Identity and Categorification

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Résumé : Dans cet article je présente une analyse critique de l'approche habituelle de l'identité mathématique dont a son origine dans les travaux de Frege et Russell, en faisant un contraste avec les approches alternatives de Platon et Geach. Après je pose ce problème dans un cadre de la théorie des catégories et montre, que la notion d'identité ne peut pas être « internalisée » par les moyens catégoriques standards. Enfin, je présente deux approches de l'identité mathématique plus spécifiques : une avec la fibration catégorielle et l'autre avec des catégories supérieures faibles.

Abstract: In this paper I consider the standard approach to identity in mathematics originating from Frege's and Russell's works against alternative approaches due to Plato and Geach. Then I put the problem in a category-theoretic setting and show that the notion of identity cannot be "internalized" with usual categorical means. Finally I present two more specific mathematical approaches to the identity problem: one based on a categorical fibration and the other involving weak higher categories.

1 Paradoxes of Identity and Mathematics

Changing objects (of any nature) pose a difficulty for the metaphysically-minded logician known as the *Paradox of Change*. Suppose a green apple becomes red. If A denotes the apple when green, and B when it is red then $A = B$ (it is the same thing) but the properties of A and B are different : they have a different color. This is at odds with the *Indiscernibility of Identicals* thesis according to which identical things have identical properties. A radical solution - to explain away and/or dispense with the notion of change altogether was first proposed by Zeno around

Philosophia Scientiæ, 11 (1), 2007, 1–39.

500 BC and remains popular among philosophers (who often appeal to the relativistic spacetime to justify Eleatic arguments). Unlike physics, mathematics appeared to provide support for the Eleatic position : for some reason people were more readily brought to accept the idea that mathematical objects did not change than to accept a similar claim about physical objects - in spite of the fact that mathematicians had always talked about variations, motions, transformations, operations and other process-like notions just as much as physicists.

The Paradox of Change is the common ancestor of a family of paradoxes of identity which might be called *temporal* because all of them involve objects changing in time[1]. However *time* is not the only cause of troubles about identity : space is another. The *Identity of Indiscernibles* (the thesis dual to that of the *Indiscernibility of Identicals*) says that perfectly like things are identical. According to legend in order to demonstrate this latter thesis, Leibniz challenged a friend during a walk to find a counter-example among the leaves of a tree. Although there are apparently no perfect doubles among material objects, mathematics appears to provide clear instances immediately : think about two (different) points. But the example of geometrical space brings another problem : either the *Identity of Indiscernibles* thesis is false or our idea of perfect doubles like points is incoherent. In what follows I shall refer to this latter problem as the *Paradox of Doubles*. Mathematics looks more susceptible to this paradox than physics. However nowadays mathematics and physics are so closely entwined it is hardly possible to isolate difficulties in one discipline from those in the other. Were she living today, Leibniz's friend might meet his challenge by mentioning the indiscernibility of particles in Quantum Physics [2].

Category theory provides an original understanding of identity in mathematics, which takes seriously the idea that mathematical objects are, generally speaking, *variable* and handles the problem of doubles in a novel way. Category theory does not *resolve* paradoxes of identity of the above form ; rather it provides a setting where paradoxes in such a form do not arise. The Category-theoretic understanding of identity in mathematics may have important consequences in today's mathematically-laden physics and hence (assuming some form of scientific realism) for the notion of identity in a completely general philosophical setting. In this paper I explore this new understanding of identity in Category Theory, leaving its implications outside mathematics for a future study.

[1] *Chrisippus' Paradox, Stature, The Ship of Theseus* belong to this family. See [Deutsch 2002]

[2] [French 1988]

The paper is organized as follows. First, I consider some difficulties about the notion of identity in mathematics, providing details and examples. Then I briefly review some attempts to overcome these difficulties. I pay particular attention to the account of identity in mathematics proposed by Frege and afterward developed by Russell, which remains standard in the eyes of many philosophers. Then I consider an alternate approach to identity in mathematics, which dates back to Greek geometry but made a new appearance in 19th century and later developed in Category theory. I consider the issue of identity in Category theory starting with general remarks and then coming to more specific questions concerning fibred categories and higher categories. Finally I suggest a way of thinking about categories, which implies deversification of the notion of identity and revision of Frege's assumption that identities must be fixed from the outset.

2 Mathematical Doubles

The example of two distinct points A, B (Fig.1) does not, it is usually argued, refute the *Identity of Indiscernibles* because the two points have different *relational* properties : in Fig. 1, A lies to the left of B but B does not lie to the left of itself[3] :

$$A \bullet \quad \bullet B$$

Fig. 1

(The difference in the relational properties of A and B amounts to saying that the two points have *different positions*.)

However the example can be easily modified to meet the argument. Consider two *coincident* points (Fig. 2) : now A and B have the *same* position.

$$\overset{A=B}{\bullet}$$

Fig.2

It might be argued that coincident points are an exotic case, one which can and should be excluded from mathematics via its logical regimentation. But this is far from evident — at least if we are talking about

[3]These relational properties of the two points depend on their shared space : the argument doesn't go through for points living on circle. I owe this remark to John Stachel.

classical Euclidean geometry. For one of the basic concepts of Euclidean geometry is *congruence*, and this notion (classically understood) presumes coincidence of points : figures F, G are *congruent* iff by moving G (without changing its shape and its size) one can make F and G *coincide* point by point.

The fact that geometrical objects may coincide differentiates them significantly from material solids like chairs or Democritean atoms. The supposed *impenetrability* of material solids counts essentially in providing their identity conditions [Lucas 1973]. Thus, identity works differently for material atoms and geometrical points[4].

We see that the alleged contradiction with the *Identity of Indiscernibles* is not the only difficulty involved here. Indeed the whole question of identity of points becomes unclear insofar as they are allowed to coincide. Looking at Fig.2 we have a surprising freedom in interpreting "=" sign. Reading "=" as identity we assume that A and B are two different names for the same thing. Otherwise we may read "=" as specifying a coincidence relation between the (different) points A and B. It is up to us to decide whether we have only one point here or a family of superposed points. The choice apparently has little or no mathematical sense. One may confuse coincidence with identity here without any risk of error in proofs. However this does not mean that one can just assimilate the notions of identity and coincidence. For identity so conceived would be very ill-behaved, allowing for the merger of different things into one and the splitting of one into many. (Consider the fact that Euclidean space allows for the coincidence of *any* point with any other through a suitable motion.) Perhaps it would be more natural to say instead that the relations of coincidence and identity while not identical in general, coincide in this context ?

For an example from another branch of elementary mathematics consider this equation : $3 = 3$. Just as in the previous case there are different possible interpretations of the sign "=" here. One may read "=" either as identity, assuming that 3 is a unique object, or as a specific relation of equality which holds between different "doubles" (copies) of 3. Which option is preferable depends on a given context. There is a unique natural number x such that $2 < x < 4$; $x = 3$. Here "=" stands for identity. But when one thinks about the sum $3 + 3$ or the sequence

[4]This fact shows that Euclidean geometrical space cannot be viewed as a realistic model of the space of everyday experience as is often assumed. One needs the third dimension of physical space to establish in practice the relation of congruence between (quasi-) 1- and 2-dimensional material objects through the application of a measuring rod or its equivalent.

