle term *D* "between" *A* and *B* and making a similar syllogism with the latter three terms. Aristotle tries to explain why Premise 1 is immediate using the traditional Platonic distinction between intrinsic features, which a given thing has "by itself" (kath'auto) and extrinsic features, which a given thing may have in virtue of some other things. Aristotle's argument is this: the two-right-angles property (*2R*) of triangle (*T*) belongs to the very essence of triangle, so it cannot be proved by introducing any new intermediate term *S* such that *2R* would be an essential property of *S* and *S* would be an essential property of *T* (Note 61). However Aristotle definitely knows that Premise 1 can be proved as a geometrical theorem (E1.32 in the *Elements*). So what he says here sounds like a sheer absurdity from the point of view of Aristotle's own logic: we have got an immediate premise, which is matter of demonstration! The problem, to which Aristotle points us here, is that the syllogistics turns to be incapable to grasp this latter demonstration. The best what it can do about E1.32 is to show how this general Theorem is applied to a special case. In order to show where lies the problem let me quote E1.32 in full:

"For any triangle, (if) one of the sides (is) produced, (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.

Let ABC be a triangle, and let one of its sides BC have been produced to D. I say that the external angle ACD is equal to the (sum of the) two internal and oppo site angles CAB and ABC, and the (sum of the) three
internal angles of the triangle ABC, BCA, and CAB is equal to two right-angles.

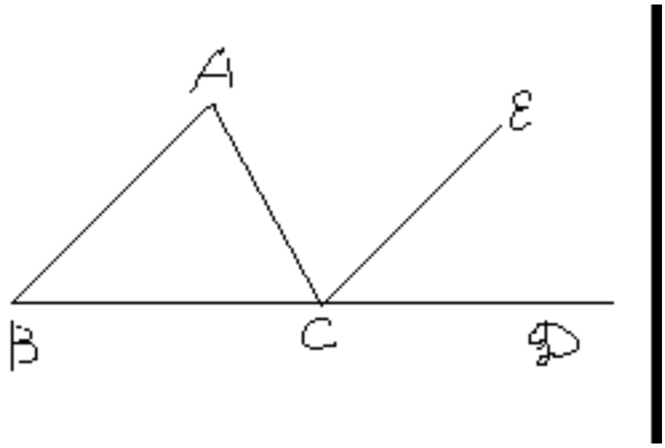


Fig.7

For let CE have been drawn through point C parallel to the straight-line AB [Prop. 1.31].

And since AB is parallel to CE, and AC has fallen across them, the alternate angles BAC and ACE are equal to one another [Prop. 1.29]. Again, since AB is parallel to CE, and the straight-line BD has fallen across them, the external angle ECD is equal to the internal and opposite (angle) ABC [Prop. 1.29]. But ACE was also shown (to be) equal to BAC. Thus, the whole angle ACD is equal to the (sum of the) two internal and opposite (angles) BAC and ABC.

Let ACB have been added to both. Thus, (the sum of) ACD and ACB is equal to the (sum of the) three (angles) ABC, BCA, and CAB. But, (the sum of) ACD and ACB is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) ACB, CBA, and CAB is also equal to two right-angles.

Thus, for any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles. (Which is) the very thing it was required to show."

In this Theorem one can immediately identify the six part of Proclus' analysis. A general difficulty of application of Aristotle's syllogistic to Euclid's reasoning is that there is no obvious way to fit these six parts into this logical form. The best one can hope to achieve by applying syllogistics in this case is to render the *proof* of E1.32 into a syllogistic form. But this *proof* is preceded by the *construction* of the new straight lines *CD* and *CE*, and essentially depends on properties of the emerging construction *ABCDE*, not only on properties of the given triangle *ABC*. This shows two important things. First, it shows that Aristotle is quite

right assuming that there is no hope to deduce E1.32 from a definition of triangle whatever this definition might be. Second, it shows that Aristotle is totally wrong assuming that the *2R* property of triangle is itself an intrinsic property, which can be considered as a basis for definition of triangle. For it becomes evident only through considering new "external" elements but not through a deepening of our grasp of triangle's intrinsic nature. Anachronistically one could say that the *2R* property of triangles actually reflects an essential property of Euclidean plane, namely the fact that this plane is flat. This news, however, would hardly help an Aristotelian to render the proof of E1.32 into syllogisms.
A more specific observation is this. Aristotle's syllogistics uses premises of the form *all A are B* (in fact the universal quantifier "all" is a modernisation but let me skip this point). In other words it operates only with *properties*. But assumptions used in mathematics more often have the logical form of *relations* like the relation of being parallel. That's why the modern logic, where the notion of property (one-place predicate) is seen as a particular case of a more general notion of relation (*n*-place predicate) has more power to support a mathematical reasoning. But in spite of this significant difference between Aristotelian and modern logic most of modern approaches to logical analysis and logical foundations of mathematical reasoning seem to assume - most often uncritically - few fundamental feature of Aristotle's approach, which I am now going to describe.
From Aristotle's viewpoint mathematics should assume, first, universal logical axioms (like any other science) and, second, certain special principles in the form of definitions and/or particular basic propositions (immediate premises). Then it should deduce further propositions from the basic propositions according to the assumed logical axioms. This is how a ready-made mathematical theory should look like accordingly to Aristotle's Classical Model of Science. I claim that in spite of significant modifications of logic this Aristotelian scheme had a great influence on mathematics and philosophy of mathematics of 20th century. Moreover the were no serious attempts to rebuild mathematics after the Classical Model of Science before 20th century. An obvious reason for it is that in spite of its ancient origin this Model clashes with the traditional mathematics (as we know it after Euclid) dramatically. I shall point here only to two major differences between Euclid's mathematics and the notion of mathematical theory implied by the Classical Model of Science.
The first difference concerns Axioms: in Euclid they cover only mathematical disciplines (geometry and arithmetic) while in Aristotle they cover all sciences including natural sciences. This implies that Euclid's mathematical reasoning prima facie is not based on any system of logic - if by logic we understand a normative theory of *formal* reasoning, i.e. a

theory of reasoning, which is neutral with respect to the *content* of this reasoning. Euclid's mathematics has only a tool of reasoning, which is neutral with respect to particular kinds of mathematical content but which is not supposed to be applicable outside mathematics. I mean, of course, the scheme of mathematical reasoning determined by Euclid's five Axioms.

The second difference, which in my view is even more significant, concerns Euclid's Postulates, which has no counterpart in the Classical Model of Science. We have seen that Postulates are non-propositional fundamentals of Euclid's mathematics, which play a central role in it. Postulates are supposed to generate the whole of mathematical universe, and we have seen more precisely how it works in Euclid's *Elements*. One can disguise Postulates by calling them "axioms" but one cannot get rid of them (as Classical Model of Science actually requires) without destroying basics of Euclid's mathematical reasoning completely. The Classical Model of Science requires objects of study to be already there, it doesn't have anything like the notion of generation allowed by the Platonic scheme. This is why it implies in mathematics what today is (inappropriately) called Mathematical Platonism. This feature of the Model survives replacements of one system of logic for another.

I shall return to this important issue but already at this point I would like to encourage the reader to think about Classical Model of Science critically. Notice that modern physics began when this Aristotelian Model was largely abandoned in this domain. So at the very least we should not take the idea to apply it in mathematics as self-obvious.

