CATEGORIES WITHOUT STRUCTURES

ANDREI RODIN

[1]

Abstract. The popular view according to which Category theory provides a support for Mathematical Structuralism is erroneous. Category-theoretic foundations of mathematics require a different philosophy of mathematics. While structural mathematics studies "invariant form" (Awodey) categorical mathematics studies covariant and contravariant transformations which, generally, dont have any invariants. In this paper I develop a non-structuralist interpretation of categorical mathematics and show its consequences for history of mathematics and mathematics education.

1. Introduction

Some time ago there was a discussion in *Philosophia Mathematica* about Hellman's question "Does Category Theory Provide a Framework for Mathematical Structuralism?" [11]. Awodey [2] answered "yes, obviously"; a version of Categorical Structuralism (i.e., Mathematical Structuralism developed in a category-theoretic framework) was later described by MacLarty [31]. Independently of this discussion a structuralist view on Category theory is argued for in [26]; a structuralist view on Category theory also underlies the recent monograph [27] (even if its author doesn't discuss in this book Mathematical Structuralism explicitly). In the paper I propose a different answer to Hellman's question by arguing that Category theory leads to a new non-structuralist view on mathematics and its foundations.

[1] *Acknowledgment*: I am very grateful to Sergei Artemov, Haim Gaifman, René Guitart, Daniel Isaacson, William Lawvere, John Mayberry, Colin McLarty, Richard Pettigrew, Irina Starikova, Michael Wright, Noson Yanofsky and two anonymous referees for their comments on earlier drafts of this paper and valuable discussions.

Mathematical Structuralism is usually opposed to various forms of Substantialism about mathematical objects. As the reader shall see the view on mathematics that I oppose to Structuralism in this paper is not of this latter sort.

In order to formulate my claims I need to make explicite some of my general assumptions concerning the aim and the scope of philosophy of mathematics. I hold a traditional view according to which the principle aim of philosophy of mathematics is to provide mathematics with appropriate foundations. Saying this I have in mind the notion of foundation described by Lawvere as follows:

> A foundation makes explicit the essential general features, ingredients, and operations of a science, as well as its origins and generals laws of development. The purpose of making these explicit is to provide a guide to the learning, use, and further development of the science. A "pure" foundation that forgets this purpose and pursues a speculative "foundations" for its own sake is clearly a nonfoundation.[21]

Following Lawvere I shall not discuss in this paper what he calls "pure" and "speculative" foundations, i.e., foundations detached from the current mathematical practice.

Further, I assume that foundations of mathematics is a subject to permanent dialectical revision and historical change; I believe that such a continuing renewal of foundations is essential for progress in mathematics (and likewise for progress in science in general). While the progress in most parts of mathematics amounts to building upon earlier acquired knowledge the renewal of its foundations work differently: it amounts to refutation of older foundations and positing of new foundations. This process represents the dialectical development of basic ideas about mathematics, which reflect, support, motivate and lead contemporary mathematical practice. My aim in this paper is to push this dialectical development further forward.

Since a general discussion about the nature and the purpose of foundations is out of place here I shall only say how the above assumptions contribute to my further claims concerning

Structuralism and Category theory. I shall speak about Structuralism as a way of building foundations mathematics, not merely as a doctrine about the nature of mathematics. I shall criticize Structuralism without meaning that Structuralism is boldly wrong. My claim is that Structuralism is not wrong but outdated. It has been successful in twentieth-century mathematics but it is no longer appropriate as a foundation for todays and future mathematics. I present in this paper an alternative foundational project related to Category theory and explain its advantages. I also explain how this categorical foundational project relates to Structuralism and why it doesnt qualify as a variety of Structuralism.

The rest of this paper is organized as follows. First, I briefly discuss Mathematical Structuralism, its historical origins and its relation to Set theory and Category theory. Here I explain reasons why MacLane, Awodey, and some other people believe that Category theory provides a support for Mathematical Structuralism. Then I provide my critical arguments against this latter view arguing that the notion of category should be viewed as generalization of that of structure rather than as a specific kind of structure. Further I analyze Lawvere's paper [20] on categorical foundations and show that the author begins this paper with a version of structuralist foundations but then proceeds in a different direction. I conclude with an attempt to outline the new categorical view on mathematics explicitly.

2. Mathematical Structuralism

Before discussing Structuralism as a philosophical view about mathematics I would like to point to an example of mathematical structure given by a working mathematician for a philosophical reader:

> All infinite cyclic groups are isomorphic, but this infinite group appears over and over again - in number theory, in ornaments, in crystallography, and in physics. Thus, the "existence" of this group is really a many-splendored

> matter. An ontological analysis of things simply called "mathematical objects" is likely to miss the real point of mathematical existence. [22]

The point stressed by Mac Lane with this example is this: things like (algebraic) groups should be thought of as structures (abstract or instantiated in various ways) rather than individual objects. Let me now for the sake of my further argument modify Mac Lane's example as follows: I replace the words "infinite cyclic group" by the words "number three" and the word "isomorphic" by the word "equal":

All threes are equal but this number appears over and over again - in number theory, in ornaments .... Thus the "existence" of this number is really a many-splendored matter.

This modification reveals a traditional aspect of Structuralism, which often remains unnoticed when people stress the novelty of this approach. Indeed the familiar number three is just as promiscuous as the infinite cyclic group or perhaps even more promiscuous. The number three equally "appears" (to use MacLane's word) both inside and outside mathematics: in a trio of apples, a trio of points, a trio of groups, a trio of numbers or a trio of anything else. As in Mac Lane's original example, there is a systematic ambiguity between the plural and the singular forms of nouns in our talk about numbers. (Notice Mac Lane's talk about "all infinite cyclic groups" and "this infinite group" in the same sentence; in my paraphrase I talk similarly about a number.) This shows that the notion of "many-splendored existence" (i.e., of multiple instantiation) is not specific for the way of mathematical thinking developed in the first half of twentieth century and usually described as "structural". Thus in order to understand what is specific for this thinking one should look elsewhere. Comparing Mac Lane's example with its modified version one can see that in Mac Lane's example the notion of isomorphism plays the same role that the notion of equality (as distinguished from identity) plays in traditional mathematics. The idea that isomorphic objects can be treated as equal is, in my view, crucial for Structuralism - at least if we are talking about Structuralism as a historical trend in mathematics rather than a philosophical theory about mathematics. [2]

This point has been made clear by Hilbert in his often-quoted letter to Frege of December 29, 1899. Stressing the "many-splendored" nature of structural theories (as we would call them today) Hilbert says:

> [E]ach and every theory can always be applied to the infinite number of systems of basic elements. One merely has to apply a univocal and reversible one-to-one transformation [to the elements of the given system] and stipulate that the axiom for the transformed things be correspondingly similar (quoted by [9], underlining mine)

Notice that the reversibility condition stressed here by Hilbert implies that the given transformation is an isomorphism.

