Axiomatic Method and Category Theory

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Theorem 1.5 of Euclid's ELEMENTS :

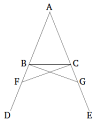
[*enunciation* :]

> *For isosceles triangles, the angles at the base are equal to one another, and if the equal straight lines are produced then the angles under the base will be equal to one another.*

Theorem 1.5 (continued) :

[*exposition*] :

> *Let ABC be an isosceles triangle having the side AB equal to the side AC ; and let the straight lines BD and CE have been produced further in a straight line with AB and AC (respectively). [Post. 2].*



Theorem 1.5 (continued) :

[*specification* :]

> *I say that the angle ABC is equal to ACB, and (angle) CBD to BCE.*

Theorem 1.5 (continued) :

[*construction* :]

> *For let a point F be taken somewhere on BD, and let AG have been cut off from the greater AE, equal to the lesser AF [Prop. 1.3]. Also, let the straight lines FC, GB have been joined. [Post. 1]*

Theorem 1.5 (continued) :

[*proof* :]

> *In fact, since AF is equal to AG, and AB to AC, the two (straight lines) FA, AC are equal to the two (straight lines) GA, AB, respectively. They also encompass a common angle FAG. Thus, the base FC is equal to the base GB, and the triangle AFC will be equal to the triangle AGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) ACF to ABG, and AFC to AGB. And since the whole of AF is equal to the whole of AG, within which AB is equal to AC, the remainder BF is thus equal to the remainder CG [Ax.3]. But FC was also shown (to be) equal to GB.*

Theorem 1.5 (continued) :

So the two (straight lines) BF, FC are equal to the two (straight lines) CG, GB respectively, and the angle BFC (is) equal to the angle CGB, while the base BC is common to them. Thus the triangle BFC will be equal to the triangle CGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. Thus FBC is equal to GCB, and BCF to CBG. Therefore, since the whole angle ABG was shown (to be) equal to the whole angle ACF, within which CBG is equal to BCF, the remainder ABC is thus equal to the remainder ACB [Ax. 3]. And they are at the base of triangle ABC. And FBC was also shown (to be) equal to GCB. And they are under the base.

Theorem 1.5 (continued) :

[*conclusion* :]

> *Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required <u>to show</u>.*

Problem 1.1 of Euclid's ELEMENTS :

[*enunciation :*]

To construct an equilateral triangle on a given finite straight-line.

Problem 1.1 (continued) :

[*exposition* :]

Let AB be the given finite straight-line.

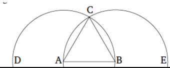
[*specification* :]

So it is required to construct an equilateral triangle on the straight-line AB.

Problem 1.1 (continued) :

[*construction* :]

> *Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another, to the points A and B [Post. 1].*



Problem 1.1 (continued) :

[*proof* :]

> *And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [Axiom 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.*

Problem 1.1 (continued) :

[*conclusion* :]

> *Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.*

3 Kinds of First Principles in Euclid's ELEMENTS : :

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  play the role similar to that of logical rules restricted to mathematics : cf. the use of the term by Aristotle
- Postulates :
  non-logical constructive rules

Common Notions

A1. Things equal to the same thing are also equal to one another.
A2. And if equal things are added to equal things then the wholes are equal.
A3. And if equal things are subtracted from equal things then the remainders are equal.
A4. And things coinciding with one another are equal to one another.
A5. And the whole [is] greater than the part.

Common Notions (continued)

Euclid's Common Notions hold *both* for numbers and magnitudes (hence the title of "common") ; they form the basis of a regional "mathematical logic" applicable throughout the mathematics. Aristotle transform them into laws of logic applicable throughout the *episteme*, which in Aristotle's view does not reduce to mathematics but also includes *physics*.

Postulates 1-3 :

P1 : to draw a straight-line from any point to any point.
P2 : to produce a finite straight-line continuously in a straight-line.
P3 : to draw a circle with any center and radius.