To conclude this paragraph I would like to quote another Aristotle's passage where he analyses the same mathematical example from a very different viewpoint:

"Diagrams are devised by an activity, namely by dividing-up. If they had already been divided, they would have been manifest to begin with; but as it is this [clarity] presents itself [only] potentially. Why does the triangle has [the sum of its internal angles is equal to] two right angles? Because the angles about one point are equal to two right angles. If the parallel to the side had been risen [in advance], this would be seen straightforwardly" (*Met.* 1051a21-26, my translation) (Note 62)

The chapter of *Methaphysics* from which I took this quote concerns Aristotle's ontological distinction, which I only briefly mentioned above, namely one between *actual* Being (energeia) and *potential* Being (dunamis). At the first look this distinction seems analogous to Plato's distinction between Being and Becoming. But the above mathematical example shows that in fact it is quite different. From this new Aristotle's viewpoint one can indeed think of geometrical constructions as pre-existent. However this pre-existence is only potential unless it is actualised through an act of mathematical thought. Aristotle considers such pre-existence as ontologically deficient; on his account *actual* Being determines *potential* Being rather than the other way round. This new view allows Aristotle to give a full justice of geometrical constructions: as the above quote clearly shows he is wholly aware about the fact that *construction* is a crucial element of geometrical theorems. Although Platonism equally allows for taking seriously the generic aspect of mathematics it cannot get rid with the basic principle according to which the generic aspect of mathematics is fully determined by its ontic aspect. But Aristotle turns here the Platonic hierarchy upside down: unless a construction is realised through an on-going mathematical activity it doesn't exist in the proper sense at all (Note 63). We see that Aristotle's distinction between the actual and the potential Being suggests a view on mathematics, which differs dramatically from one implied by Aristotle's Classical Model of Science and his syllogistics. We encounter very different Aristotles in these two cases. It is a pity that this alternative modal approach to mathematics has been never further developed either by Aristotle himself or by the following tradition. At least it has been never developed up to a point where it could serve for rebuilding of foundations of mathematics. The idea is still waiting to be further developed in the future.

E) Proportion, Metabasis and Universal Mathematics

In Euclid's *Elements* the theory of proportion is developed separately for numbers (in Books 7-8) and for geometrical magnitudes (in Books 5 and 10). More precisely, the *Elements* don't contain such thing as *the* theory of proportion at all but do contain a theory of proportion of numbers and a theory of proportion of magnitudes. Obviously Euclid's theory of proportion of magnitudes cannot reduce to his theory of proportion of numbers because the former contains the case of incommensurable magnitudes. However Euclid also treats separately proportion of numbers and proportion of commensurable magnitudes. For this reason some Propositions in the *Elements* look nearly identical. Compare, for example, these two:

E10.13: "To find the greatest common measure of two given commensurable magnitudes."

E7.2: "To find the greatest common measure of two given numbers (which are) not prime to one another."

I provide here only *enunciations* of these Problems and leave it to the reader to check that the rest is equally very similar: in both cases Euclid applies the same method known today as *Euclidean algorithm*. Euclid's quasi-geometrical proofs, which he uses in arithmetic (see paragraph 1.3C3) makes this similarity even more apparent. The following pair of Propositions looks as another example of the same sort but in fact it is different because here the second Proposition holds for incommensurable magnitudes too.

E7.13: "If four numbers are proportional then they will also be proportional alternately."

E5.16: "If four magnitudes are proportional then they will also be proportional alternately."

The rest of the last two Theorems is not quite the same because they rely on essentially different Definitions of proportion: E7.13 refers to D7.20, which defines proportional numbers while E5.16 refers to D5.6. which defines proportional magnitudes and applies (in particular) to the case of incommensurable magnitudes.
The fact that Euclid uses in both cases the same term "analogon" (proportion) and doesn't avoid the textual identity of Propositions concerning numbers and magnitudes shows that the analogy between the two theories of proportion couldn't be left unnoticed. From a modern point of view it looks very unnatural that Euclid didn't develop this obvious analogy into a genuine generalisation and didn't include into his *Elements* an universal proposition, which in modern algebraic notation would read

if $a:b = c:d$ then $a:c = b:d$

and cover E7.13 and E5.16 as special cases. But at the same time this feature of Euclid's mathematics perfectly complies with Aristotle's Classical Model of Science, which, let me remind, assumes a fixed domain of studied objects and doesn't allow for switching between domains or merging them freely. Aristotle's views on this issue will help us to understand why Euclid didn't merge his two theories of proportion together like we do this today. But before I come to a more precise analysis of these Aristotle's views I would like to stress that the situation with the two theories of proportion in the *Elements* is in fact more involved than just described. The two theories are indeed first developed separately but at certain point Euclid makes a link between them. Namely Euclid includes into Book 10 of his *Elements* five

Propositions, which involve *both* numbers and magnitudes. Here is the first of them, which is worth to be fully quoted: (Note 64):

E10.5:
"Commensurable magnitudes have to one another the ratio which (some) number (has) to (some) number.

Let A and B be commensurable magnitudes. I say that A has to B the ratio which (some) number (has) to (some) number.

For if A and B are commensurable (magnitudes) then some magnitude will measure them. Let it (so) measure (them), and let it be C. And as many times as C measures A, so many units let there be in D. And as many times as C measures B, so many units let there be in E.

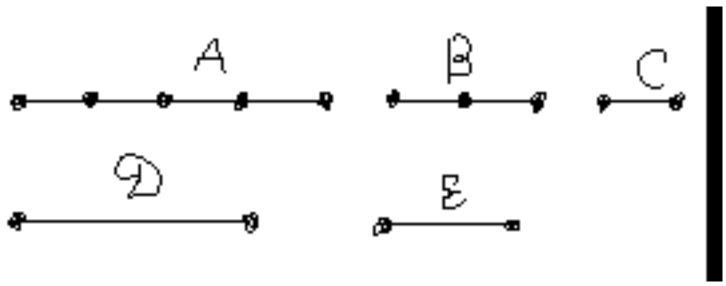


Fig.8

Therefore, since C measures A according to the units in D, and a unit also measures D according to the units in it, a unit thus measures the number D as many times as the magnitude C (measures) A. Thus, as C is to A, so a unit (is) to D [D5.6 or D7.20?]. Thus, inversely, as A (is) to C, so D (is) to a unit [E5.7?]. Again, since C measures B according to the units in E, and a unit also measures E according to the units in it, a unit thus measures E the same number of times that C (measures) B. Thus, as C is to B, so a unit (is) to E [D5.6 or D7.20?]. And it was also shown that as A (is) to C, so D (is) to a unit. Thus, via equality, as A is to B, so the number D (is) to the (number) E [E5.22?].

Thus, the commensurable magnitudes A and B have to one another the ratio which the number D (has) to the number E. (Which is) the very thing it was required to show."