In the current philosophy of mathematics Mathematical Structuralism is present in a number of different varieties [10], which include Categorical Structuralism [13], [31]. However a general notion of Mathematical Structuralism neutral with respect to its more specific varieties is also described in the recent literature. For my present purposes I shall refer only to this core of Mathematical Structuralism leaving aside more specific issues concerning its multiple varieties. Hellman [12] describes this core Structuralism as follows:

> Structuralism is a view about the subject matter of mathematics according to which what matters are structural relationships in abstraction from the intrinsic nature of the related objects. Mathematics is seen as the free exploration of structural possibilities, primarily through creative concept formation, postulation, and deduction. The items making up any particular system exemplifying the structure in question are of no importance; all that matters is that they satisfy certain general conditionstypically spelled out

[2] For a historical study of the structural trend in mathematics see [5].

> in axioms defining the structure or structures of interest - characteristic of the branch of mathematics in question.

Noticeably Hellman doesn't explicitly mention the notion of isomorphism in this description. In my view this is a serious default. To see this consider the example of group structure. A group is any "system" (to use Hellman's word) that consists of certain "items" a , b, ... and binary operation $\oplus$ associating with every ordered pair of such items (possibly identical) a third item (possibly identical to one of those) from the same system such that the following axioms hold:

G1: operation $\oplus$ is associative.

G2: there exists an item *1* (called unit) such that for all a $a \oplus 1 = 1 \oplus a = a$.

G3: for all a there exists a^{-1} (called inverse of a) such that $a \oplus a^{-1} = a^{-1} \oplus a = 1$.

These axioms are satisfied by many different groups. The infinite cyclic group mentioned above is just one example but there are many others. These other groups are not, generally, isomorphic to the infinite cyclic group, i.e. they are different in the *structural* sense. This demonstrates the obvious fact that axioms **G1-3** determine a class of structures of a particular *type* but not a particular structure. This example explains why Hellman talks in the above quote about "structures of interest" in plural. [3]

But in order to give a sense to the expression "type of structures" one needs to have the notion of structure at the first place. Axioms **G1-3**, or any other system of axioms determining some type of structure, cannot help one grasp the notion of structure unless one is

[3] We are now ready to spell out the precise definition: an infinite cyclic group is a group with an infinite number of elements and such that any of its elements is generated by some distinguished element g and its inverse g^{-1}. A group is said to be *generated* by a set of its distinguished elements called *generators* when every element of this group is a product of the generators. A canonical example of an infinite cyclic group is the additive group of whole numbers, which is generated by numbers 1 and -1. For example of a group non-isomorphic to the infinite cyclic groups consider the group of permutations of three letters A,B, C with the composition of permutations as group operation.

already aware of the fundamental role of isomorphism. For the idea of a general description satisfied by different mathematical objects is obviously not unique to Structuralism; Euclid's axioms do the same job with respect to numbers and magnitudes. Stressing the higher importance of structures with respect to "systems", the irrelevance of "intrinsic nature" and relevance of "structural relationships" cannot clarify the notion of mathematical structure by itself.

3. Isomorphisms and "Invariant Forms"

A non-structuralist may observe that axioms **G1-3** are satisfied by a number of "particular systems" (not structures so far!) called groups. Let now $\boldsymbol{G}$ be a class of such systems (i.e. groups), and consider the situation when some of these, say G_1 and G_2 are isomorphic. This actually means two things:

I1: elements of G_1 are in one-to-one correspondence with elements of G_2; by "one-to-one correspondence" between elements of two given sets A, B I understand here a set C of non-ordered pairs (a, b) such that $a \epsilon A$, $b \epsilon B$ and that every element of A is a member of exactly one of these pairs and similarly every element of B is a member of exactly one of these pairs;

I2: for all elements a_1, b_1, c_1 from G_1 such that $a_1 \oplus b_1 = c_1$ the corresponding elements a_2, b_2, c_2 from G_2 satisfy $a_2 \otimes b_2 = c_2$ where $\oplus$ is the group operation in G_1 and $\otimes$ is the group operation in G_2.

A one-to-one correspondence between elements of two given groups that satisfies **I2** is called (group) isomorphism. Groups are isomorphic if and only if there exists isomorphism between them. Notice that, given two isomorphic groups, there are, generally, many different isomorphisms between them. One should not confuse isomorphism as a particular correspondence between elements of two groups and isomorphism as an equivalence relation defined on some class of groups. Isomorphism in the latter sense holds between two given groups if and only if there exists an isomorphism in the former sense between these groups. As we can see, this terminology is slightly confusing but it is too common to try to change it.

Since isomorphism is an equivalence relation it divides class $\boldsymbol{G}$ into sub-classes containing only isomorphic groups. One may ignore differences between isomorphic groups and get through this act of abstraction various notions of groups-*qua*-structures (not to be confused with the general notion of group as a type of structure!), in particular, the notion of infinite cyclic group. To facilitate the language and provide this reasoning with some intuitive support one may talk and think about any particular structure as a thing "shared" by all members of the corresponding isomorphism class. On this basis one may claim that "the items making up any particular system exemplifying the structure in question are of no importance" (as does Hellman in the above quote). This claim describes the aforementioned abstraction, which can be called *structural* abstraction. However, one cannot forget about these exemplifying systems completely because this would destroy the whole reasoning bringing about the notion of mathematical structure. Noticeably Hellman needs the auxiliary notion of system in order to describe what is a mathematical structure. One might think that this additional notion (no matter what one calls it) plays a role in philosophical talk about structural mathematics but plays no role in structural mathematics itself. In the next Section I shall argue that this is not the case.

There is yet a different way of thinking about isomorphism (this will be already the third meaning of the term by our account!), which is common in current mathematical practice and particularly pertinent for categorical mathematics, as we shall later see. One may think about a one-to-one correspondence between elements of groups G_1 and G_2, which satisfies condition **I2**, as a map or transformation $i\colon G_1 \to G_2$ of one group into another group. Since a one-to-one correspondence is a symmetric construction the choice of G_1 as the source and G_2 as the target of this transformation is in fact arbitrary. In other words one and the same isomorphism-*qua*-correspondence gives rise to two isomorphisms-*qua*-transformations $i\colon G_1 \to G_2$ and $j\colon G_1 \to G_2$, which run in opposite directions and cancel each other on both sides. This latter property means precisely the following: the composition transformation $i \circ j$ resulting from the application of transformation j after transformation i sends every element of G_1 into itself and composition transformation $j \circ i$ sends every element of G_2 into itself (beware that none of the two conjuncts implies the other). Given these conditions each of transformations i and j is called the *inverse* of the other. Hence this definition: a transformation is called an isomorphism when it has an inverse. See Footnote 9 for a more precise categorical version of this definition. [4]

Thinking about isomorphism as a reversible transformation allows one to think of a structure shared by given transformed systems as an "invariant form", i.e. a form invariant under the given transformation. Then the structural abstraction can be described in these alternative terms: only the invariant form matters, transformed systems don't. As we shall see in what follows, the notion of isomorphism-*qua*-transformation, which may seem redundant in the context introduced so far, becomes indispensable in categorical mathematics. Noticeably Hilbert in the above quote (Section 3) talks about isomorphism as transformation, not as a symmetric one-to-one correspondence.

4. Structures versus Abstract Objects; Collections versus Transformations

Given an equivalence relation defined for a class of mathematical objects, Frege [8] considered the possibility of replacing each obtained equivalence class by a single object through an act of abstraction. [5]

[4]Notice that the order in which transformations are composed matters. I use here the so-called *geometrical* notation where the composition is written in the "direct" order. According to another notation called *algebraic* the composition is written in the inverse order.