Postulates 1-3 (continued) :

Postulates 1-3 are NOT propositions ! They are not first truths. They are basic (non-logical) *operations*.

Operational interpretation of Postulates

Postulates	input	output
P1	two points	straight segment
P2	straight segment	straight segment
P3	straight segment and its endpoint	circle

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- Postulates 1-3 and *enunciations* of Problems are NOT propositions but (non-logical) operations.
- Euclid's mathematics aims at *doing* AND *showing* but not only at showing and moreover not only to proving certain propositions

Hilbert on Euclid

Euclid's axiomatics was intended to be contentual and intuitive, and the intuitive meaning of the figures is not ignored in it. Furthermore, its axioms are not in existential form either : Euclid does not presuppose that points or lines constitute any fixed domain of individuals. Therefore, he does not state any existence axioms either, but only construction postulates.
(Hilbert&Bernays :1934)

Foundations 1899 :

Let us consider three distinct systems of things. The things composing the first system, we will call points and designate them by the letters A, B, C,. . . ; those of the second, we will call straight lines and designate them by the letters a, b, c,. . . ; and those of the third system, we will call planes and designate them by the Greek letters α, β, γ *. [..] We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as "are situated", "between"; "parallel", "congruent", "continuous", etc. The complete and exact description of these relations follows as a consequence of the axioms of geometry. These axioms [..] express certain related fundamental facts of our intuition.*

On Truth and Existence (circa1900)

[A]s soon as I posited an axiom it will exist and be "true". [] If the arbitrarily posited axioms together with all their consequences do not contradict each other, then they are true and the things defined by these axioms exist. For me, this is the criterion of truth and existence.

Hintikka on Hilbert

[For Hilbert t]he basic clarified form of mathematical theorizing is a purely logical axiom system. (Hintikka :1997)

Consistency

How we know if a given system of axioms is consistent ? Making models provides only proofs of relative consistency. Reduction to Arithmetic works for Euclidean Geometry but not universally. Is Arithmetic consistent ?

Formalization of Logic ?

[I]t appears necessary to axiomatize logic itself and to prove that number theory and set theory are only parts of logic. This method was prepared long ago (not least by Frege's profound investigations) ; it has been most successfully explained by the acute mathematician and logician Russell. One could regard the completion of this magnificent Russellian enterprise of the axiomatization of logic as the crowning achievement of the work of axiomatization as a whole. (Hilbert :1918 p. 1113)

Formalization of Logic ?

[I]n my theory contentual inference is replaced by manipulation of signs [ausseres Handeln] according to rules ; in this way the axiomatic method attains that reliability and perfection that it can and must reach if it is to become the basic instrument of all research. (Hilbert :1927)

Formalization of Logic ?

If formalization means making explicit the *logical form* of a given theory how one can formalize logic ? The answer seems to be that formalization of logic puts logic into a *symbolic* form. By extension in the new setting this applies also to mathematical theories.

Formalization of Logic ?

Remark that in Hilbert-Ackermann symbolic logic logical symbols unlike non-logical ones cannot be re-interpreted. Symbolic logic is still contentual in a sense in which mathematical theories (expressed symbolically) are not.

Proof theory as foundation

With this new way of providing a foundation for mathematics, which we may appropriately call a proof theory, I pursue a significant goal, for I should like to eliminate once and for all the questions regarding the foundations of mathematics in the form in which they are now posed, by turning every mathematical proposition into a formula that can be concretely exhibited and strictly derived, thus recasting mathematical definitions and inferences in such a way that they are unshakable and yet provide an adequate picture of the whole science. I believe that I can attain this goal completely with my proof theory, even if a great deal of work must still be done before it is fully developed. (Hilbert :1927)

Proof theory as foundation

No more than any other science can mathematics be founded by logic alone ; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought.(Hilbert :1927)