The *construction* of E10.5 produces (by fiat) a pair of numbers, which Euclid denotes by letters "D" and "E" such that the given magnitudes A, B have the same ratio as numbers D, E. Notice that talking about the "same ratio" in the previous sentence I referred to a general notion of ratio, which applies both to numbers and to magnitudes. However the preceeding part of the *Elements* contains no Definition, which allows for the equality between a ratio of numbers and a ratio of magnitudes! Euclid never provides explicit references to Definitions, Postulates, Axioms and preceding Propositions leaving it to the reader to guess which reference would be appropriate in every particular case. Usually this task is easy but in the given case one may think of different Definitions of proportionality (D5.6 or D7.20) and still none of them fits the purpose exactly. A similar observation concerns Propositions E5.7 and E5.22, which officially apply only to magnitudes but in E10.5 seem to be applied in a situation involving both magnitudes and numbers. I cannot see any way to avoid saying that E10. 5 and the following four Propositions, which are obviously supposed to fill the gap between the two theories of proportion (one for numbers and the other for magnitudes) don't actually achieve this goal.
Euclid's attempt to find a link between the two theories of proportion shows that he might be aware about the idea of a general theory of proportion. There are some independent evidences (Granger 1976) that Eudoxus, who invented the theory of proportion presented in Book 5 of the *Elements* as a geometrical theory, considered it himself as a genuine generalisation of the old Pythagorean arithmetical theory of proportion presented in Books 7-9. Eudoxus' term "magnitude" might originally stand for a general notion of quantity including numbers and geometrical magnitudes as its special cases. Another evidence, which shows that the idea of a general theory of proportion was around in Euclid's circle, is given by Proclus:

"As for unifying bond of the mathematical sciences, we should not suppose it to be proportion, as Erathosphenes says. For though proportion is said to be, and is, one of the features common to all mathematics, there are many other characteristics that are all-pervading, so to speak, and intrinsic to the common nature of mathematics." (*Commentary* , 43.22-44.1, Murrow’s translation)

Eratosphenes of Cyrene is an Alexandrian librarian contemporary to Euclid, who was, as Proclus tells us here, enthusiastic about the idea of unification of mathematical disciplines on the basis of a generalised theory of proportion. Evidences that generalised mathematical theories of this sort were known already in Aristotle's times are found in *Metaphysics* where this author repeatedly refers to "universal mathematics":

"The question may be asked whether first philosophy is universal or deals with some particular genus or some one class of things. For not even in mathematical sciences is the method one and the same; geometry and astronomy, for instance, deal with a certain class of thing, but the universal science of mathematics is common to all branches." (1026a3-7, Heath's translation)

"For each of the mathematical sciences is concerned with some distinct genus, but universal science of mathematics is common to all" (1064b8-9, Heath's translation)

"Further some propositions are proved universally by mathematicians, which extend beyond these substances [belonging to special mathematical sciences]" (1077a9-10, Heath's translation)

"Just as the universal part of mathematics deals not with objects which exist separately, apart from extended magnitudes and numbers, but with magnitudes and numbers, not however *qua* such as to have magnitude or to be divisible, clearly it is possible that there should also be both propositions and demonstrations about sensible magnitudes, not however *qua* sensible but *qua* possessed of certain definite qualities." (1077b17-22, Ross' translation)

These passages show Aristotle's attitude to the issue : he never treats the notion of universal mathematics systematically (in his known writings) but refers to "universal mathematics" and "universal mathematical propositions" as something already known to the reader. In the first of the above passages Aristotle compares the *universal mathematics* and his *first philosophy*. This clearly concerns the issue discussed in the previous paragraph: while the *universal mathematics* covers all mathematical sciences the *first philosophy* covers non-mathematical sciences as well. In the last quoted passage Aristotle refers to universal mathematical propositions (i.e. theorems of the *universal mathematics*) in order to explain his theory of abstraction (see 1.2B): just like universal mathematical propositions take numbers and geometrical magnitudes in abstraction from the specific properties allowing for distinguishing between these two kinds of mathematical entities, mathematics in general abstracts from all sensible qualities of things and treats them *qua* mathematical objects. Most likely by universal mathematical propositions Aristotle means here certain theorems of Eudoxus' theory of proportion (conceived of as a generalisation of the Pythagorean theory of proportion rather than as a specific geometrical theory).

The fact that Euclid's separate treatment of arithmetical and geometrical proportion is compatible with the Classical Model of Science doesn't mean that this Model rules out any possibility of generalisation in this case. But it requires the following question to be clearly answered: *What* a given general theory is about? or What is its subject matter? As far as we are talking about Aristotle's original version of this Model rather than its later modifications this subject-matter is supposed to be a particular *genus*. The following passage, where Aristotle explains his notion of universal proposition using a mathematical example, shows that the aforementioned requirements has a clear logical aspect and cannot be dispensed with as a "pure metaphysics":

"Something holds universally when it is proved of an arbitrary and primitive case. E.g. having [the sum of internal angles equal to] two right angles doesn't hold universally of figures - you may indeed prove of a figure that it has two right angles, but not of an arbitrary figure, nor can you use an arbitrary figure in proving it; for quadrangles are figures but do not have angles equal to two right angles. An arbitrary isosceles [triangle] does have angles equal to two right angles - but it is not primitive: triangles are prior. Thus if an arbitrary primitive case is proved to have two right angles (or whatever else), then it holds universally of this primitive item, and the demonstration applies to it universally [...] [I]t does not apply to the isosceles [triangles] universally, but extends further." (*An.Pr.* 73b33-74a4, Barnes' translation)

For better understanding of this passage I shall analyse it into a number of separate claims:

(i) To *hold universally* amounts to being true about some case, which is both *arbitrary* and *primitive*.

This is a formal definition, which the following mathematical example is supposed to explain.

(ii) It is not the case that figures (in general) have the *2R* property *universally* .

This is in spite of the fact that

(iii) Certain figures have the *2R* property.

What Aristotle tells us here is clear: some figures, namely, triangles, do have the *2R* property while some other figures, for example, quadrangles, don't. However Aristotle renders this conclusion in somewhat different form:

(iv) It is not the case that an *arbitrary* figure has the *2R* property because certain figures, for example quadrangles, don't have this property.

In order to understand why Aristotle uses here the term "arbitrary" remind the six-part structure of Problems and Theorems described in 1.3C above. Suppose one wants to prove that a triangle has the *2R* property. Then one proceeds as follows: takes an arbitrary triangle (*exposition*), applies the *enunciation* of the given theorem to this chosen triangle (*specification*), makes an appropriate *construction*, which allows for the wanted *proof*, and finally comes to the desired *conclusion*. The Theorem is tantamount to the claim that an *arbitrary* triangle has the *2R* property. "Arbitrary" refers to the *exposition* of this Theorem and to what Proclus calls "first conclusion" (see again Note 52). Thus saying that an *arbitrary* triangle has the *2R* property translates into saying that any triangle (or all triangles) has (have) this property. Saying that an arbitrary figure doesn't have *2R* property translates into saying that not any figure has the *2R* property.
Checking with (i) shows that to "be true about an arbitrary case" is indeed a necessary condition for being true (holding) universally. Since the proposition "a figure has the *2R* property" is not true about an arbitrary figure it doesn't hold universally. However, as we shall now see, this condition is not sufficient.

(v) An arbitrary isosceles triangle has the *2R* property.

This is not surprising: all isosceles triangles are triangles and all triangles have the *2R* property. Hence all isosceles triangles have the *2R* property.

(vi) Proposition "isosceles triangles have the *2R* property" doesn't hold universally; the corresponding proof doesn't apply to isosceles triangles universally.

This sounds surprising for one who expects that Aristotle uses his term "universally" in the sense of universal quantifier. But in fact he doesn't. Proposition "isosceles triangles have the *2R* property" is true for any isosceles triangle but it nevertheless doesn't hold universally because

(vii) An isosceles triangle [in the given context] is not primitive, a [general] triangle is prior.

(vii) complies with (i): the condition of being true *primitively* turns out to be essential. We should now understand what it amounts to. For this end let me now make explicit one further assumption, which Aristotle makes here only tacitly:

(viii) A [general] triangle [in the given context] is primitive.

The last element needed for understanding of "primitive" is this:

(ix) Proposition "an isosceles triangle has the *2R* property" extends further, [namely, to triangles in general].