[5]Frege's example is the concept of direction built, as follows. One considers the class of all straight lines on a Euclidean plane and the equivalence relation "is parallel". Then one associates a single abstract concept called *direction* with each isomorphism class of parallel lines.

Frege calls the result of this procedure an abstract object, not a structure, and indeed he doesn't think about this outcome as a structure. So we need a further effort for distinguishing structural abstraction from other types of mathematical abstraction. To this end, let us first consider this question: What are elements of a group-*qua*-structure? For the reason that I have already explained we don't want these elements to have anything like an "intrinsic nature". So they should be just "items" or "abstract elements"; the predicate "abstract" refers here to the act of abstraction through which the notion of group-*qua*-structure is obtained. However, we still need to make some assumptions about these things. We want them to be many and form (or belong to) well-distinguishable collections. Since we want to use the same notion of collection for different purposes we don't want the collected elements to be related in a specific way. This will give us the freedom to stipulate any relation between elements by fiat using the same notion of collection.

This is an important point where Structuralism meets Set theory. Having a notion of set at our disposal we are in a position to give the standard structural definition of group as a "structured set", namely a set provided with a binary operation satisfying axioms **G1-3** given above. There is a standard way to account for algebraic operations as relations that I shall not explain here.

As we have seen, the notion of isomorphism plays a crucial role in structural abstraction, which brings about new mathematical objects (namely, new mathematical structures). Importantly isomorphisms do not disappear when a given act of structural abstraction is accomplished and a new mathematical structure is well-defined. Mathematicians think about abstract groups and other abstract structures as given in an indefinite number of isomorphic copies, not as unique objects. As I have already stressed, people think similarly about numbers in traditional arithmetic (see Section 3). This, in my view, is the principal point where Frege's notion of abstraction fails to account for structural abstraction as this latter notion has been developed in twentieth-century mathematics. Reasoning "up to isomorphism" doesn't amount to the strict identification of isomorphic structures; it rather amounts to replacement of traditional equality by isomorphism in appropriate contexts. From a mathematical (as distinguished from logical and philosophical) viewpoint the question whether or not two isomorphic structures are identical is just as pointless as the question whether or not two equal numbers are identical. A sound mathematical question about two given numbers is whether or not they are equal. A sound mathematical question about two given structures is whether or not they are isomorphic. [6]

Set theory makes the talk of isomorphism as transformation redundant because the notion of one-to-one correspondence may be analyzed set-theoretically in terms of pairs of elements. However in many important mathematical contexts the notion of transformation is widely used anyway: groups of (reversible) transformations are abundant and geometry and also in physics. As far as foundations of mathematics are concerned we have an important choice here: either to (i) consider the notion of collection as more fundamental than that of transformation and reduce the latter to the former or to (ii) consider the notion of transformation as more fundamental and reconstruct the notion of collection on this basis. The former option brings (some version of) set-theoretic foundations of mathematics. The idea of categorical (i.e. category-theoretic) foundations amounts to taking the latter option. However the project turns to be non-viable unless one takes into the account other transformations than isomorphisms.

5. Homomorphisms

Given a type of structures it is always possible to define a general notion of map between structures of the given type. I shall discuss first the case of general maps between groups; such maps are called *homomorphisms* or more precisely *group homomorphisms*. Then I shall say few words about general maps between structures of different types. The term "homomorphism" is traditionally reserved for groups (apparently because this case was studied first), although, as its etymology suggests, it could also be used for structures

[6] I elaborate on this issue in [32] In particular, I discuss in this paper the idea of "weakening" equalities by replacing them with appropriate equivalences in n-categories.

of different types like the term "isomorphism". So in what follows I shall use the term "homomorphism" in the sense of general map between structures of some given type.

The notion of group homomorphism generalizes upon that of group isomorphism in the following way: instead of one-to-one correspondence between elements of groups G_1, G_2, one considers a more general kind of correspondence that is allowed to be many-to-one (but not one-to-many). In other words, one considers a *function* (in the set-theoretic sense of the term) $f\colon S_1 \to S_2$ from the set S_1 of elements of G_1 to the set S_2 of elements of G_2. Condition **I2** from Section 4 remains the same; notice that it can be satisfied when elements a_1, b_1 are different but elements a_2, b_2 are the same.

Group homomorphism and similar general maps between structures of other types are colloquially called "structure preserving". This is somewhat misleading because if such maps preserve anything at all it is a type of structure but not a particular structure. Think about this trivial example: for all groups G_1, G_2 there exist a homomorphism h: $G_1 \to G_2$ which sends every element of G_1 to the unit of G_2. This homomorphism "destroys all information" about G_1 reducing its image to a single element; it doesn't provide any information about G_2 either.

Actually the example of group homomorphism doesn't straightforwardly generalize to maps betweens structures of different types. For given a type of structure there are, generally, different ways to define maps between structures of the given type (some of which may be reasonable and some other not). Such maps can be of different kinds. Usual maps between topological spaces, i.e., general continuous transformations, do not *preserve* topological structure (in the same sense in which group homomorphisms are said to preserve group structure) but *reflect* it: the *inverse* image of any open set under a given continuous transformationis is always open while the direct image of an open set can be closed. In the case of isomorphism the difference between reflection and preservation of structural properties disappears. This fact shows that thinking about homomorphisms as "imperfect isomorphisms" can be misleading; at the very least one should not forget that a given structural isomorphism may "loose its perfection" in two different ways.

I shall now argue that homomorphisms, generally, don't allow for invariants in anything like the same sense in which isomorphisms do so. Let us try to replace isomorphisms by homomorphisms in the process of structural abstraction described in Section 4 and see what happens. One might expect to get in this way a generalized notion of structure but this doesn't work. Recall the first step: given class $\boldsymbol{G}$ of groups we have divided it into equivalence subclasses of isomorphic groups. Two groups are isomorphic if and only if there exists isomorphism (i.e., a reversible transformation) between them; clearly this is an equivalence relation. Let me (for the sake of argument) call two groups *homomorphic* if and only if there is a homomorphism between them. Although this latter relation is also an equivalence, one can see the difference: since *all* groups are homomorphic (see the above example of group homomorphism) one cannot use this equivalence for dividing $\boldsymbol{G}$ into equivalence subclasses! Saying that two given groups are homomorphic is tantamount to saying that the given groups are groups. So the relation of homomorphism just introduced (not to be confused with the standard notion of homomorphism as transformation) doesn't make sense.

In order to see the reason of this failure, note that the existence of homomorphism of the form $G_1 \to G_2$ doesn't imply the existence of homomorphism of the form $G_2 \to G_1$. This means that in the case of homomorphism (unlike that of isomorphism) the difference between the source and the target of the given transformation matters. But the relation of homomorphism tentatively introduced above doesn't take this difference into account. It forgets the difference between isomorphic and non-isomorphic groups and thus confuses their structural properties and offers no replacement.

A more reasonable choice of relation associated with a given homomorphism $h\colon G_1 \to G_2$ would be that of non-symmetrical relation $>$ such that $G_1 > G_2$ holds just in case there is a homomorphism of the form $h\colon G_1 \to G_2$. However, since $>$ is asymmetric it is not an equivalence and so doesn't allow one to proceed further with the structural abstraction or anything similar.