Proof theory as foundation

If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction. This is the basic philosophical position that I regard as requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. (Hilbert :1927)

Proof theory as foundation

And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable. This is the very least that must be presupposed ; no scientific thinker can dispense with it, and therefore everyone must maintain it, consciously or not. (Hilbert :1927)

Real and Ideal Mathematical Objects

Just as, for example, the negative numbers are indispensable in elementary number theory and just as modern number theory and algebra become possible only through the Kummer-Dedekind ideals, so scientific mathematics becomes possible only through the introduction of ideal propositions. (Hilbert :1927)

Metamathematics

When we now approach the task of such an impossibility proof, we have to be aware of the fact that we cannot again execute this proof with the method of axiomatic-existential inference. Rather, we may only apply modes of inference that are free from idealizing existence assumptions.(Hilbert&Bernays :1934)

Formal Axiomatic Method

Formal Axiomatic Method splits into *genetic* part (building formulas) and *axiomatic-existential* part (interpreting formulas as proofs) : Doing is Showing ! Real mathematical objects are only symbolic expressions built after the model of alphabetic linguistic expressions. In Metamathematics one needs to *show* (prove) things differently !

Hilbert Program

failed but the method survived

Axiomatic Set theory

It is a *metamathematics* of ZFC and the like. Ex. : Continuum hypothesis

Boorbaki on Axiomatic Method

After more or less evident bankruptcy of the different systems [..] it looked, at the beginning of the present [20th] century as if the attempt had just about been abandoned to conceive of mathematics as a science characterized by a definitely specified purpose and method [..] Today, we believe however that the internal evolution of mathematical science has, in spite of appearance, brought about a closer unity among its different parts, so as to create something like a central nucleus that is more coherent than it has ever been. The essential aspect of this evolution has been the systematic study of the relation existing between different mathematical theories, and which has led to what is generally known as the "axiomatic method."

Boorbaki on Axiomatic Method

[E]very mathematical theory is a concatenation of propositions, each one derived from the preceding ones in conformity with the rules of a logical system [..] It is therefore a meaningless truism to say that this "deductive reasoning" is a unifying principle for mathematics. [..] [I]t is the external form which the mathematician gives to his thought, the vehicle which makes it accessible to others, in short, the language suited to mathematicians ; this is all, no further significance should be attached to it. (Bourbaki :1950)

Ex : Axiomatic Group theory

It is a *model theory* of the Axiomatic Group theory in the (axiomatic ?) Set theory ! Axioms of Group theory by themselves imply very little !

G1 : $x \circ (y \circ z) = (x \circ y) \circ z$ (associativity of $\circ$)

G2 : there exists an item *1* (called *unit*) such that for all x $x \circ 1 = 1 \circ x = x$

G3 : for all x there exists x^{-1} (called *inverse* of x) such that $x \circ x^{-1} = x^{-1} \circ x = 1.$

Kant on the role of geometrical constructions

Give a philosopher the concept of triangle and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on his concept as long as he wants, yet he will never produce anything new. He can analyze and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts.

Kant on the role of geometrical constructions

But now let the geometer take up this question. He begins at once to construct a triangle .... In such a way through a chain of inferences that is always guided by intuition, he arrives at a fully illuminated and at the same time general solution of the question. (Critique of Pure Reason)

Mathematics and Metamathematics

"Real" mathematics (= mathematics as it is practiced) qualifies as metamathematics ! (Whether we are talking about formal studies or the mainstream "informal" mathematics.)