Proposition "a triangle has the *2R* property" holds universally because it holds for an arbitrary triangle and an arbitrary triangle is a *primitive* case with respect to this proposition. Thus "holds universally about *T*s" in Aristotle's language means roughly this: "true of all *T*s and of nothing else".
This is, of course, another rendering of Aristotle's notion into the modern extensional language. Aristotle himself thinks of general triangles and isosceles triangles not in terms of logical extensions of these notions but in terms of their "generic cases" appearing in *expositions* of corresponding Problems and Theorems. The *arbitrary* character of a given case guarantees that every proposition *P*, which is true in this arbitrary case *A*, also holds for its genus *G*. Another relevant issue is whether or not *P* holds for *G* "by itself" or it holds in virtue of some other genus G'. The former situation occurs if and only if the case *A* is primitive with respect to *P*. An arbitrary isosceles triangle is not primitive with respect to the *2R* property because this property belongs to its genus (of isosceles triangles) in virtue of another genus (that of triangles). In terms of syllogistics this is tantamount to saying that the premise "triangle has the *2R* property" is immediate while the premise "isosceles triangle has the *2R* property" is not (see again *An.Pr.*48a33-37 discussed in the previous paragraph).

Let's now see what kind of constraints this Aristotle's notion of being universal imposes onto possible generalisations of mathematical theories. Let me first illustrate this with a dummy example. Imagine that the *2R* property of triangles has been noticed and proved only in some very special cases, say, only for regular triangles and for isosceles right-angled triangles (these two kinds of triangles appear in Plato's *Timaeos*). Then in Aristotle's view it wouldn't be appropriate to call these two kinds of triangles by some general name (say, that of *simple* triangles) and to state a general theorem according to which all *simple* triangles have the *2R* property. For this general theorem would anyway reduce to considering the two known cases. In this sense *simple* triangle wouldn't constitute a genus. Then imagine that a further research brought into the light the fact that the *2R* property belongs to all *isosceles* triangles but not only to the isosceles triangles of the two aforementioned kinds. Although this looks like a real progress Aristotle wouldn't consider the corresponding general theorem about isosceles triangles as a sound generalisation either. For he already knows that the *2R* property belongs to isosceles triangles not in virtue of their proper genus but in virtue of a "higher" (i.e. more general) genus of triangles! So Aristotle would dismiss the proposed generalisation as incomplete.

This dummy example shows, in my view, what Aristotle could feel about generalised theories of proportion and the "universal mathematics". A mere merging of the two existing theories of proportion, on Aristotle's account, couldn't produce anything deserving the name of a theory. Calling numbers and magnitudes by a common name like *quantity* and stating general theorems about *quantities*, in Aristotle's view, can be only misleading unless a general theory of quantity is developed independently. Only in this latter case *quantity* could be considered as a *genus* and earlier known theories about numbers and magnitudes could be reduced to a general theory of quantities. Until the higher genus of quantity is properly understood and its properties are effectively used for proving theorems about numbers and magnitudes one should avoid the talk of general theory of proportion and stick to the traditional separation of mathematical disciplines.

As a matter of fact Aristotle (as we know him after his preserved writings) doesn't say much about the universal mathematics or about generalisation in mathematics. For mathematics, as I have already argued, was Aristotle's starting point rather than his field of study. It seems that Aristotle considered the generalisation aimed at by the universal mathematics as insufficient, and for this reason not worth trying. Aristotle's project of "first philosophy" was similar to (by Aristotle's own word) but by far more ambitious than that of universal mathematics: it aimed at covering everything that there is but not only what is mathematically treatable. Anyway the idea to find for mathematics its proper *genus*, and develop various mathematical disciplines on the basis of a single general theory treating this proper genus, remained pertinent during the whole history of mathematics until today. In Modern times, as we shall see, people identified the universal mathematics with algebra, while in 20th century people tried to base the whole of mathematics on Set theory. This pattern greatly influenced the current notion of foundations of mathematics.

Conclusion of Episode 1

In the above philosophically-laden presentation of Euclid's *Elements* I told very little about its mathematical content; its detailed analysis can be found elsewhere, in particular in Heath's edition of the *Elements*. But I hope I managed to present the reader some philosophical aspects of Euclid's *Elements*, which, in my view, remain pertinent for today's mathematics. We have seen that Plato's and Aristotle's philosophy provide rather different perspectives on Euclid's mathematics. Aristotle's approach is by far more revisionary with respect to his contemporary mathematics than Plato's approach. This is why Plato's philosophy and the later Platonic philosophical tradition turn to be more helpful for understanding Euclid than Aristotle's philosophy and its tradition. However when one studies the history of Greek mathematics in a wider historical perspective Aristotle's approach turns to be at least equally important since many of later trends in foundations of mathematics followed the Aristotelian line. This in particular concerns the notion of *logical* foundations of mathematics, which played a central role in foundations of mathematics in 20th century.

It is difficult to imagine that writing his *Elements* Euclid intended to produce a book, which would be later qualified by anyone as *the* foundations of mathematics. For Proclus' *Commentary* and some other sources tell us about a variety of concurrent approaches to the issue existing in Euclid's times. However this is exactly how Euclid's *Elements* were conceived of at certain point in the later tradition, which tried to improve upon Euclid rather than treat the issue of foundations of mathematics independently. The history of these improvements is very telling and could be a subject of a special study (which was never systematically done so far) but it lies out of the scope of this present book. The principle issue of this present study is the renewal of foundations, not their progressive development. Even if the difference between the renewal and the progressive development is a matter of degree, and even if it is too often confused by authors' rhetoric, I shall not make an attempt to build a narrative covering the whole history of foundations of mathematics from ancient times to today. I shall jump instead immediately into 17th century and consider several mathematical works which explicitly aim at the renewal of foundations. I don't want to hide that this Early Modern spirit of renewal greatly motivates the notion of foundations developed in this book.

Endnotes

Note 1

The case of cosmology is particularly interesting. Taking the possibility of empirical check as a necessary condition of science Kant qualified cosmology as a part of metaphysical speculation. For it seemed him obvious that the past of the universe we are living in is beyond any possible human experience. However it turned out that new observational methods together with new theoretic backgrounds allow for empirically-grounded claims about the remote past of our universe. Kant's views can be probably salvaged by arguing that cosmology in his sense is not today's empirical cosmology. This example shows that boundaries between the philosophical speculation and the empirical science cannot be determined once and for all.

Note 2

How this works in a given community depends on how it works in minds of its individual members, and how it works in an individual mind obviously depends on how it works globally in a community in which this given mind participates. I think about individual minds and communities of minds as complex systems without trying to reduce one level of organisation to the other and to claim that one of them is fundamental while the other is not.

Note 3

The difference between a "genuine understanding" of an earlier tradition and a "genuine invention", which brakes an existing tradition and starts a new one, is often conventional and sometimes even merely rhetorical. In some epochs like late Antiquity and Middle Ages new developments are usually presented as improvements on the common understanding of older sources; in some other epochs like ours it is more common to express oneself through refuting older doctrines and putting forward new ones. One may often switch between the two strategies by playing with identity conditions of older and new doctrines.

Note 4

One may ask where proposition T comes from. This is a matter of discovery, which I leave here aside. In any event it doesn't come first through a deduction from given premises, except trivial cases. All interesting theorems are first conjectured and only then proved. Remarkably this order is preserved in academic papers and textbooks, that is, in standard presentations of a ready-made knowledge.