We see that homomorphisms cannot do the same job as isomorphisms: the reversibility condition stressed by Hilbert in the above quote (Section 3) turns out to be crucial for structural abstraction. One cannot reason "up to homomorphism" in anything like the same way in which people reason up to isomorphism doing structural mathematics. Since "invariant" in the given context is just another word for structure it is clear that homomorphisms, generally, don't have invariants in anything like the same sense in which isomorphisms and groups of isomorphisms do so. [7]

6. Structuralist Motivations behind Category Theory

The emergence of Category theory in he 1940s and its further development in the context of structural mathematics was related to a growing awareness of the role of general maps (not only isomorphisms). I shall not explain here the precise mathematical context in which this theory first proved useful but only mention that the notion of category generalizes upon such examples as the class of all sets and all functions, all groups and all group homomorphisms, all topological spaces and all continuous maps (not only reversible ones!) between topological spaces. This is a simple theorem [22] that a class of structures of any fixed type provided with an appropriate notion of general map form a category. Generally, a category comprises a class of objects and a class of composable maps (called in Category theory *morphisms*) for every ordered pair of objects, which are subject to few

[7] I mention here groups of isomorphisms (not to be confused with isomorphisms of groups!) because they are very important in geometry and physics. I mean groups of geometrical transformations of a given space. Only *reversible* geometrical transformations, i.e. geometrical isomorphisms, of a single object (the given space) form groups (with the composition of transformations as group operation) because in this case the reversibility is equivalent to the existence of inverse elements. So the talk of invariants of groups, which is so important for structural approaches in physics, concerns only reversible transformations and doesn't apply to geometrical (or other) transformations in general. A non-mathematical reader may skip the reference to groups of isomorphisms in this part of the paper. I shall explain the idea of group of isomorphisms more clearly in categorical terms in Section 8.

natural axioms. Given two different categories one defines a notion of map between categories. Such maps are called *functors*; the usual definition of functor is based on the same idea as the definition of group homomorphism given in the previous Section: a functor sends each object of the source category into an object of the target category and each morphism of the source category into a morphism of the target category in such a way that composition of morphisms is "preserved" in the same sense in which the group operation is said to be preserved by a group homomorphism. Using the notion of functor one may consider various categories of categories, i.e. categories such that their objects are themselves categories. One may also consider categories objects of which are functors. The above standard description of basic categorical concepts is structuralist in spirit. In Section 9 I shall describe functors and categories anew from a foundational and "more categorical" viewpoint. [8]

The idea of categorical foundations as viewed from a structuralist perspective amounts to recovering all the relevant properties of any structure of any given type through properties of the category of (all) structures of this given type. In the case of the category of sets this provides an alternative (category-theoretic) Set theory: one first conceives of sets as abstract objects and stipulates that they form a category; then one stipulates desired properties of this category, which make this category "into" the intended category of sets. This result (see [19]) shows that a reasonable notion of collection (set) can be developed on the basis of that of transformation (morphism of sets) but not only the other way round.

The growing popularity of Category theory as a common (albeit certainly not unique) "language" of contemporary mathematics as well as the continuing efforts of building categorical foundations of mathematics are generally seen as a further step of the structuralist project briefly described above. I agree with this view so far as it does not require preserving the basic principles of Mathematical Structuralism (as specified above) in the new categorical setting. In my understanding, these developments diverge from Mathematical

[8] For a detailed historical account of early days of Category theory see [16].

Structuralism and tend towards a very different view on mathematics and science in general. Before I describe this new view, let me explain reasons why categorical foundations appear to many as a version of structural foundations. In the next Section I shall show that this impression is wrong.

As I have explained in Section 4, the notion of set plays a special role in structural mathematics. This explains why Set theory itself is rarely seen as a structural theory on equal footing with, say, Group theory. As Hellman [12] rightly remarks:

> [D]espite the multiplicity of set theories (differing over axioms such as well-foundedness, choice, large cardinals, constructibility, and others), the axioms are standardly read as assertions of truths about "the real world of sets" rather than receiving a structuralist treatment.

The structural notion of group explained above is usually construed as a "set with a structure" or "structured set" rather than a pure structure (whatever this might mean); the underlying set of a given group is thought of as a background supporting the structure rather than a part of this structure. This way of thinking in mathematics is reminiscent of Aristotle's metaphysics of Matter and Form. The need for the set-theoretic Matter to do structural mathematics becomes clear from our analysis given in Section 4, but the presence of this ingredient doesn't fully comply with the philosophy of Mathematical Structuralism, which purports to make mathematical objects into pure forms (structures) and leave anything like their "background" outside mathematics. The desired "purely structural" mathematics would deal only with the "invariant Form" and require no set-theoretic Matter. Historical evidence of such an attitude can be found in what Dieudonne (under the name of Bourbaki) says in his structuralist manifesto [4] about set-theoretic difficulties:

> The difficulties did not disappear until the notion of set itself disappears ... in the light of the recent work on the logical formalism. From this new

> point of view mathematical structures become, properly speaking, the only "objects" of mathematics.

I don't believe that Dieudonne's claim concerning the alleged "disappearance" of sets is justified but the quote clearly demonstrates such an intention.

In this context the idea of accounting for relevant properties of mathematical structures only in terms of structure-preserving maps between these structures independently of any set-theoretic background, i.e., the idea of categorical foundations, indeed may look like a further step in the structuralist direction. Hence the popular view according to which categorical mathematics is the desired purely structural mathematics.

Remarkably, Category theory did never make it into Bourbaki's *Elements* [3], which is the most systematic attempt to develop structural mathematics ever undertaken. This is in spite of the fact that both founders of Category theory, Eilenberg and MacLane, were eventually involved in the Bourbaki group, so all the principal members of this group were well aware about their work. This fact is often seen as a historical puzzle but in my view it is not. For, as we shall shortly see, categorical foundations of mathematic are not and cannot be anything like the structural foundations developed by Bourbaki in his fundamental work. So in order to incorporate Category theory into their *Elements* Bourbaki would need to abandon his basic structuralist principles and engage himself into a very different foundational project.

One may agree that Bourbaki's version of Structuralism is incompatible with categorical foundations of mathematics but argue that some other variety of Structuralism is appropriate for building such foundations. For this reason I would like to stress once again that my following arguments concerning Structuralism and Category theory refer to the general notion of Structuralism described in Section 2 but not only to simple Bourbaki-like examples of structures and maps between structures. One may also argue that this notion of Structuralism is in fact too restrictive and doesn't really reflect the structural character of modern mathematics in full. Even if in this case the issue may look merely terminological

I would stress the need to define one's general notions of structure and Structuralism explicitely and precisely. Distinguishing between multiple varieties of Structuralism doesn't help one to meet this requirement unless one addresses the question What these different varieties are varieties of? What I want to stress in this paper is a conceptual difference between the "classical" structuralist thinking exemplified by [14] and [3], on the one hand, and some developments in Category theory, on the other hand. Leaving terms "structure" and "Structuralism" without any precise definition and using them in the broadest possible sense can hardly be helpful for showing such a difference. If Category theory indeed brings about a new philosophy of mathematics this new philosophy needs a new vocabulary. [9]

7. Categories versus Structures; Embodiement of Mathematical Concepts

Categories of structures like the category of groups, topological spaces, etc. capture the notion of type of structure, not the notion of singular structure. Particular structures (identified up to isomorphism) may be often also rendered as categories but in this case their morphisms are no longer structure-preserving maps. For example, a particular group (like the infinite cyclic group mentioned above) can be presented as a category with just one object such that all of its morphisms (going from this object to itself) are isomorphisms. The group operation is given by composition of morphisms; the existence of unit follows from the definition of a (general) category and the existence of inverse elements follows from the fact that all morphisms of the given category are reversible. [10]

[9] This paragraph is written after a very valuable discussion with Colin McLarty over an earlier draft of this paper.