Venn on Logical Diagrams

The relative cumbousness of such a mode of expression [in a natural language] is obviously the real measure of our need for a reformed or symbolic language. [..] That such a scheme]of symbolic representation] is complete there can be no doubt. But unfortunately, owing to this very completeness, it is apt to prove terribly lengthy. [..] This then is the state of thing which a reformed scheme of diagrammatic notation has to meet. It must correspond in all essential respects to that regular system of class subdivision which has just been referred to under its verbal and its literal or symbolic aspect. Theoretically, as we shall see, this is perfectly attainable. (Venn :1881)

Venn on Logical Diagrams

It has been said above that this question of the Universe only arises when we apply our formulas. Now diagrams are strictly speaking a form of application, and therefore such considerations at once meet us when we come to make use of diagrams. I draw a circle to represent X, then what is outside of that circle represents not-X, but the limits of that outside are whatever I choose to consider them. (Venn :1881)

Tarski on Topological Interpretation of Propositional Calculus

(both Classical and Intuitionistic)

> *The present discussion seems to me to have a certain interest not only from the purely formal point of view ; it also throws an interesting light on the content relations between the two systems of the sentential calculus and the intuitions underlying these systems [zugrundeliegende Intuitionen]. (Tarski :1956)*

Combinatory Logic

The successes that we have encountered thus far on the road taken encourage us to attempt further progress. We are led to the idea, which at first glance certainly appears extremely bold, of attempting to eliminate by suitable reduction the remaining fundamental notions, those of proposition, propositional function, and variable [..]. To examine this possibility more closely and to pursue it would be valuable not only from the methodological point of view that enjoins us to strive for the greatest possible conceptual uniformity but also from a certain philosophic, or, if you wish, aesthetic point of view. (Schönfinkel :1926)

Combinatory Logic

Combinatory logic is a branch of mathematical logic which concerns itself with the ultimate foundations. Its purpose is the analysis of certain notions of such basic character that they are ordinarily taken for granted. These include [(i)] the process of substitution, usually indicated by the use of variables ; and also [(ii)] the classification of the entities constructed by these processes into types or categories, which in many systems has to be done intuitively before the theory can be applied. It has been observed that these notions, although generally presupposed, are not simple ; they constitute a prelogic, so to speak, whose analysis is by no means trivial. (Curry&Feys&Craig :1958)

Curry-Howard-Lambek

Isomorphism between typed lambda-calculi and formal deductive systems and cartesian closed categories

Axiomatic Method and Foundations of Mathematics

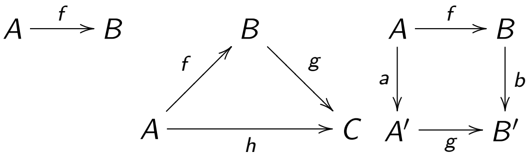
Replacement of Set theory by Category theory in Foundations does not necessarily requires a modification of the Formal Axiomatic Method : ex. ETCS (Lawvere 1964). But.

Lawvere on foundations

In my own education I was fortunate to have two teachers who used the term "foundations" in a common-sense way (rather than in the speculative way of the Bolzano-Frege-Peano-Russell tradition). [..] The orientation of these works seemed to be "concentrate the essence of practice and in turn use the result to guide practice". I propose to apply the tool of categorical logic to further develop that inspiration.
Foundations is derived from applications by unification and concentration, in other words, by the axiomatic method. Applications are guided by foundations which have been learned through education. (Lawvere :2003)

Lawvere : Categories

The formalism of category theory is itself often presented in "geometric" terms. In fact, to give a category is to give a meaning to the word morphism and to the commutativity of diagrams like



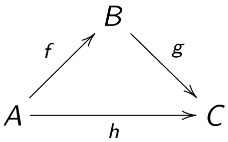
which involve morphisms, in such a way that the obvious associativity and identity conditions hold,

Lawvere : Categories

..as well as the condition that whenever

$$A \xrightarrow{f} B\ ,\ B \xrightarrow{f} C$$

are commutative then there is just one h such that



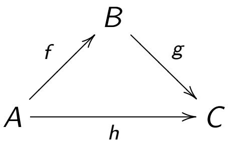
is commutative.

Lawvere : Categories

To save printing space, one also says that A is the domain, and B the codomain of f when

$$A \xrightarrow{f} B$$

is commutative, and in particular that h is the composition $f.g$ if



is commutative.