$3, 3, 3, \ldots$ it is convenient to think of the 3s as many. In this latter case $3 = 3$ still holds but now "=" is being read as equality rather than identity. Again the choice looks like a matter of convenience rather than of theoretical importance.

Similarly, in one sense *cube* is a particular geometrical object, while in a different sense there exist (in some suitable sense of "exist") many cubes. When one proves that there exist exactly 5 different regular polyhedrons, and says that the *cube* is one of them, one speaks about the cube in the first sense. When one considers a geometrical construction, which comprises several cubes, one thinks about the cubes in the second sense. However no distinction between the two meanings of the term "cube" can be found in standard textbooks, and it is not even clear whether such distinction can be sharply made. In fact in geometry the situation is even more complicated. For there is a sense in which the "same figure" means a figure of the same shape and the same size, and there is another sense in which it means only a figure of the same shape, and the notion of "same shape" can itself also be specified in different ways. In addition geometry unlike arithmetic allows for the identification of its objects (of geometrical figures) by directly naming them, usually through naming of their most important points. This allows us to distinguish two different triangles ABC and $A'B'C'$ which are the "same" in any of above senses. There is apparently no clear argument, which would allow us to choose one of these senses of "the same" as basic and eliminate the others as an abuse of the language!

The above examples might make one think that the notion of identity simply plays no significant role in mathematics. $2 \times 2 = 4$ remains true independently of whether the sign "=" is read as equality, or as identity, whether equality is treated as identity, or identity is weakened to equality. It looks as if here one may choose one's interpretation according to personal taste or preferred philosophical position. However such a liberal attitude to identity in mathematics looks suspicious from the logical point of view. Claims of existence and uniqueness of mathematical objects satisfying given descriptions (definitions) play an important role in mathematics. Such a claim means that a given description indeed picks out (identifies) an object, not just a property. The standard definition of the *unit* of a given group G (also often called the *identity of* G) is an example[5]. Obviously a claim that such-and-such an object is unique

[5]The unit of a group G is defined as the element $1 \in G$ such that for any element $x \in G$(including 1 itself) $1 * x = x * 1 = x$, where $*$ is the group operation. The existence of 1 is guaranteed by definition but its uniqueness is proved. Suppose $1'$ is another element of the group satisfying the same condition : $1 * x{=}x * 1' = x$. Then

makes sense only if its identity conditions are fixed. But as we have seen they may in fact be very loose. It is clear that 3 is the only natural number bigger than 2 and smaller than 4 but it is not clear that 3 indeed refers to an unique object. But how can mathematics hang together as a body of knowledge if it apparently does not meet Quine's "no entity without identity" requirement ?

There are several ways to approach this problem. I now explore them.

3 Types and Tokens

The remedy, which readily comes to mind on the part of anyone familiar with contemporary Analytic metaphysics, is that of the type/token distinction. Consider another example, which prima facie looks very like the above mathematical cases. There are 26 letters in the English alphabet, and the letter a is one of them. In the last phrase the letter a is referred to as a particular thing, namely a particular letter of the alphabet. But in this phrase itself there are five such things. Hence the letter a is not a particular thing. The standard way of dissolving this puzzle is to say that here we have one a-type and five a-tokens. In explaining the distinction, one starts from tokens : an a-token is a piece of paper with typographic pigment, or another material object (e.g. a piece of printer's type) representing the letter a. Obviously a-tokens are many. The second step is to explain what the a-type is. Intuitively it is what all and only a-tokens share in common (typically a certain shape). To explain the notion of type better than this is not an easy task, and it involves old and hard metaphysical questions as well as complicated logical problems, which I shall not enter into here. Let me show instead that the type/token distinction doesn't fix the problem of identity of mathematics anyway : whatever mathematical types might be they do not correspond to well-distinguishable tokens.

The natural number **3** indeed looks like a type but the 3s, which we find in the series 3,3,... or in the formula 3+3 do not look like tokens from the viewpoint of standard examples (like particular chairs). For formula 3+3 may be applied to many different situations : one might add 3 chairs to 3 chairs, 3 points to 3 points, or even (taking a liberal attitude) 3 chairs to 3 points[6]. Arguably such application amounts to instantiation of both

taking first $x = 1$, and then $x = 1'$ we have $1' * 1 = 1 * 1' = 1 = 1'$. This argument justifies the use of the definite article in the expression "*the* unit of G".

[6] The last example shows that the typification certainly matters here but this does not change the argument.

3s (in formula 3+3) by certain sets of objects. That is certainly not how good tokens behave : the fact that types can be instantiated but tokens cannot is essential ; if we allow for the instantiation of tokens by other tokens we either lose the type/token distinction or must provide it with a new relational sense (which looks like an interesting project but I cannot pursue it here).

The case of points (or more structured geometrical figures like triangles) at first sight looks more promising. Apparently points are well-distinguishable tokens of the same type. Unlike the case of numbers it is common in mathematics to denote different point-tokens by different labels such as A and B. However this works only until coincident points are taken into consideration. For in the case of coincident points we cannot distinguish a singular point-token from a "stock" of point-tokens. It is tempting in this case to think of the stock of points as a "place" occupied by a family of singular point-tokens. But this again involves a reiteration of the type/token distinction on another level as in the case of 3-tokens. Point-locations initially considered as tokens can themselves be instantiated by second-order tokens stocked there. Once again this destroys the usual distinction between point-tokens and the point-type. It is a condition of acting as a (classical) token that the object so acting have determinate identity conditions - as concrete symbols like printed numerals do. But our hypothetical number- and point-tokens do not meet this condition. So the type/token distinction (at least in its usual form) does not help us to handle the identity issue in mathematics. (This also makes me doubt how well it works outside mathematics.)

4 Frege and Russell on The Identity of Natural Numbers

Frege considered it a principal task of his logical reform of arithmetic to provide absolutely determinate identity conditions for the objects of that science, i.e. for numbers. Referring to the contemporary situation in this discipline he writes in the Introduction to [Frege 1884] :

"How I propose to improve upon it can be no more than indicated in the present work. With numbers . . . it is a matter of fixing the sense of an identity." (quoted by [Frege 1964], p. Xe)

Frege makes the following important assumption : identity is a general logical concept, which is not specific to mathematics :

"It is not only among numbers that the relationship of identity is found. From which it seems to follow that we ought not to define it specially for the case of numbers. We should expect the concept of identity to have been fixed first, and that then from it together with the concept of number it must be possible to deduce when numbers are identical with one another, without there being need for this purpose of a special definition of numerical identity as well." (quoted by [Frege 1964], p.74e)

In a different place Frege says clearly that this concept of identity is absolutely stable across all possible domains and contexts :

"Identity is a relation given to us in such a specific form that it is inconceivable that various forms of it should occur" ([Frege 1903] ; quoted by [Frege 1962, 254])

Frege's definition of natural number, as modified in [Russell 1903] later became standard[7]. I present it here informally in Russell's simplified version. Intuitively the number 3 is what all collections consisting of three members (trios) share in common. Now instead of looking for a common form, essence or type of trios let us simply consider all such things together. According to Frege and Russell the collection (class, set[8]) of all trios *just is the number* 3. Similarly for other numbers. Isn't this construction circular ? Frege and Russell provide the following argument which they claim allows us to avoid circularity here : given two different collections we may learn whether or not they have the same number of members without knowing this number and even without the notion of number itself. It is sufficient to find a one-one correspondence between members of two given collections. If there is such a correspondence, the two collections comprise the same number of members, or to avoid any reference to numbers we can say that the two collections are *equivalent*. I shall follow current usage in calling this equivalence *Humean*[9]. Now we check that this relation is indeed an equivalence in the usual sense, and define natural numbers as equivalence classes under this relation.