[10] Categorical definition of isomorphism resembles the definition of reversible transformation given in the end of Section 3. However, it doesn't involve a reference to elements. Think about groups G_1, G_2 as objects of a category and modify the definition of Section 3 in this way: $i \circ j = 1_{G_1}$ and $j \circ i = 1_{G_2}$ where 1_{G_1} is the identity morphism of G_1 and 1_{G_2} is the identity morphism of G_2. The rest of the definition remains the same.

This simple example shows that categorical morphisms can but should not be structure-preserving maps. Moreover, the above categorical presentation of group, unlike its standard set-theoretic presentation, is not structuralist in character. For the standard structuralist presentation involves this idea: an abstract group can be "exemplified" by what Hellman calls "particular systems", like systems of numbers, systems of geometrical motions and so on and so forth. Of course, when one pictures elements of a given group as loops rather than dots this does not produce any conceptual change by itself. But given the above categorical presentation of a group, and using standard category-theoretic means, one can do something other than keep saying that morphisms of the given category (i.e., the given group) stand or may stand for something else than themselves. Namely, one may consider functors from the given group-category into some other categories, which in their turn present (rings or fields of) numbers, geometrical spaces, etc. This provides a much more precise idea of "standing for" in each particular case than the general structuralist rhetoric. In the structuralist setting the notion of *exemplification* remains meta-theoretical and escapes a precise mathematical treatment. But in the categorical setting this notion becomes a proper part of the given mathematical construction. Instead of saying that A stands for B one considers functors of the form $A \to B$ and treats these functors on equal footing with "internal" morphisms of A and B. [11]

In my understanding, this latter type of mathematical thinking has little if anything to do with structural abstraction. A principal epistemic strategy of Structuralism is to capture what various "particular systems" share in common, namely their "shared structure". The corresponding categorical strategy can be described in this way: look how particular systems translate into each other. Unlike the structuralist strategy this categorical strategy doesn't make the particular systems less important. Given morphism $A \to B$ there is, generally, no reason to think of A and B "up to" some equivalence and dispense with A

[11] A further step of such categorical analysis amounts to considering the full category of functors of the given form; such a functor category provides a precise information about how A translates into B.

and B in favor of their shared structure or anything else. As I have already shown in Section 5 the notion of thinking "up to homomorphism" is plainly unsound.

Let us now consider the case when a category presents a type of structure rather than a singular structure. To analyze this case I shall use the notion of *embodiment*, which I have introduced elsewhere [33]. As we have seen in Section 4 a mathematical structure cannot be identified with its corresponding abstract concept: something else is needed in order to make a given concept into a mathematical object. Kant would call this additional element an intuitive construction; I use the word "embodiment" for a similar purpose but in a different mathematical context. We have seen how the notion of structure allows for making a concept describing different particular systems into a single mathematical object (single up to isomorphism, of course). As we have seen in Section 5 this structuralist method of embodiment doesn't work for *types* of structure. While the concept "infinite cyclic group" can be embodied into a single structure, the concept "group" cannot; "the group" unlike "the infinite cyclic group" is not a name of unique (up to isomorphism or otherwise) mathematical object. However the *category* of (all) groups is a single mathematical object like number 3, the infinite cyclic group or, say, the Euclidean plane. Each of these objects has a *many-splendored existence* (to use MacLane's word), so its singleness must be understood appropriately. But I want now to stress a different point: the way in which all isomorphic cyclic groups are made into a single object with the notion of structure and the way in which all groups are made into a single object with the notion of category are essentially different. While the former involves structural abstraction the latter involves a different kind of abstraction, which I shall call *categorical*. Roughly, categorical abstraction amounts to the following: one forgets about the fact that groups have elements and consider only how they map to (i.e. transform into) each other with appropriate morphisms; a relevant notion of element is recovered in this categorical setting only later on. Obviously the two kinds of abstraction are quite different. I shall say more about categorical abstraction in the Conclusion.

A category in which morphisms (including identity morphisms) form a set (in the technical sense of the term) is called *small*. Small categories can be thought of as structures on their own. The corresponding type of structures is defined straightforwardly: one takes a set of elements called morphisms, stipulates appropriate primitive relations between elements of this set, and spells out the necessary axioms (see the next Section for more details). Thus small categories like groups can be thought of as structures of a specific type. Noticeably, this straightforward approach doesn't work in the case of large categories corresponding to types of structures - think again of the category of groups or the category of all small categories. Since morphisms of such categories form proper classes they cannot be described as structured sets. Although this may look like a minor technical difficulty, which can be resolved by an appropriate generalization of the usual notion of structure, this difficulty provides additional evidence that the structural approach, generally, doesn't work in Category theory. Instead of thinking of categories as structures (or generalized structures) of a particular type, it seems to me more reasonable to reverse the order of ideas and think of structures as categories or categorial constructions of a particular type. An immediate suggestion would be to identify structures with small categories. A more elaborate suggestion by Lawvere (in person) is to identify a structure with a functor from a small category to a large "background" category, say, that of sets.

To conclude this Section, let me stress that categories don't always represent particular structures or particular types of structure. Examples of this latter kind are today so popular only because they connect the new categorical mathematics with the older structuralist mathematics. But categorical mathematics also involves concepts and constructions that were first developed in a categorical setting, for example that of Grothendieck topology. One may expect that the further development of categorical mathematics will make such "purely categorical" concepts better known and more useful in various branches of mathematics; then the link between the categorical mathematics and its structural predecessor will become a historical and philosophical rather than mathematical issue.

8. "The category of categories"

The idea of categorical foundations amounts to taking the notions of category, functor and/or some other related categorical notions as primitive and recovering the rest of mathematics on this basis. What are possible ways of realizing this project? In which precise sense can one consider category-theoretic notions as primitive? A way to do this, which immediately suggests itself, is to use in categorical foundations a modern version of Hilbert-style axiomatic method after the example of standard set-theoretic foundations.

Consider a class of things called morphisms and three primitive relations: one that associates with every given morphism its source, one that associates with every given morphism its target, and, finally, one that associates with some (ordered) pairs of morphisms a third morphism called the composition of the given two morphisms. Then we need axioms to ensure that sources and targets of morphisms behave as identity morphisms (i.e. as *objects*), that two given morphisms are composable if and only if the target of the first morphism coincides with the source of the second morphism, and some other similar axioms. Finally we should assume that the composition of morphisms is associative. For the full list of such axioms I refer the reader to [20]. The axiomatic theory just described this author calls the *elementary theory of abstract categories.*

Lawvere's paper begins as follows:

> In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of. The question thus naturally arises whether one can give a foundation for mathematics which expresses wholeheartedly this conviction concerning what mathematics is about, and in particular in which classes and membership in classes do not play any role.