Note 5

The fact that foundations of a given discipline unify and organise the given discipline into a systematic whole explains why philosophy as an art of renewal of foundations is doomed to be disorganised. In order to be organised like a science philosophy would need its proper foundations and it's own cumulative research programmes. Such organisation may be appropriate for particular philosophical projects (which can eventually develop into new scientific disciplines) but not for philosophy in general. A science-like philosophy is incapable to perform a full-scale revision of foundations of science. This is unfortunate both for science and for philosophy. Sometimes cumulative philosophical projects indeed give birth to new scientific disciplines as it happened with psychology and is about to happen with neuroscience. But far more often they degenerate into a sheer scholasticism. The birth of a new scientific discipline is a good thing but notice that to perform a sustainable progress a newly born discipline still needs a continuing philosophical questioning of its foundations.

Note 6

One may argue that non-scientific doctrines cannot possibly have foundations. I assume here that they can. Think about a well-developed religious doctrine like Christianity. What I told above about educational, conceptual and systematic foundations seems to be applicable in this case: Christianity involves basic texts and basic beliefs reproduced and transmitted through education as well as a room for further developments. What is specific for scientific foundations is, in my view, the way in which these foundations are reproduced, namely the kind of radical revision, which has no analogues in religious doctrines or elsewhere.

Note 7

The scientific consensus is based on grounds (in particular empirical evidences and theoretical conclusions) which appeal to individual minds. Because of specialisation of science any individual scientist has the full access only to grounds related to his or her limited domain of study. The global scientific consensus is possible only because scientists trust each other: although an individual scientist has only very limited access to scientific grounds outside his or her domain of study he or she trusts his or her colleagues working in different domains. This double mechanism combining the direct access to grounds and mutual trust produces the global scientific consensus.

Note 8

The discovery of Non-Euclidean geometries certainly played a role in this change of the view on the *Elements* but this role was not as crucial as it is often claimed. See Episode 3 for a further discussion.

Note 9

Cf. this passage:

"[W]e must make a distinction and ask, What is that which always is and has no becoming; and what is that which is always becoming and never is? That which is apprehended by intelligence and reason is always in the same state; but that which is conceived by opinion with the help of sensation and without reason, is always in a process of becoming and perishing and never really is." (*Tim.* 27d-28a, Jowett's translation)

Note 10

Today's term "Mathematical Platonism" stems from (Bernays 1935); for a recent account of Mathematical Platonism see, for example, (Balaguer 1998). For a recent study of Plato's philosophy of mathematics see (Pritchard 1995).

Note 11

English term "understanding" is a reasonably good translation of Greek "dianoia" but it certainly should not be taken here along the line of philosophical hermeneutics of 19-20th centuries. One may rather think of Kant's "Verstand" also usually translated as "understanding" .

Note 12

See *Seventh Letter* where Plato disqualifies written literature as an appropriate means for doing philosophy.

Note 13

In this dialog Plato (as usual through the voice of Socrates) argues that it is inappropriate to think of number 2 as the sum of two units because one may obtain 2 by the inverse operation, namely by dividing a given unit into two halves. Thus, Plato argues, one should rather think of 2 "through itself", that is, through the idea of 2.

Note 14

Thus Plato's view according to which only ideas properly exist Plato perfectly agrees with Quine's dictum "no entity without identity".

Note 15

"[T]he world has been framed in the likeness of that which is apprehended by reason and mind and is unchangeable, and must therefore of necessity, if this is admitted, be a copy of something. Now it is all-important that the beginning of everything should be according to nature. And in speaking of the copy and the original we may assume that words are akin to the matter which they describe; when they relate to the lasting and permanent and intelligible, they ought to be lasting and unalterable, and, as far as their nature allows, irrefutable and immovable-nothing less. But when they express only the copy or likeness and not the eternal things themselves, they need only be likely and analogous to the real words. As being is to becoming, so is truth to belief. If then, Socrates, amid the many opinions about the gods and the generation of the universe, we are not able to give notions which are altogether and in every respect exact and consistent with one another, do not be surprised. Enough, if we adduce probabilities as likely as any others; for we must remember that I who am the speaker, and you who are the judges, are only mortal men, and we ought to accept the tale which is probable and enquire no further."
(*Tim.* 29a-d, Jowett's translation)

Note 16

It is not quite clear whether this Aristotle's argument is directed against Plato himself or rather against some of Plato's followers, who improperly vulgarise Plato's doctrine. Many of Aristotle's arguments against Platonics can be in some form found in Plato himself; some of these arguments can be interpreteded as a defence of "true" Platonism against its vulgarisation. Arguably Plato himself didn't conceive of ideas as separate : remind his notion of partaking. I skip these nuances in my presentation and don't distinguish between Plato and early Platonics attacked by Aristotle. Although Aristotle's philosophy for the obvious historical reason has a Platonic background it certainly doesn't reduce to it.

Note 17

Ross translates Aristotle's "too logoo proteron" (literally "prior in logos") as "prior in definition" or "prior in formula". Although Greek word "logos" in a technical sense indeed means "definition", the meaning of the term is larger and also includes “reasoning”. By “prior in logos” Aristotle means prior in a theoretical order.

Ross translates "the ousia proteron" as "prior in substance" or "prior in the order of substantiality". Arguably Ross' translation is better than mine, since Aristotle indeed aims here at the ontological order (as distinguished from the epistemological order), while the term “essence” in its common today's sense refers to conceptual priority rather than to ontological priority. I opt nevertheless for "prior by essence" because “essence” is the standard English translation (derived from the Latin translation tradition) of Aristotle's "ousia", which is widely used by Ross and other translators in other contexts. I feel very uneasy about translating a term like "ousia", which clearly stands in the original text for a single notion, by different terms depending on a given context, even if I realise that such equivocacity can be hardly always avoided.

Note 18

Aristotle's physics comprises all of the natural science but not only physics in today's more restricted sense. The principle Aristotle's example of physical entity is a living organism rather than an inanimate object like a stone.

Note 19

Compare this notion of hypostatising abstraction with Plato's notion of hypothesis, which helps him to distinguish between mathematics and dialectics (see the previous section).

Note 20

The modern history of mathematical physics begins only with Galileo's attempts to change the scholastic pattern of doing physics developed by Schoolmen after (their reading of) Aristotle. It is not surprising that Galileo looked for a support of Platonic rather than Aristotelian philosophy.

Note 21

In Greek we have two different terms here : “stoiceia” for the title of the book and “archai” for the general notion, which (according to Aristotle in *Posterior Analytics*) comprises Definitions, Postulates and Axioms. I translate the former term as “foundations” and the latter term as “fundamentals”.

Note 22

It is a plausible historical speculation that Euclid copied the list of definitions of Book 1 from an older source and didn't pay attention to the fact that some of these definitions are useless as far as further propositions are concerned. But this speculation doesn't explain why Euclid could do this (provided he was a rational being). My point is that he could consider this list of definition as having an independent epistemic value, not as an auxiliary device for proving theorems.

Note 23

Postulate 3 allows for drawing a circle with the center in one of the two ends of a given straight segment and the radius equal to this segment. The Postulate doesn't immediately allow for drawing a circle with a given centre and the radius equal to any given straight segment. However this latter construction is also doable on the basis of the complete set of Postulates and Axioms as it is shown in Proposition 2 of Book 1.