Lawvere : Categories

We regard objects as co-extensive with identity morphisms, or equivalently with those morphisms which appear as domains or codomains. As usual we call a morphism which has a two-sided inverse an isomorphism. (Lawvere 1969)

Lawvere : Categories

[In a category w]e identify objects with their identity maps and we regard a diagram

$$A \xrightarrow{f} B$$

as a formula which asserts that A is the (identity map of the) domain of f and that B is the (identity map of the) codomain of f. Thus, for example, the following is a universally valid formula

$$A \xrightarrow{f} B \Rightarrow A \xrightarrow{A} A \wedge A \xrightarrow{f} B \wedge B \xrightarrow{B} B \wedge Af = f = fB$$

Lawvere on Logic

The term "logic" has always had two meanings - a broader one and a narrower one :
(1) All the general laws about the movement of human thinking should ultimately be made explicit so that thinking can be a reliable instrument, but
(2) already Aristotle realized that one must start on that vast program with a more sharply defined subcase.

Lawvere on Logic

The achievements of this subprogram include the recognition of the necessity of making explicit
(a) a limited universe of discourse, as well as
(b) the correspondence assigning, to each adjective that is meaningful over a whole universe, the part of that universe where the adjective applies. This correspondence necessarily involves
(c) an attendant homomorphic relation between connectives (like and and or) that apply to the adjectives and corresponding operations (like intersection and union) that apply to the parts "named" by the adjectives.

Lawvere on Logic

When thinking is temporarily limited to only one universe, the universe as such need not be mentioned ; however, thinking actually involves relationships between several universes. [...] Each suitable passage from one universe of discourse to another induces

Lawvere on Logic

(0) an operation of substitution in the inverse direction, applying to the adjectives meaningful over the second universe and yielding new adjectives meaningful over the first universe. The same passage also induces two operations in the forward direction :
(1) one operation corresponds to the idea of the direct image of a part but is called "existential quantification" as it applies to the adjectives that name the parts ;
(2) the other forward operation is called "universal quantification" on the adjectives and corresponds to a different geometrical operation on the parts of the first universe.

Lawvere on Logic

It is the study of the resulting algebra of parts of a universe of discourse and of these three transformations of parts between universes that we sometimes call "logic in the narrow sense". Presentations of algebraic structures for the purpose of calculation are always needed, but it is a serious mistake to confuse the arbitrary formulations of such presentations with the objective structure itself or to arbitrarily enshrine one choice of presentation as the notion of logical theory, thereby obscuring even the existence of the invariant mathematical content.

Lawvere on Logic

In the long run it is best to try to bring the form of the subjective presentation paradigm as much as possible into harmony with the objective content of the objects to be presented ; with the help of the categorical method we will be able to approach that goal.
(Lawvere&Rosebrugh :2003)

Adjunction

An *adjoint situation* (called also an *adjunction*) is a pair of categories A, B with two functors f, g going in opposite directions :

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

provided with *natural transformations* $\alpha : A \to fg$ and $\beta : gf \to B$ such that $(g\alpha)(\beta f) = g$ and $(\alpha f)(g\beta) = f$. This latter condition is tantamount to saying that the following diagrams commute :

$$g \underset{\beta g}{\overset{g\alpha}{\rightleftarrows}} gfg$$

$$f \underset{f\beta}{\overset{\alpha f}{\rightleftarrows}} fgf$$

Quantifiers as Adjoints

Let us for simplicity think of these universes as *sets* but have in mind that the functorial construction of logical quantifiers does not require this assumption. Suppose now that we have a one-place predicate (a property) P, which is meaningful on set Y, so that there is a subset P_Y of Y (in symbols $P_Y \subseteq Y$) such that for all $y \in Y$ $P(y)$ is true just in case $y \in P_Y$. Now using these data (together with morphism f as above) we can define a new predicate R on X as follows : we say that for all $x \in X$ $R(x)$ is true when $f(x) \in P_Y$ and false otherwise. So we get subset $R_X \subseteq X$ such that for all $x \in X$ $R(x)$ is true just in case $x \in R_X$.