This definition reduces the question of identity of numbers to that of identity of classes. This latter question is settled through the axiomatization of set theory in a logical calculus with identity. Thus Frege's project is realized : it has been seen how the logical concept of identity applies to numbers. However a closer look reveals some problems which call the success of Frege's project into question.

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[7]See, for example [Fraenkel 1966, 10].

[8]Following [Russell 1903] I use here words *class, collection*, and *set* interchangeably ignoring their technical meanings if any. The standard distinction between sets and classes is discussed in the next section.

[9][Hume 1978], book 1, part 3, sect. 1.

5 Logical Identity at Work

In an axiomatic setting "identities" in Quine's sense (that is, identity conditions) of mathematical objects are provided by an axiom schema of the form

$$\forall x \forall y (x = y \Leftrightarrow \_\_) \qquad \text{(IS)}$$

called in [Keranen 2001] the *Identity Schema* (IS). A paradigmatic example of IS is the Extensionality Principle (EP) for classes, according to which "two" classes are the same if and only if they consist of the same members, or in symbols :

$$\forall x \forall y (x = y \Leftrightarrow \forall z (z \in x \Leftrightarrow z \in y)) \qquad \text{(EP)}$$

To see how EP helps to dispense with "mathematical doubles" consider this example which is on a par with the examples of equal natural numbers and of coinciding points given above. Namely, consider the Cartesian square A^2 of a given class A, that is, the class of all ordered pairs (a, b) where a, b are members of A. In particular A^2 contains pair (a, a). Do we have the same a "taken twice" here, or two equal but still different "copies" of a, or something else ? Now EP provides a definite answer, which rules out the colloquial talk about "copies" and "repetitions". To see this define ordered pair (a, b) as follows :

$$(a, b) =_{\text{def}} \{\{a\}, \{a, b\}\}$$

where $\{x, y, ..\}$ stands for the class having exactly x, y, .. as its members. The rationale behind this definition is this : EP implies that

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} \text{ if and only if } a = c \text{ and } b = d$$

so our ordered pairs are "extensional". Then apply this definition to pair (a, a) and use EP again :

$$(a, a) =_{\text{def}} \{\{a\}, \{a, a\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}$$

Thus the "doubling" of a is explained away.

The set-theoretic formalism used in this example doesn't support the usual intuition about copying. This creates a tension between the formalism and the intuition, which is hardly helpful for doing mathematics. Assuming that the formalism is correct one should rather abandon the intuition about copying and develop another intuition supporting this formalism. Alternatively the intuition of copying could be kept and supported by a different formalism. The former option is problematic, on the one hand, because the intuition of copying is virtually ubiquitous in today's mathematics, and on the other hand, because the difference between an object a, the class $\{a\}$ (singleton), and class $\{\{a\}\}$ ("double singleton") has a very poor intuitive appeal if any. The latter option is also problematic because no formalism justifying the idea of copying of mathematical objects is presently known. (The two other options — doing mathematics purely intuitively and purely formally — I don't consider as viable.)

A tentative application of the same method to a more general situation, in particular for spelling out the Russell's definition of natural number as an equivalence class of classes brings more serious problems. Remind the idea : one considers *class F of all finite classes*, then splits its up into equivalence classes by Humean relation, and finally identifies the equivalence classes with natural numbers. The problem is that reasoning with classes like F easily leads to contradictions as Russell's famous paradox shows. I shall not trace here the history of struggling against this problem but only mention what is today commonly viewed as a reasonable solution[10]. One distinguishes between well-behaved classes called *sets* and ill-behaved classes called *proper classes*. Sets are subject of a system of restrictive axioms (like ZF) safeguarding them from the known paradoxes ; proper classes are also allowed but classes of proper classes are not. This latter restriction (which doesn't apply to sets) allows for avoiding the known paradoxes and suggests the following informal explanation of the distinction : unlike sets proper classes are *overcomprehensive*, that is, "too big" to behave properly.

I believe that the idea that the bad behavior of proper classes is caused by their "size" is just wrong. I think that the notion of "class of all sets" and other similar notions are ill-formed in a different sense : elements of such collections are not properly individuated, hence such collections are not extensional, hence they are not classes. An evidence for this is the following : unlike sets proper classes don't have any definite cardinality, so strictly speaking they don't have any definite size at all. In other words elements of proper classes cannot be "counted" even in the generalized sense of the term in which elements of any set — and not only of "countable" set (in the usual technical sense) — *can* be counted by the corresponding cardinal. The idea that given a predicate P one also gets for free uniquely defined class EXT_P (called the *extension* of P) of "all" individuals x such that $P(x)$, in my view, cannot be justified. Recall that an analogous claim for sets (the Separation axiom) is much weaker. Given a set S and a predicate P such that for any element x

[10] See [Bernays 1958]

of S $P(x)$ is either true or false this axiom guaranties the existence of subset T of S such that $P(x)$ holds for every x from T. So the Separation axiom allows for distinguishing between individuals (elements of S) which are previously given while its counterpart for proper classes is supposed to bring such individuals about. Even naive examples show that this doesn't work automatically. Think about the property of being a human. The expression "all humans" is usually taken to designate the class of all presently living humans. This is, of course, a well-defined finite set. However one might wish also to take all (or some) previous and future generations of humans into the account. Whether this is possible or not it is clear that the accounting for "all humans" is an empirical matter : we can imagine a possible world in which there is no humans just like in ours there is no unicorns. In logic and mathematics we need a general scheme applicable to different empirical situations. Hence the idea to collect all x such that $P(x)$ living in all possible worlds[11] in one big proper class EXT_P . But this is a misleading idea, in my view. For it doesn't take it into account that any such x must be an individual to begin with, so something like Keranen's *Identity Schema* or another kind of individuating mechanism is required in every particular case. In different worlds (different theoretical settings) such mechanisms can be different. So as a general scheme appropriate for mathematics and logic we should rather conceive of the extension of a given predicate P as function $EXT_P(W)$ depending on possible world W (in the simplest case — *i.e.*, on a given "universal" set S). In other words the extension of a given predicate should be thought of as variable (except particular cases when the extension is provably constant, in particular, empty). I cannot develop this point further in this paper but only remark that the suggested view justifies the traditional strict distinction between concepts and their extensions.