Note 24

Here is the appropriate diagram:

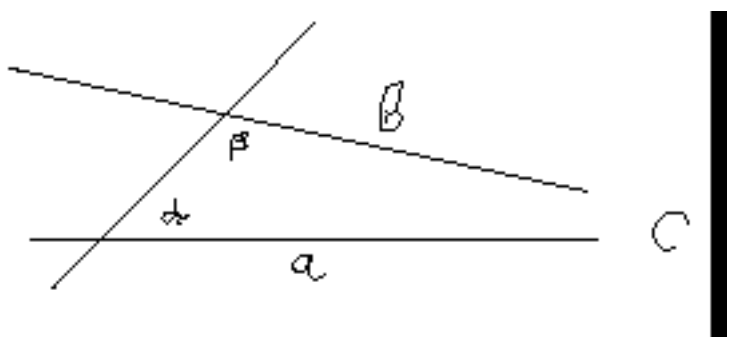


Fig.N1

P5 says that if ALFA+BETA <2d (=two right angles) then straight lines *a* and *b* intersect at certain point C. This famous postulate distinguishes Euclidean, i.e. in modern word "flat", geometry among a wider family of geometries developed in 19th century.

Note 25

A comprehensive history of attempts to prove P5, which eventually resulted into the discovery of Non-Euclidean geometry, can be found in (Bonola 1955)

Note 26

P1-P3 determine a precise mathematical sense in which the straight line and the circle generate the universe of geometrical objects dealt with in Euclid's Elements. But the notion of straight line itself is not given in the Elements any precise mathematical treatment. Instead Euclid mentions somewhat cryptic D1.4, which appeals to the intuition of "straight", "even" and the like.

Note 27

Cf. in Proclus:

"The drawing of a line from any point to any point follows from the conception of the line as the flowing of a point and of the straight line as its uniform and undeviating flowing. For if we think of the point as moving uniformly over the shortest path, we shall come to the other point and so shall have got the first postulate without any complicated process of thought. And if we take a straight line as limited by a point and similarly imagine its extremity as moving uniformly over the shortest route, the second postulate will have been established by a simple and facile reflection. And if we think of a finite line as having one extremity stationary and the other extremity moving about this stationary point, we shall have produced the third postulate." (*Commentary* 185.8-2, Morrow's translation)

Note 28

In A5 the copula is made explicit by the translator. I think this is perfectly justified; I can see no point in elaborating on the fact that there is no explicit copula in the original text here.

Note 29

Think about "equal copies" 2=2=2=... of the "ideal number" **2**. This ideal number can be thought of as a type shared by all equal 2s.

Note 30

I owe this remark to John Mayberry and Michael Wright

Note 31

Consider, for example, Proposition 7.1:

"Two unequal numbers (being) laid down, and the lesser being continually subtracted, in turn, from the greater, if the remainder never measures the (number) preceding it, until a unit remains, then the original numbers will be prime to one another.

For two [unequal] numbers, AB and CD, the lesser being continually subtracted, in turn, from the greater, let the remainder never measure the (number) preceding it, until a unit remains. I say that AB and CD are prime to one another—that is to say, that a unit alone measures (both) AB and CD.

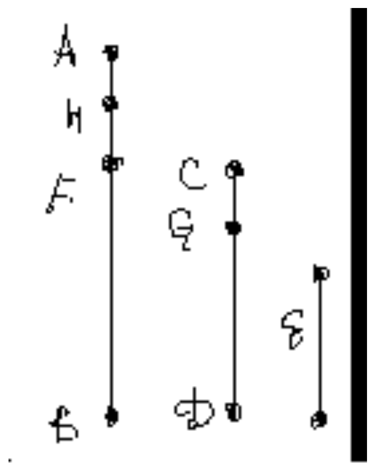


Fig.N2

For if AB and CD are not prime to one another the some number will measure them. Let (some number) measure them, and let it be E. And let CD measuring BF leave FA less than itself, and let AF measuring DG leave GC less than itself, and let GC measuring FH leave a unit, HA.

In fact, since E measures CD, and CD measures BF, thus also measures BF. And (E) also measures the whole of BA. Thus, (E) will also measure the remainder and AF measures DG. Thus, E also measures DG. And (E) also measures the whole of DC. Thus, (E) will also measure the remainder CG. And CG measures FH. Thus, E also measures FH. And (E) also measures the whole of FA. Thus, (E) will also measure the remaining unit AH, (despite) being a number. The very thing is impossible. Thus, some number does not measure (both) the numbers AB and CD.

Thus, AB and CD are prime to one another. (Which is) the very thing it was required to show."

The talk of representation of numbers by Euclid straight segments from a Platonic viewpoint requires some reservations. Remind the Platonic theory according to which geometrical objects are "distorted images" of numbers, while material drawings are distorted images of geometrical objects. This theory provides an explanation of Euclid's method used in the arithmetical Books of the *Elements*, which doesn’t involve the modern notion of representation. First of all it justifies the practice of using the same drawings for numbers and for geometrical magnitudes: since the Platonic relation “to be an image of” is assumed to be transitive one and the same picture can stand both for a number and for a straight line. This allows one to think of images used in arithmetic as images of numbers ignoring the fact that in a different context the same pictures can refer to lines.
Beware that all drawings found in modern editions the *Elements* are made by editors. But there is little doubt that geometrical diagrams were widely used in Euclid’s times too. See (Netz 2005).

Note 32
After discussing A1-A5 Proclus considers an additional axiom proposed by some people, which reads as follows:

"Two straight lines don't contain a space"

The axiom says that the two lines making part of the construction below cannot be straight lines:

Fig.N3

Proclus rules out this additional axiom on the ground that it doesn't have any arithmetical meaning but applies only in geometry (while axioms should apply universally). This shows that he believes that A4 like any other axiom from Euclid's list applies both in geometry and arithmetic.

Note 33

Remind that the equivalence of the equicomposability and the sameness of areas of polygons doesn't generalise to the case of polyhedra: polyhedra having the same volume are not always equicomposable.

Note 34

It seems me suggestive to think of the content of Books 1-2 as a theory which aims at the "regularisation" of any given polygon, i.e. at the construction of a square equal to a given polygon. In the end of Book 1 (E1.45) Euclid makes the first important achievement toward this strategic goal, namely he constructs a rectangle equal to an arbitrary polygon. After studying rectangles in Book 2 Euclids resolves in E2.14 (the last Proposition of this Book) the problem of construction a square equal to an arbitrary polygon. One may speculate that an ultimate aim of this theory was to construct a circle equal to a given polygon, which would require to square a circle (or, better, to say, to "circle a square"). Remarkably in Books 3-4 Euclid studies circles and their relations to polygons.

Note 35

One may notice that conic sections were given a precise treatment in Greek geometry. But this is exactly because there is a sense in which they are generated by the straight line and the circle involved into the construction of a cone!

Note 36

Cf. "Plato himself dislikes Eudoxus, Archytas, and Menaechmus for endeavouring to bring down the doubling the cube to mechanical operations; for by this means all that was good in geometry would be lost and corrupted, it falling back again to sensible things, and not rising upward and considering immaterial and immortal images, in which God being versed is always God." (Plutarch 1909)

Note 37

This formal criterion works perfectly in the geometrical Books of the *Elements*. In the case of the arithmetical Books 7-9 there arises an interesting problem concerning this criterion, which I discuss in paragraph C3 of the same section. So far I follow Proclus and limit my discussion to the geometrical case. Beware that in spite of its very general philosophical *Prologue* Proclus' *Commentary* technically concerns only the Book 1 of the *Elements*, and leaves out of its scope the arithmetical Books and all specific questions concerning arithmetic.

Note 38

Greek terms are respectively - "protasis", "ekthesis", "diorismos", "kataskeuhe", "apodeixis", "sumperasma". In order to stress that the corresponding English terms have in the given context a specific technical sense I always write them in *italic*.