We see that Lawvere embraces Mathematical Structuralism here but at the same time rejects set-theoretic (and even more general class-based) foundations of mathematics. Since the Hilbert-style axiomatic method is essentially structural (see Section 2) Lawvere's method of building his *elementary theory of abstract categories* perfectly fits his stated purpose. After the introduction of the axioms of the *elementary theory* and providing some definitions on their basis Lawvere says:

> By a category we of course understand (intuitively) any structure which is an interpretation of the elementary theory of abstract categories, and by a functor we understand (intuitively) any triple consisting of two categories and a rule T which assigns, to each morphism x of the first category, a unique morphism xT of the second category in such a way that ...

(follow the conditions of being structure-preserving). A problematic aspect of this first part of the paper concerns Mayberry's argument that Lawvere's *elementary theory* like any other theory built with the Hilbert-style axiomatic method requires some primitive (non-axiomatic) notion of collection, which cannot be identified with that of category [28]. The argument implies that the *elementary theory* and the corresponding elementary notion of category cannot be a genuine foundation. I agree with Mayberry on this point (this follows from my understanding of the relationships between Structuralism and Set theory explained in the beginning of Section 4), but unlike Mayberry I think that such a primitive notion of collection is dispensable in foundations of mathematics along with the Hilbert-style structural axiomatic method itself. In what follows I shall sketch a different version of axiomatic method that seems to me more appropriate for categorical foundations. Let me now return to Lawvere [20].

Lawvere's *elementary theory* is a preparatory step towards another theory of categories, which Lawvere calls *basic theory*. My claim is that unlike the elementary theory the *basic theory* is not structural, at least not in a similar sense. If I am right this shows that the main content of [20] in fact doesn't agree with the structuralist agenda announced by the author in the beginning of his paper: Lawvere begins with structural reasoning but then proceeds with a very different agenda, which can be described as genuinely categorical.

The *basic theory* begins with a re-introduction of the notion of functor:

> Of course, now that we are in the category of categories, the things denoted by the capitals will be called categories rather than objects, and we shall speak of functors rather than morphisms.

This may sound like a mere terminological convention (rather than an alternative definition) but in fact it signifies a sharp change of perspective. The idea is now the following: given a preliminary notion of category (through the *elementary theory*), conceive of category $\boldsymbol{C}$ of *all* categories; then pick up from $\boldsymbol{C}$ an arbitrary object A (i.e., an arbitrary category) and finally specify A as a category by internal means of $\boldsymbol{C}$, stipulating additional properties of $\boldsymbol{C}$ when needed. More precisely it goes as follows (I omit details and streamline the argument). Stipulate the existence of terminal object $\mathit{1}$ in $\boldsymbol{C}$, i.e., the object with exactly one incoming functor from each object of $\boldsymbol{C}$. Then identify objects (= identity functors) of A as functors in $\boldsymbol{C}$ of the form $\mathit{1} \to A$. Stipulate also the existence of initial object $\mathit{0}$, i.e. the object with exactly one outgoing functor into each object of $\boldsymbol{C}$. Then consider in $\boldsymbol{C}$ object $\mathit{2}$ of the form $\mathit{0} \to \mathit{1}$ and stipulate for it some additional properties among which is the following: $\mathit{2}$ is a universal generator which means that:

G (generator): for all f, g of the form:

$$A \overset{f}{\underset{g}{\rightrightarrows}} B$$

and such that $f \neq g$ there exist x such that:

$$\mathit{2} \xrightarrow{x} A \overset{f}{\underset{g}{\rightrightarrows}} B$$

and $xf \neq xg$.

U (universal): if any other category N has the same property, then there are y, z such that:

$$\begin{tikzcd} A \arrow[r, shift left, "y"] & B \arrow[l, shift left, "z"] \end{tikzcd}$$

and $yz = \mathit{2}$.

This allows Lawvere to identify functors (morphisms) of A as functors of the form $\mathit{2} \rightarrow A$ in C. The fact that $\mathit{2}$ is the universal generator (it is unique up to isomorphism as follows from the above definition) assures that categories are determined "arrow-wise": two categories coincide if and only if they coincide on all their arrows. This new definition of functor also allows one to make sense of the notion of a component of a given functor of the form h: $A \rightarrow B$, which in the elementary theory is understood as a map m sending a particular morphism f of A into a particular morphism g of B . In the basic theory, m turns into this commutative triangle: [12]

[12] A categorical diagram is said to commute or be commutative when the compositions of all morphisms shown at the given diagram produce other morphisms shown at the same diagram in appropriate places, so that any ambiguity about results of the compositions is avoided. For example, saying this triangle

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

is commutative is simply tantamount to saying that $fg = h$. Morphisms resulting from composition of shown morphisms can be omitted at a commutative diagram when this doesn't lead to an ambiguity. For example, saying this square

$$\begin{tikzcd} A \arrow[r, "g"] & B \\ C \arrow[u, "f"] \arrow[r, "h"'] & D \arrow[u, "i"'] \end{tikzcd}$$

is commutative is tantamount to saying that $fg = hi$.

$$\begin{tikzcd} & \mathit{2} \arrow[dl] \arrow[dr] & \\ A \arrow[rr] & & B \end{tikzcd}$$

This, once again, significantly changes the whole perspective: categories and functors are no longer built "from their elements" but rather "split into" their elements when appropriate. Although the notion of functor as a structure-preserving map can be recovered in this new context it no longer serves for defining the very notion of functor. Rule T used by Lawvere for defining functors in the *elementary theory* disappears in the *basic theory* without leaving any trace.

Further consider this triangle which Lawvere denotes $\mathit{3}$:

$$\begin{tikzcd} & \mathit{0} \arrow[dl] \arrow[dr] & \\ \mathit{1} \arrow[rr] & & \mathit{2} \end{tikzcd}$$

(It should satisfy a universal property which I omit). $\mathit{3}$ serves for defining composition of morphisms in our "test-category" A as a functor of the form $\mathit{3} \to A$ in C. Finally, in order to assure the associativity of the composition Lawvere introduces category $\mathit{4}$, which is pictured as follows:

$$\begin{tikzcd} & \mathit{3} & \\ \mathit{0} \arrow[ur] \arrow[rr] \arrow[dr] & & \mathit{2} \arrow[ul] \\ & \mathit{1} \arrow[uu] \arrow[ur] & \end{tikzcd}$$

(The associativity concerns here the path $\mathit{0} \to \mathit{1} \to \mathit{2} \to \mathit{3}$.)

This construction provided with appropriate axioms makes A into an "internal model" of the elementary theory in the following precise sense: If F is any theorem of the *elementary theory*, then "for all A, A satisfies F" is a theorem of the *basic theory*. [13]

The following analogy with the set-theoretic mathematics helps to clarify the role of categories of categories in foundations. As long as the notion of set is not supposed to provide a foundation for mathematics, one thinks of sets after examples of sets of numbers, sets of points, and the like. But in a foundational axiomatic theory of sets like ZF there are no other sets but sets of sets, and every mathematical object like a number or a point is supposed to be a set. Similarly in a foundational axiomatic theory of categories there are no other categories but categories of categories and every mathematical object is ultimately a category.