Since the "class of all finite classes" F is a proper class but not a set Russell's definition of natural number doesn't go through (using this definition one couldn't ever talk about classes of numbers). The standard replacement is this : one first defines *ordinal* numbers as particular sets constructed by means of ZF or another similar theory, and then identifies *cardinal* numbers with particular ordinal numbers. In the finite case every ordinal number is also a cardinal number, so the difference between ordinals and cardinals can be neglected. Thus every natural number n is identified (by definition) with a particular set. Then one says that finite set s "has n elements" (or "has cardinality n") when sets s and n are

[11]"All possible worlds" is another example of ill-defined class.

equivalent (in the sense of the Humean equivalence) [12].

This standard definition differs strikingly from what Frege looked for in [Frege 1884]. To define the number 5, for example, as $\{\{\{\{\{\emptyset\}\}\}\}\}$ or alternatively as $\{\emptyset, \{\emptyset, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\}\}$ is not very unlike defining 5 as the set of fingers at the right hand of the Pope. Definitions of this sort certainly cannot be seen as realizing Frege's intention to grasp the meaning of n as a common feature of all sets having n elements. Since this intention has been given up the fact that theories like ZF allow for different choices of the definition of natural number (like the two just mentioned) is hardly surprising[13].

Thus in spite of the fact that Frege's project of providing mathematics with solid logical foundations triggered very important mathematical and philosophical developments in 20$^{\text{th}}$ century, the initial goals of this project has not been achieved. I shall not go here for general philosophical pros and contras Frege's basic assumption of the universal character and the uniqueness of the identity concept. But I shall challenge this assumption presenting some alternative views, including those associated with some recent mathematical developments. Before doing this let me mention another Frege's idea relevant to the present discussion.

6 Definitions by Abstraction

To pursue his project of reducing the various informal meanings of "the same" in mathematics to a standard notion of identity captured in a universal logic Frege proposed the method of "definition by abstraction". In [Frege 1884] the author gives the following example of such definition :

"The judgment "line a is parallel to line b", or, using symbols $a//b$, *can be taken as identity.* If we do this, we obtain the concept of direction, and say : "the direction of line a is identical with the direction of line b". Thus we replace the symbol $//$ by the more generic symbol $=$, through removing what is specific in the content of the former and dividing it between a and b." (quoted by [Frege 1964], p. 74$^{\text{e}}$, italic mine)

Notice that the procedure as described here by Frege involves a change of notation : in the formula a=b the symbols a, b no longer stand for lines but denote the same direction. Calling this formal procedure *definition by abstraction* Frege suggests its interpretation. The idea is that the procedure picks out a property common to all members of a given equivalence class. As our earlier quotations from [Frege 1884] clearly

[12] For details see [Bernays 1958] or any modern textbook on the Set theory.
[13] [Benaceraff 1965]

show, in treating an equivalence E "as identity" Frege does not mean to replace identity by something else. He aims at the exact opposite : to introduce identity where mathematicians usually use only equivalencies.

Definition by abstraction is problematic from the logical point of view[14] (Frege himself gives it up after a thorough analysis). But I want to stress a different point. Even if definition by abstraction were justified logically it would not provide what a mathematician normally looks for. Frege's "direction" (not to be confused with orientation !) is hardly an interesting mathematical notion ; this concept might play at most an auxiliary role in geometry. The idea of a *family* of parallel lines does the same job as Frege's abstract direction but is more convenient and more intuitive[15]. Similarly it is more convenient to think of a natural number as a family of equal "doubles" rather than a unique abstract object. Such abstract numbers would be much like Plato's *ideal numbers* as distinct from the usal mathematical numbers. Let me shortly present a reconstruction of Plato's view on mathematics which gives an interesting alternative to the Frege's view.

7 Plato

The usual talk about "copies" of mathematical objects (see section **5** above) carries echoes from Plato. A glance at Plato's philosophy of mathematics[16] shows some features which might be attractive for a mathematician resistant to the logical regimentation of talk of identity in different contexts proposed by Frege and Russell. If I understand Plato correctly, according to him identity applies only to the immutable *ideas*, and only ideas *exist.* (So Plato's view in this respect is in accord with Quine's dictum about "no entity without identity".) Material things don't exist but *become* (they change, come into and go out of being) and hence have no proper identities : this is another possible way out of the Paradox of Change. Mathematical things occupy an intermediate position between material stuff and ideas : they involve a weaker sort of becoming and a softer form of identity. In the case of numbers such "soft identity" is equality. Things in the three layers of Plato's ontology are partially ordered by "distorted copying" where ideas are the maximal

[14] See [Scholz & Schweitzer 1935] for a historical survey and further references.

[15] In section **13** below I shall show that *family* in this context may mean something else than just a class.

[16] Not to be confused with "Mathematical Platonism" in the sense of [Balaguer 1998] and many others, which has little if anything to do with historical Plato. For an introduction to Plato's philosophy of mathematics see [Pritchard 1995].

elements, mathematical objects are distorted copies of ideas, and material objects are distorted copies of mathematical objects (and hence also of ideas). The distortion of self-identical ideal numbers amounts to their replacement by families of equal mathematical numbers. For example, there is a unique ideal number **3** and an indefinite number of its equal mathematical copies. To put it in the current jargon numbers in mathematics are defined up to equality but not up to identity[17]. I cannot provide here a full justification of this reading of Plato and give only the following hints referring to my [Rodin 2003a] for a systematic treatment. There are multiple passages in Plato where he speaks of "X itself", "X (thought of) through itself" ($\kappa\alpha\vartheta'\alpha\nu\tau\omega$) and "Idea of X" interchangeably or explains the latter through the former. For example in *Symposium* (210-211) Plato does this with the notion of Beauty, and in *Phedon* (96-103) with number 2[18]. I interpret these passages in the sense, which seems me straightforward : the notion of "being identical to itself" applies to ideas but not to material things, nor to mathematical objects. To see that Plato's "idea of 2" is indeed something else than mathematical number see last chapters of Aristotle's *Metaphysics* where the author criticizes the "Unwritten doctrine" developed by Plato in the later period of his life [Findlay 1974]. Here the distinction between ideal and mathematical numbers is made explicit. Aristotle stresses the fact that each ideal number is unique while their mathematical copies are many (*Met.* 987b) and the fact that ideal numbers are not a subject of arithmetical operations (*Met.* 1081a–1082b).

Thus unlike Frege Plato does not suppose that the notion of identity applies to whatever there is (or whatever occurs) indiscriminately. Instead Plato thinks of identity as a specific property of things he calls *ideas* and points to the fact that in mathematics the identity requirement is relaxed. In what follows I shall show that this Platonic insight is particularly appealing in the context of our contemporary Category-theoretic mathematics.

[17] So Plato hints at the following division of labor : mathematicians work on equalities whilst philosophers take care of identities. This sounds like a veritable description of what mathematicians and philosophers like Frege or myself are doing for centuries !

[18] In this dialog Socrates rejects the view that 2 is essentially the sum of two units pointing to the fact that 2 can be equally obtained through division of a given unit into two halves. Since each of the two operations is the reverse of the other none of them can be viewed as bringing 2 about. So one needs first to think of 2 "as it is" independently of operations of this sort.