Note 39

Exceptions from this rule are of the following two types. (i) Some simple geometrical Theorems of the *Elements* lack any *construction* (called today *auxiliary* construction), or they involve only a minimal *construction*, which is difficult to separate from the following *proof*. Such is, for example, E1.8. This situation is more common for Euclid's arithmetical Propositions, see, for example, E7.4. (For a clear example of arithmetical *construction* see E7.1.) Thus (i) presents a reduced form of the same six-part pattern rather than a different pattern. (ii) Some Problems like E4.10 lack both an *exposition* and a *specification*. This deviation from the basic pattern is more profound and I shall discuss it in what follows in the main text. However I don't think that these deviations are sufficiently systematic for talking about the presence of a different pattern of reasoning in certain Euclid's Propositions.

Note 40

I owe this remark to Svetlana Mesjatz

Note 41

Here is the Greek text: ADD QUOTE

Note 42

A more popular translation of this Greek term is *proposition* (through Latin *propositio* which translates the Greek word literally). I don't use it here because this would make the terminological situation yet more complicated.

Note 43

Beware that P3 assures only the construction of a circle with the centre at one of the two endpoints of a given straight line. So the compass used for geometrical constructions is not supposed here to preserve the distance between its legs when one moves it from one place to another. E1.2 shows that this latter assumption, which looks stronger than P3, doesn't in fact bring a stronger theory.

Note 44

Notice that P1 assures the construction of straight line by its endpoints, not the construction of straight line without any qualification. One might argue on this basis that E"1.2 is impossible rather than trivial. However as one can see at the example of E1.5 Euclid in fact admits the arbitrary choice of points as a justified constructive step. Thus to chose a couple of points and then use P1 would qualify for Euclid as a solution of E"1.2.

Note 45

Notice however that in the *conclusion* of E1.1 Euclid uses the full expression "the given straight line *AB* ", which from a notational viewpoint is redundant. One can explain this redundancy by saying that Euclid reminds here the reader what the name "*AB* " stands for. In what follows I shall provide another explanation of this apparent redundancy.

Note 46

Here is Proclus' account of how the six-part structure is realised in E1.1:

"Let us view the things that have been said by applying them to this our first problem. Clearly it is a problem, for it bids us devise a way of constructing an equilateral triangle. In this case the enunciation consists of both what is given and what is sought. What is given is a finite straight line and what is sought is how to construct an equilateral triangle on it. The statement of the given precedes and the statement of what is sought follows, so that we may weave them together as "If there is a finite line, it is possible to construct an equilateral triangle on it." If there were no straight line, no triangle could be produced, for a triangle is bounded by straight lines; nor could it if the line were not finite, for an angle can be constructed only at a definite point, and an unbounded line has no end point.

Next after the enunciation is the exposition: "Let this be the given finite line". You see that the exposition itself mentions only the given, without a reference to what is sought. Upon this follows the specification: "It is required to construct an equilateral triangle on the designated finite straight line". In a sense the purpose of the specification is to fix our attention; it makes us more attentive to the proof by announcing what is to be proved, just as the exposition put us in a better position for learning by producing the given element before our eyes. After the specification comes the construction [...]. In general the postulates are contributory to constructions and the axioms to proofs. Next comes the proof [...]. "The three lines therefore are equal, and the equilateral triangle [ABC] has been constructed". This is the first conclusion following upon the exposition. And then comes the general conclusion: "An equilateral triangle is therefore been constructed upon the given straight line". For even if you make the line double that set forth in the exposition, or triple, or of any other length greater or less than it, the same construction and proof would fit it."

(*Commentary* 208.1-210.16 Morrow's translation, slightly corrected)

In the end of this passage Proclus distinguishes between a preliminary conclusion given in the end of the *proof* and the general *conclusion*, which concludes the whole Problem. While this preliminary conclusion concerns only the particular straight line *AB* identified in the *exposition* the general conclusion applies to any given straight line. Beware that Proclus doesn't quote here Euclid's text but paraphrases it. About the two conclusions see the next Note.

Note 47

Here is what Proclus says about it:

"[M]athematicians are accustomed to draw what is in a way a double conclusion. For when they have shown something to be true of the given figure, they infer that it is true in general, going from the particular to the universal conclusion. Because they do not make use of the particular qualities of the subjects but draw the angle or the straight line in order to place what is given before our eyes, they consider that what they infer about the given angle or straight line can be identically asserted for every similar case. They pass therefore to the universal conclusion in order that we may not suppose that the result is confined to the particular instance. This procedure is justified, since for the demonstration they use the objects set out in the diagram not as these particular figures, but as figures resembling others of the same sort. It is not as having such-and-such size that the angle before me is bisected, but as being rectilinear and nothing more. Its particular size is a character of the given angle, but its having rectilinear sides is a common feature of all rectilinear angles. Suppose the given angle is a right angle. If I used its rightness for my demonstration, I should not be able to infer anything about the whole class of rectilinear angles; but if I make no use of its rightness and consider only its rectilinear character, the proposition will apply equally to all angles with rectilinear sides. (*Commentary* 207.4-25, Morrow's translation)

In the last part of the quote Proclus takes as an example a *rectilinear* angle, that is, a "usual" angle sides of which are straight lines. In the *Elements* rectilinear angles are defined in D1.9 while D1.8 defines a more general notion of angle, which allows sides of an angle to be curve. Curvilinear angles appear in E3.16.

I shall not elaborate on Proclus' notion of "double conclusion" but in what follows (in the main text) propose my own "Platonic" account of this phenomenon, which is partly based on Proclus'.

Note 48

For Aristotle's example of a construction, which makes a non-obvious theorem nearly obvious see 1.4D below in the main text.

Note 49

As Proclus' rightly points out (see his *Commentary* 203-204) the situation is not wholly symmetric: some Theorems lack any (auxiliary) *construction* but each Problem without exception involves a *proof*. This remark only strengthens my point: *proofs* are not only important but also indispensable in Problems. About Theorems lacking *construction* see also Note 39 above.

Note 50

A more involved but in sometimes the only available way to proceed in the same situation is to consider a number of special cases separately. Then one should make sure that the list of cases is complete, i.e. that the considered cases exhaust all the possibilities. In such situations the use of special properties is not any longer forbidden but still tightly controlled. Naturally this strategy can work only when the number of special cases is finite.

Note 51

Concerning the question of first principles (fundamentals) remind section 1.1: on Plato's view the first principles cannot be adequately grasped by mathematical reasoning (which Plato calls "understanding") alone; it requires a different epistemic capacity and a different theoretical activity (viz. dialectics).

Note 52

All Aristotle's mathematical references are collected, translated and commented in (Heath 1949).

Note 53

Cf. "Division is a sort of weak deduction; for it postulates what it has to prove. ... But at first this escaped the notice of all those who made use of it..." (*An. Pr.* 46a 32-7, Barnes' translation). By "division" Aristotle means here the standard procedure of getting a definition through dividing a given genus into species.

Note 54

In Aristotle's own writings one finds a proof, which involves the proposition *a point has no parts* as a premise. In his *Physics* (226b) Aristotle defines a binary relation "in continuity with" as follows: *A* is in continuity with *B* iff boundaries of *A* and *B* are the same. Then (231a) he uses this definition for proving that two points cannot be in continuity, and hence that no collection of points can constitute a continuous line (or any other continuum). For suppose they can and consider two neighbouring point. Since the line is supposed to be continuous the two points are in continuity. The above definition implies that boundaries of these points are the same. But since a point has no parts it is identical with its own boundary. Hence the two points are the same. Hence any collection of points such that each point is in continuity with at least one (possibly the same) point from this collection reduces to a single point.