9. Functorial Semantics, Sketch Theory and Internal Language

In order to see that Lawvere's basic theory unlike his elementary theory is not based on structuralist principles, and then to get an idea of non-structuralist principles behind this theory, it is instructive to take into consideration two similar approaches: *Functorial semantics* developed by the same author elsewhere [18] and *Sketch theory* founded by Ch. Ehresmann in the 1960s and later developed by other people (see [35] for an overview and further references).

Functorial semantics involves the presentation of mathematical theories as categories of a special sort; models of a given theory are functors from the theory to the background category of sets or another appropriate topos. The very idea of "interpretation" or "realization" of a given theory in a set-theoretic background obviously comes from the standard (due to Tarski) Model theory. Lawvere's functorial semantic can be seen as a category-theoretic version of the same basic construction. However, as we shall now see, this technical update

[13] Isbell in his review [15] of Lawvere's [20] points to a technical flaw in Lawvere's proof of this theorem . This flaw is fixed, in particular, in [30].

comes with a significant revision of the structuralist background of Tarski's Model theory inherited from Hilbert's notion of axiomatic method.

In order to determine a theoretical structure, an axiomatic theory should be *categorical* , i.e., to have models that are all isomorphic. (Beware that this older sense of the term "categorical" has nothing to do with Category theory!) True, not all axiomatic theories built by the standard method satisfy this requirement; also true, non-categorical theories are usually not disqualified solely on this basis. Anyway, in the standard setting the categoricity of axiomatic theory is commonly (and usually as a matter of course) viewed as an epistemic gain while the lack of categoricity is viewed as a problem. As long as one commits oneself to Structuralism such an attitude is understandable: when a set of axioms fails to specify a model up to isomorphism it fails to specify a structure. Saying that a non-categorical theory determines many structures rather than one structure is somewhat misleading because such a theory, strictly speaking, doesn't specify any structure at all (cf. Section 2).

In the case of Lawvere's functorial semantics, the structuralist pursuit of categoricity turns into an absurdity. For the purpose of this construction is to produce a workable category of models rather than just one model up to isomorphism. In the functorial setting a theory determines a category, not a structure. This makes the structuralist thinking behind the axiomatic method as expressed by Hilbert in the above quote (Section 2) irrelevant. In the new setting:

> The theory appears itself as a generic model [18].

This means that the older structuralist distinction between abstract "formal" axiomatic theories, on the one hand, and their semantics, on the other hand, doesn't apply; what distinguishes a theory form its (other) models is its *generic* character rather than formal or abstract character.

The setting of Sketch theory is similar to that of Lawvere's Functorial semantics but in the former case generic categories are designed as "generic shapes" or "generic figures" rather than axiomatic theories. Unlike the case of Functorial semantics such generic categories are not supposed to have logical properties; in some approaches sketches are not even categories but directed graphs with an additional structure. It seems natural to think of sketches as "proto-structures" but this is somewhat misleading insofar as the usual notion of structure is concerned. A sketch doesn't represent a bunch of isomorphic systems but generates non-isomorphic systems (its models). These systems share their generic shape not in the same sense in which different systems are said to share the same mathematical structure. In fact they share a shape in a more straightforward sense: a given sketch is a common source of all of its models (i.e. specific functors from this given sketch to a background category). To "have the same source" is obviously an equivalence relation but this equivalence relation doesn't support anything like the structural abstraction. Unlike a shared structure a shared sketch is concrete (it is usually even supposed to be finite and easily pictured) while things generated by a sketch can be indeed described as abstract structures in the older sense because they are usually distinguished only up to isomorphism! Thus Sketch theory turns Structuralism upside down and in certain aspects reminds of more traditional ways of doing mathematics. Euclid's geometrical universe is generated by two generic figures, namely, the straight line and the circle, which is tantamount to saying that every geometrical object is constructed by ruler and compass. The analogy seems to me straightforward. [14]

Whether or not the new categorical approach to theory-building - differently realized in Functorial semantics, Sketch theory, and the *basic theory* of [20] - can compete with the standard Hilbert-style structural approach remains an open question. The considered constructions don't allow one to claim that this new approach can work independently: we have seen that Lawvere's *basic theory* depends on the structural *elementary theory*, Functorial semantics is developed by this author similarly in two steps, and Sketch theory in its

[14] Does this mean that Ehresmann misconceived of his own invention when he thought of Sketch theory as a general theory of structure? I don't think so. A general theory of structure should not be necessarily a structural theory and should not provide a support for Structuralism as a philosophical view about mathematics.

existing form uses Set theory and usually doesn't make foundational claims at all. However there is no reason either to claim that the pre-theoretical notion of collection involved in the standard set-theoretic foundations is indispensible in foundations of mathematics (cf. Mayberry's argument in Section 8). It can be replaced by a primitive pre-theoretic notion of category that involves common intuitions about processes (transformations) and their composition. What remains a problem is how to upgrade this pre-theoretic notion of category to a theoretical one without using other means but properly categorical.

Which means and constructions may qualify as "properly categorical" in a foundational context also remains an open question but I think that the standard machinary of first-order logic used in [20] and later in [30] for writing down axioms of Category theory after the example of Set theory does not qualify as such. Category theory suggests a change of the traditional conception of logic, which is analogous to the change of the traditional conception of geometry that occurred in the 19th century when people stopped thinking about "the" geometrical space as a universal container of geometrical objects and learned to think about spaces as objects and about objects as spaces with the notion of *intrinsic geometry* of a given geometrical object. In the first half of the 20th century people learned to think about systems of logic as objects living in larger meta-logical frameworks. Category theory showed how one can think about objects (i.e., appropriate categories) as systems of logic with the notion of internal language of a given category [17]. This reciprocal move that allows one to avoid the bad infinity of meta-meta.....-logics and meta-meta....-mathematics in foundations of mathematics has immense philosophical importance and I think that it has to be taken into account in categorical foundations. This is why the presence of a self-standing system of logic representing alleged universal laws of reasoning seems me inappropriate in categorical foundations. A candidate for replacement can be a version of Sketch logic developed in the vein of [23], [24], [25], [34] and [6]. I leave this issue for a further study.

10. Conclusion: a categorical perspective in and on mathematics

I hope to have convinced the reader that the project of categorical foundations requires a new philosophical view on mathematics, which the traditional Structuralism cannot possibly provide. Let me now try to summarize this new categorical view by contrasting it with the structuralist view. What matters in the categorical mathematics is how mathematical objects and constructions transform into each other, not what (if anything) remains invariant under these transformations. So categorical mathematics is a theory of abstract transformation, not a theory of abstract form. A theory in categorical mathematics is a generic model (Lawvere) rather than a scheme (Hilbert). In the end of his [1] Awodey puts forward the following structuralist slogan:

> The subject matter of pure mathematics is invariant form, not a universe of mathematical objects consisting of logical atoms.

I suggest instead this alternative slogan:

The subject matter of pure mathematics is covariant and contravariant transformation, not invariant form.

The categorical view on mathematics - as distinguished from categorical foundations of mathematics in the sense articulated in the previous Section - suggests a new understanding of the role of history of mathematics in mathematics itself. Consider these two versions of the Pythagorean theorem.