8 Relative Identity

A more recent alternative to Frege's view on identity is given by the theory of Relative Identity due to Geach ([Geach 1972], ch.7). Remarkably this theory is motivated by the same sort of mathematical examples as Frege's definition by abstraction. Like Frege Geach seeks to give a logical sense to mathematical talk "up to" a given equivalence E through replacing E by identity but unlike Frege he purports, in doing so, to avoid the introduction of new abstract objects (which in his view causes unnecessary ontological inflation). The price for the ontological parsimony is Geach's repudiation of Frege's principle of a unique and absolute identity for the objects in the domain over which quantified variables range. According to Geach things can be same in one way while differing in others. For example two printed letters *aa* are *same as a type* but different *as tokens*. In Geach's view this distinction does not commit us to a-tokens and a-types as entities but presents two different ways of describing the same reality. The unspecified (or "absolute" in Geach's terminology) notion of identity so important for Frege is in Geach's view incoherent[19].

Geach's proposal appears to account better for the way the notion of identity is employed in mathematics. It meshes particularly well with how the notion of identity is usually understood in Category theory :

"In a category, two objects can be "the same in a way" while still being different." ([Baez & Dolan 1998], p.7)

But from the logical point of view the notion of relative identity remains highly controversial. Let x, y be identical in one way but not in another, or in symbols : $\mathbf{Id}(x, y) \& \neg \mathbf{Id}'(x, y)$. The intended interpretation assumes that x in the left part of the formula and x in the right part have the *same* referent, where this last (italicized) *same* apparently expresses absolute not relative identity. So talk of relative identity arguably smuggles in the usual absolute notion of identity anyway. If so, there seems good reason to take a standard line and reserve the term "identity" for absolute identity.

We see that Plato, Frege and Geach propose three different views of identity in mathematics. Plato notes that the sense of "the same" as applied to mathematical objects and to the *ideas* is different : properly speaking, sameness (identity) applies only to ideas while in mathematics sameness means equality or some other equivalence relation. Although Plato certainly recognizes essential links between mathematical objects

[19] For recent discussion see [Deutsch 2002]

and Ideas (recall the "ideal numbers") he keeps the two domains apart. Unlike Plato Frege supposes that identity is a purely logical and domain-independent notion, which mathematicians must rely upon in order to talk about the sameness or difference of mathematical objects, or any other kind at all. Geach's proposal has the opposite aim : to provide a logical justification for the way of thinking about the (relativized) notions of sameness and difference which he takes to be usual in mathematical contexts and then extend it to contexts outside mathematics :

"Any equivalence relation ... can be used to specify a criterion of relative identity. The procedure is common enough in mathematics : e.g. there is a certain equivalence relation between ordered pairs of integers by virtue of which we may say that x and y though distinct ordered pairs, are one and the same rational number. The absolute identity theorist regards this procedure as unrigorous but on a relative identity view it is fully rigorous." ([Geach 1972], p.249)

Let me now present a different view on identity suggested by mathematics.

9 Relations versus Transformations

The replacement of the equivalence xEy by the identity $x = y$ proposed by Frege allows for a stronger interpretation than Frege gave in his account of abstraction. Namely, E can be interpreted as a *reversible transformation*, which turns x into y and vice versa, and the identity $=$ as identity *through* this transformation. In the case of congruence the transformation is (Euclidean) *motion :* y is the *same* object x but subject to translation and/or rotation in Euclidean space. Here x and y are said to be the *same* in the same sense in which, for example, an adult yesterday and today is the same person. So we think here of a given triangle in much the way we think of a *substantial continuant* — as an entity capable of changing its states and/or positions. Such a "substantialist" interpretation works also for Frege's example of parallel lines[20].

The substantialist reinterpretation of mathematical relations may look like an exercise in old-fashioned metaphysics but it appears surprisingly fruitful from the mathematical point of view. Given an equivalence xEy there are, generally speaking, *many* distinguishable transformations turning x into y while xEy only says that one such transformation exists. So here the underlying naive metaphysics matters mathematically.

[20]For a modern account of the notion of substance and of identity through change see [Wiggins 1980]

The difference becomes particularly evident in the case of (global) reversible transformations of a given geometrical space. In the language of relations the existence of such transformations amounts only to the claim that a given space is equivalent to itself. But in fact such transformations contain the most basic information about the corresponding space as it was first recognized by Klein[21].

It is not the notion of a "substantial form" surviving through transformations that is the major issue in the new framework for the study of geometrical structure proposed by Klein. Rather there is something of a different sort, which also remain unchanged through the transformations. That something is the structure(s) or *forms* of the transformations themselves. I refer to the fact that reversible geometrical transformations like Euclidean motions form algebraic *groups* under composition. This fact remains completely hidden from view when one speaks about equivalences as relations. Thus the traditional metaphysics of substance and form fulfills a mathematical need which the new Frege-Russell metaphysics does not — whatever might be said in favor of the latter against the former for philosophical reasons.

Let me next specify some terminology, which will be useful for what follows. We have considered three different ways of thinking of what is involved in operating with an (arbitrary) equivalence relation xEy.

1) Extension Consider equivalence classes formed of those things equivalent under the relation E.
2) Abstraction : Replace the relation xEy by identity $x = y$, and read x, y anew as standing for a (relational) property common to all and only members of the same equivalence class under E.
3) Substantiation. Think of the given relation as a reversible transformation of relata into each other, and read E as identity *through* this transformation.

In the case of Humean relation H one may proceed from 1) to 3) through the following steps. Given certain class of classes x, y, ... equivalent by H

- think of the one-one correspondences between elements of given classes x, y as reversible transformations (isomorphisms) f, g ... turning elements of x into elements of y and conversely (reversibility implies that different elements of x turn into different elements of y and vice versa)[22];

[21] [Klein 1872]

[22] Noticeably such a reading is found already in [Cantor 1895] where he says : "es verwandelt sich dabai M in N" to the effect that elements of a source set M are replaced one-by-one by elements of an equivalent target set N.

- think of x, y as different states of the same underlying substratum X, and think of (auto)morphisms f, g, ... as changes of X ;
- similarly identify all classes equivalent to x and y with X.

A non-trivial fact, which makes mathematical sense of this metaphysical exercise, is that the automorphisms of X form a group called its permutation group or symmetric group. To see better what we gain and what we might lose in switching from relations to transformations consider the following table :

Extensional reading	Substantional reading
we write $x = y$ for "class x is equivalent (isomorphic) to class y"	we write $f : X \longrightarrow X$ or simply f for an isomorphism from a class X to itself (automorphism)
$=$ is an equivalence relation ; this means that :	Automorphisms of X form a group ; this means that :
$=$ is transitive : $x = y$ and $y = z$ implies $x = z$.	given automorphisms f, g there exists a unique automorphism gf resulting from the application of g after f.
$=$ is reflexive : every class x is isomorphic to itself : $x = x$	there exist an identity automorphism 1 such that $1f = f1 = f$ for any f
$=$ is symmetric : if $x = y$ then $y = x$	every atomorphism f has an inverse f^{-1} such that $ff^{-1} = f^{-1}f = 1$

The analogy between the two columns of the table characterizing a conceptual shift between the "language of relation" and the "language of transformations" is, of course, informal and incomplete[23]. Taking a more formal line one can note that everything told above about transformations (and in particular the above definition of group) can be coded into the language of relations : one doesn't need E for it but does need a three-place relation between elements of the group (i.e. transformations). The converse is perhaps less known but not less straightforward : the notion of n-place relation can be defined in terms of transformations (morphisms) in the standard category-theoretic settings[24]. However since it is not immediately clear what these formal remarks bring to our understanding of the conceptual difference between relations and transformations

[23]Remark the lack of the associativity condition in the right column : it is not clear what counterpart it might have on the left side.