Note 55

Cf. "[I]t is impossible that there should be demonstration of absolutely everything, for there would then be an infinite regress, so that even then there would be no proof" (*Met.* 1006a7-9, Ross' translation). Even if Aristotle recognises a semantic regress in definitions, i.e. the fact that a definitions explains the *meaning* of its definiendum in terms of the meaning of its definiens, this is, in Aristotle's eyes, only a secondary function of definition. This is why he never discusses the notion of "semantic primitive" stopping the semantical regress.

Note 56

Proclus says this definitely about P5. He is less sure about P4 but also quotes its known proof, which he tends to accept. See the *Commentary* on P4-P5. Talking about attempts to prove P5, which in a long while led to the discovery of non-Euclidean geometries, it should be stressed that at least ancient mathematicians and their commentators like Proclus didn't look specifically for a proof of P5 based only on the rest of the Euclid's fundamentals. They would be quite ready to embrace a proof P5 like that proposed by Wallis in 1663 (Bonola 1955), which proves P5 on the basis of an additional principle having a stronger intuitive appeal than P5 itself.

Note 57

This fact is related to different ontologies of the two thinkers. Both consider mathematical objects as ontologically deficient but not in the same sense: in Plato's view they are "intermediate" while in Aristotle's view they are abstract. While Plato's basic ontological distinction between Being and Becoming can be well interpreted within mathematics Aristotle's distinction between abstract and non-abstract is rather meta-mathematical than mathematical. The fact that mathematical objects exist only in a specific sense of "exist", which is appropriate for abstract objects, doesn't have any direct impact on mathematical reasoning itself; it is relevant only when mathematics is applied elsewhere, notably in physics.

Note 58

I refer here after Cantor to the scholastic distinction between potential and actual infinity, which is indeed helpful in the given context. But it must be stressed that the modern notion of infinity developed in mathematics of 20th century is free from *any* associated modality - be it potentiality or actuality. I shall explain this issue providing more details in section I.3.

Note 59

For attempts to rewrite Euclid's proofs in terms of Aristotle's syllogisms see (Euclid 1845) and (Euclid 1848).

Note 60

Let me demonstrate my earlier claim that Aristotle's letter notation derives from Euclid's (I mean, of course, from an earlier geometrical practice, which is similar to Euclid's). After specifying referents of symbols "*A* " and "*B* " Aristotle introduces the new name "*AB*", which stands for the proposition formed out of *A*, *B* and the copula – similarly to a geometer who uses the name"*AB* " for a line drawn between points *A* and *B*. This observation suggests that Aristotle took an earlier established system of geometrical notation and used it for a different purpose.

Note 61

Remind how it works in this schoolish example. One needs to prove that Socrates is mortal. One assumes that (i) Socrates is a man (i.e. it is Socrates' essential property) and (ii) that all men are mortal (once again it is an essential property of men). Then the desired conclusion follows by the perfect syllogism. The proof can be described as the introduction of the middle term "man" between terms "Socrates" and "mortal". Premises (i) and (ii) are supposed to be immediate, which means that they cannot be proved similarly by introduction of new additional terms between terms "Socrates" and "man" and between terms "man" and "mortal".

Note 62

Here is the original text ADD QUOTE

Heath translates it is follows:

"Propositions too in mathematics are discovered by an activity; for it is be a process of dividing-up that we discover them. If the division had already been performed, the proposition would have been manifest; as it is they are present only potentially. Why does the triangle imply two right angles? Because the angles about one point are equal to two right angles. If, therefore, the straight line parallel to the side had been drawn upwards, the reason why would at once have been clear."

Ross translates it as follows:
"It is by actualisation also that geometrical relations are discovered; for it is by dividing the given figures that people discover them. If they have been already divided, the relations would have been obvious; but as it is the divisions are present only potentially. Why are the angles of the triangle equal to two right angles? Because the angles about one point are equal to two right angles. If, then the line parallel to the side had been already drawn, the theorem would have been evident to anyone as soon as he saw the figure."

It is not easy to be as laconic in English as Aristotle manages to be in Greek but in my proposed translation I tried to avoid as far as possible any additional words and expressions and particularly logically charged terms like "proposition", "relation" or "imply" absent in the original. Explaining his translation of Aristotle's "diagrammata" by "propositions" Heath writes:

"I feel no doubt that "ta diagrammata" in 1051a22 are, as in *Categ.*14a39 and *Metaph.* 1014a36, geometrical *propositions* including the proofs of the same, and not merely "diagrams" or even "constructions".

I opted for "diagrams" in spite of this warning. Heath is right that Aristotle is talking here about the process of proving a theorem, not about a diagram in a narrow sense. But he tells us about proving a theorem *with* a diagram. Aristotle, in my understanding, describes here an informal reasoning, which combines talking and drawing, rather than an accomplished result in the form found in the *Elements*. The additional line referred to by Aristotle makes the theorem obvious only in an informal sense: to prove it rigorously one still needs a solid theoretical basis like one provided by Euclid for E1.32. This is why I think that the litteral translation of Aristotle's "diagrammata" is more appropriate. Another reason for it is that in the second sentence of the quoted passage the word "diherhemena" (divided)" uncontroversially relates to "diagrammata". As Heath himself stresses quite rightly Aristotle's "diarountes" (dividing-up) should be understood here in a literal non-technical sense but not

in an abstract logical sense. Heath finds his way to avoid talking about "divided propositions" in his translation but it seems me artificial.
What Aristotle means here by "diarountes" is quite clear from the following examples: he talks about geometrical constructions made on the basis of some given figures or constructions. It is less clear why Aristotle uses here this word, which doesn't seem to be quite appropriate in the given case. I think that this fact has to do with Aristotle's theory according to which additional constructions are potentially present *in* a given figure rather that in an outer space. We can see behind this term the same Aristotle's intuition, which we have already encountered elsewhere: every essential property of a figure is discovered by an analysis of its "proper nature" rather than through its relations to anything else. The fact that additional constructions in geometry don't reduce to cutting given figures into parts suggests that Aristotle's approach doesn't actually square with his contemporary mathematics. This is not particularly surprising because this approach, as I have already stressed, is mainly motivated by physical and biological rather than mathematical examples.

Note 63
The idea to distinguish between "Being properly speaking" and various kinds of deficient "Being in a special sense" Aristotle obviously took from his teacher.

Note 64
Here are *enunciations* of all the five:

E10.5: "Commensurable magnitudes have to one another the ratio which (some) number (has) to (some) number."

E10.6: "If two magnitudes have to one another the ratio which (some) number (has) to (some) number, then the magnitudes will be commensurable."

E10.7: "Incommensurable magnitudes do not have to one another the ratio which (some) number (has) to (some) number."

E10.8: "If two magnitudes do not have to one another the ratio which (some) number (has) to (some) number, then the magnitudes will be incommensurable."

E10.9: "Squares on straight-lines (which are) commensurable in length have to one another the ratio which (some) square number (has) to (some) square number. And squares having to one another the ratio which (some) square number (has) to (some) square number will also have sides (which are) commensurable in length. But squares on straight-lines (which are) incommensurable in length do not have to one another the ratio which (some) square number (has) to (some) square number. And squares not having to one another the ratio which (some) square number (has) to (some) square number will not have sides (which are) commensurable in length either."

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