> **(1)** In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

(Proposition 1.47 of Euclid's *Elements*).

> **(2)** If two non-zero vectors x and y are orthogonal then$(y-x)^2 = y^2 + x^2$

([7], slightly modified). [15]

[15] The original version reads

What justifies saying that **(1)** and **(2)** express one and the same theorem? A structuralist's answer is: "Obviously the invariant content of these two expressions!" I claim that the answer is wrong. For there are many ways in which Euclid's geometry can be interpreted in modern terms ([7] is just one way of doing this among many others) but there is no way to spell modern geometry in Euclid's terms. We recognize **(2)** as the old Pythagorean theorem **(1)** because the latter naturally translates into the former. This translation is not just a matter of the glass bead game but it reflects the historical process of dialectical change of foundations of geometry from Euclid's times to 1960s. Crucially this translation doesn't work the other way round: our history in general and our intellectual history in particular develops from the past to the future but not from the future to the past. According to the argument given in the Section 5, this implies that no translation between **(1)** and **(2)** allows for the identification of an invariant. Thus the existence of sound translations between theories doesn't imply that these theories share anything like an invariant content. There is no essence, no conceptual core preserved by the translation of **(1)** into **(2)**. But why in this case should we count them as different versions of the same theorem?

My answer is this. The Pythagorean theorem as distinguished from its particular formulations like **(1)** and **(2)** is a conceptual entity *perduring over* (rather than *enduring through*) the historical change of foundations. The change of perspective that I suggest here is analogous to the replacement of the traditional 3-dimensional ontology in physics by the modern 4-dimensional ontology [29]. In this sense **(1)** and **(2)** can be compared with points of a trajectory in a spacetime. Importantly Pythagorean theorem doesn't reduce to some set of such points, i.e., a set of particular formulations of this theorem. For such a reduction leaves out what from a categorical viewpoint is the most important, namely translations between these different formulations. Instead of thinking in this context about a set of sentences like **(1)** and **(2)** I suggest the reader to consider a category of such things.

Two non-zero vectors x and y are orthogonal if and only if $(y-x)^2 = y^2 + x^2$

A coherent translation between them is still possible even when no invariant structure is available.

References

[1] S. Awodey. Structure in mathematics and logic: A categorical perspective. *Philosophia Mathematica*, 3(4):209 – 237, 1996.

[2] S. Awodey. An answer to g. hellman's question "does category theory provide a framework for mathematical structuralism? *Philosophia Mathematica*, 12(3):54–64, 2004.

[3] N. Bourbaki. *Elements de mathematique*. Masson (Paris), 1939-1983.

[4] N. Bourbaki. The architecture of mathematics. *The American Mathematical Monthly*, 57(4):221 – 232, 1950.

[5] L. Corry. *Modern Algebra and the Rise of Mathematical Structures, (Science Networks Vol. 17)*. Birkhuser, 2004.

[6] Z. Diskin and U. Wolter. A diagrammatic logic for object-oriented visual modeling. *Electronic Notes in Theoretical Computer Science (ENTCS)*, 203(6):19–41, 2008.

[7] A. Doneddu. *Geometrie Euclidienne*. Plane (Paris), 1965.

[8] G. Frege. *Die Grundlagen der Arithmetik, eine logisch mathematische Untersuchung ber den Begriff der Zahl*. Breslau: W. Koebner, 1884.

[9] G. Frege. *On the Foundations of Geometry and Formal Theories of Arithmetic*. Yale University Press, 1971.

[10] G. Hellman. Three varieties of mathematical structuralism. *Philosophia Mathematica*, 2(9):184–211, 2001.

[11] G. Hellman. Does category theory provide a framework for mathematical structuralism? *Philosophia Mathematica*, 2(11):129–157, 2003.

[12] G. Hellman. Structuralism, mathematical. *Encyclopedia of Philosophy (Macmillan), Ed. Donald M. Borchert, 2nd ed.*, 9:270–273, 2006.

[13] G. Hellman. What is categorical structuralism? *The Age of Alternative Logics Assessing Philosophy of Logic and Mathematics Today (eds. Benthem et al.)*, pages 151–161, 2006.

[14] D. Hilbert. *Foundations of Geometry*. OpenCourt, 1980 (1899).

[15] J.R. Isbell. Review of lawvere:1966. *Mathematical Reviews*, 34, 1967.

[16] R. Kromer. *Tool and Object: A History and Philosophy of Category Theory*. Birkhauser, 2007.

[17] J. Lambek and Scott P. J. *Introduction of Higher-Order Categorical Logic*. Cambridge University Press, 1986.

[18] F.W. Lawvere. Functorial semantics of algebraic theories, ph.d. thesis (1963), columbia university, 1963-2004. republished in Reprints in Theory and Applications of Categories, No. 5 (2004):1-121, with a new authors Commentary; available at www.tac.mta.ca/tac/reprints/articles/5/tr5.pdf.

[19] F.W. Lawvere. Elementary theory of the category of sets. *Proceedings of the National Academy of Science*, 52(6):1506–1511, 1964.

[20] F.W. Lawvere. The category of categories as a foundation for mathematics. In *Proceedings of the La Jolla Conference on Categorical Algebra*, pages 1–21, 1966.

[21] F.W. Lawvere. Foundations and applications: Axiomatization and education. *The Bulletin of Symbolic Logic*, 9(2):213–224, 2003.

[22] S. Mac Lane. Structure in mathematics. *Philosophia Mathematica*, 4(2):174–183, 1996.

[23] M. Makkai. Generalized sketches as a framework for completeness theorems (i). *Journal of Pure and Applied Algebra*, 115:49–79, 1997.

[24] M. Makkai. Generalized sketches as a framework for completeness theorems (ii). *Journal of Pure and Applied Algebra*, 115:179–212, 1997.

[25] M. Makkai. Generalized sketches as a framework for completeness theorems (iii). *Journal of Pure and Applied Algebra*, 115:241–274, 1997.

[26] M. Makkai. Towards a categorical foundation of mathematics. *Lecture Notes in Logic*, 11:153190, 1998.

[27] J.-P. Marquis. *From a Geometrical Point of View: A Study in the History and Philosophy of Category Theory*. Springer, 2009.

[28] J.P. Mayberry. *The foundations of mathematics in the theory of sets*. Cambridge University Press, 2000.

[29] N. McKinnon. The endurance/perdurance distinction. *The Australasian Journal of Philosophy*, 80(3):288–306, 2002.

[30] C. McLarty. Axiomatizing a category of categories. *The Journal of Symbolic Logic*, 56(4), 1991.

[31] C. McLarty. Exploring categorical structuralism. *Philosophia Mathematica*, 12(1):37–53, 2004.

[32] A. Rodin. Identity and categorification. *Philosophia Scientiae*, 11(2):27–65, 2007.

[33] A. Rodin. How mathematical concepts get their bodies. *Topoi*, 29(1):53–60, 2010.

[34] Ch. Wells and A. Bagchi. Graph-based logic and sketches, 2008. arXiv:0809.3023.

[35] Charles Wells. Sketches: Outline with references. In *Dept. of Computer Science, Katholieke Universiteit Leuven*, 1994.