[24][Borceux 1994, v.2 p.101].

I leave the analogy as it stands and ask the following question : Does the approach outlined above provide indeed a viable alternative to Frege's project of settling the question of identity in mathematics by external logical means ?

On the one hand, one may claim we have here a new formal concept of identity as the *unit* of a group of transformations, which meshes well with the metaphysical intuition that any changing entity contains a core invariant through changes. But on the other hand, it is not clear whether this group-theoretic identity has anything to do with the logical notion of identity, which was Frege's concern. For one may argue that the unity of a group is just a particular mathematical object which needs identity conditions of its own. Remark that in order to define a group of transformations f, g, $h, \ldots$, and in particular to distinguish its unit 1, one still needs the "usual" equality $=$ which appears every time one writes composition of transformations (an action of group operation) $fg = h$. So one may argue that like everywhere in mathematics it is $=$ but not *1* which functions as identity here.

This is quite a serious objection. Let's see how it can be at least partly met through upgrading the concept of group to that of *category*.

10 Categorification or How to Think Circle ?

Arguably the best way to explain what is circle is to show one (see Fig. 3).

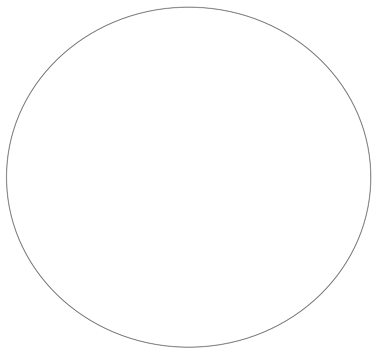

Fig. 3

However this would hardly work unless you have seen some circles before. For otherwise you wouldn't know to which features of the shown picture I'm trying to attract your attention. A way to make this clear is to show *many* circles of different size, color, etc., so your capacity for abstraction would allow you to grasp common features of all these things. (It is also helpful to demonstrate some figures like ovals, which look much like circles but are not, so you could also grasp the difference).

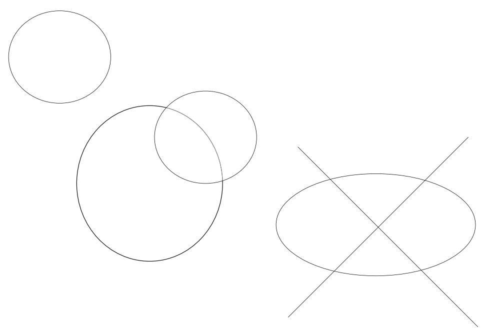

Fig.4

Computer educational media allow for a different option : instead of showing you different circles I might show a moving circle changing its color and size but not its shape (unfortunately I cannot do this in the printed paper).

The two options just mentioned correspond to what I called in the previous section *extension* and *substantiation* of a given concept. Speaking in a more abstract manner, in the former case we have a class of objects instantiating the given concept while in the latter case we have just one object of the given type provided with a group of transformations. Now remark that mathematicians usually need both these ways of representation of their concepts : they need many different objects of any given type to play with (many circles, for examples), and they also need to transform these objects. Just like in the everyday life in mathematics people deal with many changing and moving things of each kind but not just one. But to the contrary to what one might expect from a strict science ambiguities about identity of changing objects are even more common in mathematics than in the everyday life. One should be a philosopher to wonder if yesterday and today I am the same guy or invent examples like *The Ship of Theseus*. But in mathematics like in the world of Ovidius' *Metamorphoses* [Ovidius 2004] such examples are found everywhere. Given two circles C, C' one is always allowed to think of C' as a transformed version of C making no commitment about preservation of identity through the change similar to those we are usually making in the case of pets and persons. We are so accustomed to these ambiguities through the school education that rarely pay any attention to them. I can quite understand Frege's concern about this issue even if I'm not sympathetic with his attempts to improve on it.

From a metaphysical viewpoint taken in the previous section extension and substantiation look like two incompatible options[25] however there is the following obvious way to combine them mathematically. Given a class of (immutable) circles, for example, provide them with transformations (transforming them into each other and to themselves) without collapsing the circles into one. This construction involves an important choice : one should somehow distinguish between self-transformations of any given object and transformations between different objects. But in most cases like in the case of circles on Euclidean plane this choice can be made rather naturally : for example, in the case of circle one may naturally opt for considering only rotations of circles around their centers as self-transformations of the circles. Although any particular choice of this sort is questionable an important advantage of the construction is that it requires to make such choices explicitly.

To complete the definition of *category* of circles C there remains only very few things to say. The transformations (of both kinds) are composed in the usual way (so the associativity of the composition is assumed) but since the transformations involve different objects one should keep track of what is transformed : transformations f, g are composable only if the target (codomain) of f coincides with the source (domain) of g. Finally with every circle A we associate a special self-transformation 1_A called *identity of* A and having the following property : $f1_A = f$ and $1_Ag = g$ for any transformations f, g such that compositions $f1_A$ and 1_Ag exist. Denoting circles by capital letters A, B, C and transformations by arrows we get the following diagram (see Fig. 5) :

[25] Think about the ongoing ontological debate between 3D and 4D ontologies otherwise referred to as *endurantism* and *pendurantism*. For a recent discussion see [McKinnon 2002].

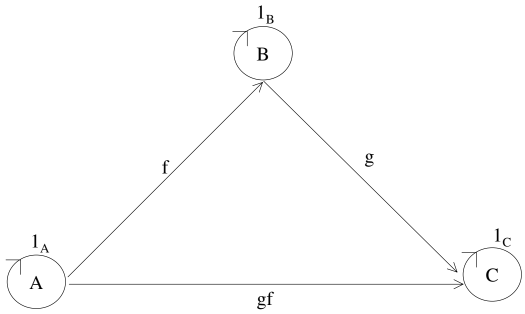

Fig 5

Notice that (to the contrary to what Fig.5 might suggest) given circles A, B there are, generally speaking many transformations (a class of transformations) from A to B. In particular (in addition to 1_A) there is a class of rotations transforming circle A into itself.