Quantifiers as Adjoints

Let us assume in addition that every subset P_Y of Y is determined by some predicate P meaningful on Y. Then given morphism ("passage") f from "universe" X to "universe" Y we get a way to associate with every subset P_Y (every part of universe Y) a subset R_X and, correspondingly, a way to associate with every predicate P meaningful on Y a certain predicate R meaningful on X. Since subsets of given set Y form Boolean algebra $B(Y)$ we get a map between Boolean algebras (notice the change of direction !) :

$$f^* : B(Y) \longrightarrow B(X)$$

Quantifiers as Adjoints

Since Boolean algebras themselves are categories (where objects are subsets and maps are their inclusions) f^* is a functor. For every proposition of form $P(y)$ where $y \in Y$ functor f^* takes some $x \in X$ such that $y = f(x)$ and produces a new proposition $P(f(x)) = R(x)$ (for a single given y it may produce a set of different propositions of this form). Since it replaces y in $P(y)$ by $f(x) = y$ it is appropriate to call f^* *substitution* functor.

Quantifiers as Adjoints

The *left* adjoint to the substitution functor f^* is functor

$$\exists_f : B(X) \longrightarrow B(Y)$$

which sends every $R \in B(X)$ (i.e. every subset of X) into $P \in B(Y)$ (subset of Y) consisting of elements $y \in Y$, such that *there exists* some $x \in R$ such that $y = f(x)$; in (some more) symbols

$$\exists_f(R) = \{y | \exists x(y = f(x) \wedge x \in R)\}$$

Quantifiers as Adjoints

In other words $\exists_f$ sends R into its *image* P under f. Now if (as above) we think of R as a property $R(x)$ meaningful on X and think of P as a property $P(y)$ meaningful on Y we can describe $\exists_f$ by saying that it transformes $R(x)$ into $P(y) = \exists_f x P'(x, y)$ and interpret $\exists_f$ as the usual existential quantifier.

Quantifiers as Adjoints

The *right* adjoint to the substitution functor f^* is functor

$$\forall_f : B(X) \longrightarrow B(Y)$$

which sends every subset R of X into subset P of Y defined as follows :

$$\forall_f(R) = \{y | \forall x (y = f(x) \Rightarrow x \in R)\}$$

and thus transforms $R(X)$ into $P(y) = \forall_f x P'(x, y)$.

Quantifiers as Adjoints

Notice that functors $\exists_f$ and $\forall_f$ are defined here as adjoints to functor f^*, i.e., quite independently from their interpretation as logical quantifiers explained above.

Grothendieck (circa 1962)

Topos : 2-fold generalization

- Topological space $\longrightarrow$ Space of sheafs (detach sets from opens)
- Poset of opens $\longrightarrow$ (small) category of opens (Grothendieck topology)

Elementary Topos : Geometry and Logic

The unity of opposites in the title [Quantifiers and Sheafs] is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. At the same time, in the present joint work with Myles Tierney there are important influences in the other direction : a Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ *have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for "the" category* $\underline{S}$ *of abstract sets to an arbitrary topos. (Lawvere : 1970)*

Lawvere

Toposes = continually variable sets

Johnstone : Sketches of the Elephant

Three Sketches : Space, Logical Framework, Universe of discourse

Features of Topos Logic

- Geometry comes first (the "leading aspect") ;
- (Internal) logic of a given topos reflects its geometrical structure ;
- Connectives, quantifiers and truth-values are internalized (the "local truth")
- Identity is not

Fundamental group

Fundamental group G^0_T of a topological space T :

- a base point P ;
- loops through P (loops are circular paths $l : I \rightarrow T$) ;
- composition of the loops (up to homotopy only ! - see below) ;
- identification of homotopic loops ;
- independence of the choice of the base point.