In order to get a general definition of category from this example we need only to replace circles by abstract objects and talk about *morphisms* rather then transformations. Thus a category comprises :

- Class of its objects A, B, C, $\ldots$;
- For each ordered pair of objects A, B class of morphisms $f : A \to B, g : A \to B, \ldots$ from A to B; given $f : A \to B$, A is called domain of f and B is called codomain of f;
- Composition gf of morphisms f, g such that the codomain of f equals the domain of g (see the above diagram); the composition is associative : $h(gf) = (hg)f = hgf$;
- Identity morphism 1_A associated with each object A and defined by the condition : for any morphisms f, g, $1_A f = f$ and $g 1_A = g$ (provided the compositions $1_A f, g\, 1_A$ exist).

When in a categorical diagram any arrow $A \to C$ equals to any other arrow between A and C obtained through composition of arrows shown at this diagram (like at Fig.5) the diagram is said to be *commutative*.

Notice that our category of circles C has the following additional property not assumed in the general definition of category : all its morphisms (transformations) are reversible . The reversibility is a basic property of all usual geometrical transformations like motion or scale transformation in virtue of which such transformations form groups. In the category-theoretic terms just introduced the reversibility of transformation (morphism) $f : A \to B$ amounts to existence of transformation (morphism) $g : B \to A$ (called the *reverse* of f) such that $gf = 1_A$ and $fg = 1_B$. In Category theory this property is taken as the definition of *isomorphism*, so isomorphisms are reversible morphisms. A category like $\boldsymbol{C}$ such that all its morphisms are isomorphisms is called *groupoid*. Thinking of objects of a groupoid "up to isomorphism" one gets a group. (So group is a category with only one object such that all its morphisms are isomorphisms.) However such identification causes a lost of information, namely the lost of distinction between morphisms of objects to themselves (automorphisms) and morphisms of objects to other objects. Thus groupoids provide an important counter-example against the widespread belief according to which in categories all isomorphic objects can be always viewed as identical (see the next section).

The full strength of the notion of category is revealed through the case when morphisms between objects are not all reversible, that is, are not all isomorphisms. A basic example is the category of sets having sets as objects and functions between sets as morphisms. Further examples are obtained through equipping sets with various structures like group structure or topological structure. Then morphisms are required to "preserve" or "respect" the corresponding structure : so in the category of groups morphisms are homomorphisms of groups, and in the category of topological spaces morphisms are continuous transformations[26].

Thus the upgrade of the notion of group up to that of category involves two independent steps : (i) introduction of multiple identities (multiple objects) instead of unique identity (unique object), and (ii) allowing for non-reversible morphisms. This upgrade can be shown with the following diagram[27] :

[26] Recall the definitions. Homeomorphism between groups G, G' is function f between underlying sets of G, G' such that if $a * b = c$ in G then $f(a) \circledast f(b) = f(c)$ in G', where $*$ and $\circledast$ are group operations in G and G' correspondingly. Continuous transformation $f : T \to T'$ between two topological spaces is a function between underlying sets of T, T' such that any inverse image of an open in T' is open in T. Notice that the talk of 'preservation of structure' is at least partly misleading because it too easily makes one think about homomorphisms as if they were isomorphisms. Consider the case of group homomorphism f such that $f(a) = f(b) = f(c) = 1_{G'}$: the "structure" of group G is not preserved here in anything like the usual sense of the word but reduced to group unit. The talk of "respect" of structure is less popular but in my view better fits its intended meaning.

[27] In the standard set-theoretical setting monoid is defined as set M provided with a binary operation and unit. Unlike the case of group the existence of inverse elements is not required.

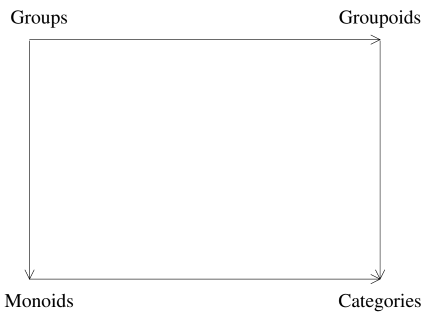

Fig 6

Examples of categories given so far are *concrete* categories. This means that objects of such categories are specified in advance (usually this means that they are construed à la Bourbaki as structured sets), so a category could be seen as a structure over and above a class of its specific objects. However the Category theory allows for a different approach : starting with the general notion of category one specifies its algebraic properties to the effect that the structure of morphisms between objects and their compositions determines properties of these objects. (A specification of a given abstract category amounts to the requirement that certain morphisms exist and certain diagrams commute.) In particular a properly specified abstract category "becomes" the category of sets [Lawvere 1964] in the sense analogous to that, in which logical variables in axiomatic systems like ZF "become" sets under its intended interpretation. I cannot go here for details but mention these facts in order to stress that the idea of "replacement of relations by transformations" is pushed much further forward in Category theory than in Group theory. So the argument according to which transformations unlike relations have nothing to do with logic and with identity in the category-theoretic context doesn't go through, or at least doesn't go through straightforwardly.

11 Identity and Isomorphism

The mathematical notion of category just introduced makes paradoxes of identity of mathematical objects discussed in the beginning of this paper more explicit than usual. Consider the category of (all) groups $\boldsymbol{G}$, for example, and take S_2 (symmetric group with two elements : unit and involution) as an example of group. Outside $\boldsymbol{G}$ one may think about S_2 either as a particular object (*the* symmetric group with two elements) or as *kind* of (isomorphic) objects dependently on a given context just like one does it with numbers, circles and whatnot. However since $\boldsymbol{G}$ is supposed to comprise *all* groups (whatever this might mean) the switching between different senses of "the" cannot any longer remain unnoticed. Similar problems arise in abstract categories. The notion of *terminal object* defined as object having exactly one incoming morphism from each object of a given category (including itself) is a typical example. This definition immediately implies that any two terminal objects are isomorphic, and moreover that there is exactly one isomorphism between any two such objects. In any reasonable context (I don't know about exceptions) terminal objects can be identified "up to unique isomorphism", and this is exactly what people do. This identification cannot be hidden by switching to a new context and should be mentioned explicitly. Having no suitable theory of identity in hands category-theorists often justify their liberal use of the equality sign by remarks like this one taken from [Fourman 1977]. Referring to a formula involving equality the author makes the following reservation :

"Strictly speaking the "canonical" isomorphisms... are necessary (instead of equality — *A.R.*) ... Having realized this it is best, in the interests of clarity, to forget them." (p. 1076)

The fact that isomorphic objects are often (albeit not always) regarded as identical in categorical contexts was used by some philosophers as an evidence supporting the claim that Category theory provides "a framework for mathematical structuralism" (see [Landry & Marquis 2005] for a recent summary of continuing discussion on this issue in *Philosophia Mathematica*). Mathematical structuralism is, roughly, the view according to which the identity up to isomorphism is the only kind of identity available for mathematical objects. This view squares well with what mathematicians say in informal remarks like the following :

"The recursive weakening of the notion of uniqueness, and therefore of the meaning of "the", is fundamental to categorification." ([Baez & Dolan 1998], p.24)

or

"The basic philosophy is simple : *never mistake equivalence for equality*" (*ibid.*, p.46, italic of the authors)

Notice that the "philosophy" suggested by Baez & Dolan here is in accord with my reconstruction of Plato's views given above and exactly the opposite to Frege's attempts aiming to *strengthening* "the meaning of "the"" in mathematics[28]. I shall not discuss here mathematical structuralism and its relationships with Category theory but remark that mathematically speaking the issue is far from being straightforward. Notice that in the standard categorical setting explained above the identity "up to isomorphism" doesn't apply to all morphisms. To define the notion of terminal object and the very notion of isomorphism (as reversible morphism) one needs to know precisely which morphisms are equal and which are not. So equalities in categories cannot be simply dispensed with and replaced by isomorphisms in any obvious way.