Fundamental (1-) groupoid

G^1_T :

- all points of T (no arbitrary choice) ;
- paths between the points (embeddings $s : I \rightarrow T$) ;
- composition of the *consecutive* paths (up to homotopy only ! - see below) ;
- identification of homotopic paths ;

Since not all paths are consecutive G^1_T contains more information about T than G^0_T !

Path Homotopy and Higher Homotopies

$s : I \longrightarrow T$, $p : I \longrightarrow T$ where $I = [0, 1]$: paths in T
$h : I \times I \longrightarrow T$: homotopy of paths s, t if $h(0 \times I) = s$, $h(1 \times I) = t$
$h^n : I \times I^{n-1} \longrightarrow T$: n-homotopy of $n-1$-homotopies h_0^{n-1}, h_1^{n-1} if $h^n(0 \times I^{n-1}) = h_0^{n-1}$, $h^n(1 \times I^{n-1}) = h_1^{n-1}$;
Remark : : Paths are zero-homotopies

Higher Groupoids and Omega-Groupoids

- all points of T (no arbitrary choice) ;
- paths between the points ;
- homotopies of paths
- homotopies of homotopies (2-homotopies)
- higher homotopies up to n-homotopies
- higher homotopies ad infinitum

G_T^n contains more information about T than G_T^{n-1} !

Composition of Paths

Concatenation of paths produces a map of the form $2I \rightarrow T$ but not of the form $I \rightarrow T$, i.e., not a path. We have the whole space of paths $I \rightarrow 2I$ to play with ! But all those paths are homotopical. Similarly for higher homotopies (but beware that n-homotopies are composed in n different ways !)
On each level when we say that $a \oplus b = c$ the sign $=$ hides an infinite-dimensional topological structure !

Grothendick Conjecture :

G_T^ω contains all relevant information about T ; an omega-groupoid is a complete algebraic presentation of a topological space.

Martin-Löf

Constructive Intensional Type theory (with dependent types). Is it really a “logic” ?

Voevodsky : Univalent Foundtions

(circa 2008)
Higher (homotopical) groupoids model higher identity types in Martin-Löf's Intensional Constructive Type theory !
Identities (i.e. identity types of all orders) get internalized.
Disclaimer : 1-groupoids model Martin-Löf's theory too but they kill higher identity types ("extensionality one level up")
Only omega-groupoids provide "fully instensional" models ;
Voevodsky's *Univalence Axiom* rules out all groupoid models except the omega-groupoid model.

Lurie : Higher Topos Theory

(2011)
Unification of Topos theory and Higher-categorical Homotopy theory ?

Kant-Friedman on Euclid

Euclidean geometry [..] is not to be compared with Hilbert's axiomatization [of Euclidean geometry], say, but rather with Frege's Begriffsschrift. It is not a substantive doctrine, but a form of rational representation : a form of rational argument and inference.[..] There remains a serious question about Euclid's axioms, of course ; when pressed, Kant would most likely claim that they represent the most general conditions under which alone a concept of extended magnitude - and therefore a rigorous conception of an external world - is possible (see A163/B204). And, of course, we now know that Kant is fundamentally mistaken here. (Friedman :1992)

Quod Erat Faciendum

- Doing and Showing go closely together again ;
- Foundations are no longer purely logical. Operations with symbols represent operations with mathematical (geometrical) objects but not just operations with linguistic expressions ;
- No straightforward distinction between *real* and *ideal* mathematical objects ;
- Geometrical intuition plays a role in foundations.

Back to Euclid ?

- No uniqueness but a broad unification anyway ;
- A new mathematical language for the Book of Nature ?

Logic is rather a feature of a physical system or region of spacetime than the universal ambient environment unchanged whatsoever the scales of energies are considered. (Krol, circa 2010)

movie

THE END