Another part of the same problem concerns isomorphism *of categories*. It has been widely observed that although this notion is easily definable it is quite "useless" ([Gelfand & Manin 2003], p.70). Take category $\boldsymbol{G}$ of (all) groups for example. Isomorphic copy $\boldsymbol{G'}$ of $\boldsymbol{G}$ cannot be anything else but the (?) category of groups. But as far as $\boldsymbol{G}$ is supposed to comprise *all* groups (including all isomorphic groups) the talk of isomorphic copies of $\boldsymbol{G}$ comprising all these groups once again doesn't make sense (or even is contradictory is "all" is taken seriously). For this reason equivalence of categories is defined as a weaker relation than isomorphism. To give strict definitions we need the notion of *functor*, which is morphism *between categories* respecting the basic categorical structure in the same sense in which homomorphisms of groups respect the basic group structure. Then isomorphism of categories is defined as usual (as reversible functor). To define the *equivalence* between categories we need also the notion of *natural transformation*, which is morphism *between functors* sharing domain and codomain. A natural isomorphism is reversible natural transformation. Now functor $F : A \rightarrow B$ is called *equivalence* if there exist functor $G : B \rightarrow A$ (called *quasi-inverse* of F) such that GF is *isomorphic* to the identity of A and FG is isomorphic to the identity functor of B[29].

The equivalence of categories so defined preserves isomorphisms in categories but doesn't preserve identities. This suggests the following view : the "real" sameness of objects in a category is isomorphism but not equality and the "real" sameness of categories is their equivalence but not isomorphism. However we need equality and identity morphisms (in particular, identity functors) in order to define these notions. So a more

[28]Since Frege interpreted this replacement of equivalences by equality as abstraction, this gives an interesting possibility to account for abstraction in terms of *decategorification* introduced in [Baez & Dolan 1998] further on.

[29]For details see Gelfand & Manin 2003, ch. 2 or any introductory text in Category theory.

precise analysis is in order before making any philosophical judgement about identity in categories.

12 Equality Relation and Identity Morphisms

Just like a group a category comprises two very different identity-related elements : the "usual" mathematical equality and identity morphisms. (Remind that groups can be viewed as a special case of categories, namely as categories having only one object and such that all their morphisms are isomorphisms.) As I mentioned in the end of section **10** in a category-theoretic context a relevance of the notion of identity morphism to logic cannot be ruled out on a general ground. However in order to claim such relevance we need to be more specific. Let me try to show what is going on with logic in categories without entering into details.

With a suitable category $\boldsymbol{T}$ (noticeably with a topos) one may associate logical calculus L called *internal language* of $\boldsymbol{T}$ to the effect that each formula provable in L corresponds to certain commutative diagram in $\boldsymbol{T}$ (soundness)[30]. $\boldsymbol{T}$ is a semantic for L in the usual sense but $\boldsymbol{T}$ may also represent certain features of L that the usual (Tarskian) semantic doesn't, for example, the truth-values. This gives reason for the reversal of the usual point of view on semantic and syntax and explains the term "internal" : L may be viewed as a secondary structure (or even just a symbolic convention) associated with $\boldsymbol{T}$ and reflecting specific features of $\boldsymbol{T}$ rather than a self-standing syntactic construction waiting for an interpretation. L brings with itself identity predicate $\equiv$ while the construction of $\boldsymbol{T}$ comprises the "external" ("usual") equality $=$ from the outset. The "adjustment" of L to $\boldsymbol{T}$ makes $\equiv$ and $=$ interchangeable. However this doesn't mean that $=$ (or $\equiv$) gets "internalized" in the same sense in which people speak about internalization of truth-values and logical connectives : the internalization of logic in a category amounts to representation (or replacement) of the usual logical syntax by categorical constructions while $=$ is *not* a categorical construction but just the "usual" mathematical equality! Identity morphisms of $\boldsymbol{T}$ are not used for representing $\equiv$. Thus we can see that the standard "internalization of logic in a topos" with an internal language has indeed no bearing on the identity issue. Although the idea to account for identity in categorical terms cannot be ruled out on a general ground the standard

[30] See for example [McLarty 1992].

device of "internal logic" doesn't realize this idea. Let's look for different possibilities.

13 Fibred Categories

The following discussion is based on [Bénabou 1985]. The idea is the following. Recall that categories have been introduced in section **10** as *classes* of a certain kind. Which properties of classes are used in the "naive" Category theory ? Let category $\boldsymbol{C}$ be our "object of study" and category $\boldsymbol{B}$ be our "optical instrument" for studying $\boldsymbol{C}$. $\boldsymbol{B}$ can be thought of as category $\boldsymbol{S}$ of sets however we can also consider different possibilities, in particular abstractly defined toposes. Following Bénabou I shall call objects of $\boldsymbol{B}$ sets (remembering that they could be somewhat different than usual sets) and call classes of morphisms or objects of $\boldsymbol{C}$ *families*. (In what follows *families* will reappear as multiplicities of a different sort than classes.) Now given a set I (an object of $\boldsymbol{B}$) we may master category $\boldsymbol{C}(I)$ called *fiber over* I whose objects and morphisms are families of objects and morphisms of $\boldsymbol{C}$ *indexed* by elements of I, that is, families of the form $X = (X_i)$ and $f = (f_i : X_i \to Y_i)$ where $i \in I$. Bénabou remarks that speaking about categories naively we assume more than this, so we cannot just fix some sufficiently large set I and use it for indexing every time when this is needed. Namely, we also assume the possibility of *re-indexing* : given families $X = (X_i)$, $Y = (Y_j)$ in $\boldsymbol{C}$ where $i \in I$, $j \in J$ and morphism $u : J \to I$ in $\boldsymbol{B}$ we assume that family of objects $X_{u(j)}$ and family of morphisms $f = (f_j : Y_j \to X_{u(j)})$ is uniquely defined and "behaves properly". This allows us to extend $\boldsymbol{C}(I)$ through introducing new category $\boldsymbol{Fam(C)}$ of *families of* $\boldsymbol{C}$ where objects are families of objects of $\boldsymbol{C}$ indexed by different sets and morphisms are pairs of the form (u, f) where u and f are as just described. Morphisms of the form $f = (f_i : X_i \to Y_i)$ we identify with (id_I, f) where id_I is identity morphism of I in $\boldsymbol{B}$. The composition of morphisms in $\boldsymbol{Fam(C)}$ is defined in the obvious way. We equipe the construction with projection functor $\boldsymbol{p}_C$ which sends every family of objects of $\boldsymbol{C}$ to the set by which this family is indexed and every morphism (u, f) between families to morphism u between sets : $\boldsymbol{p}_C : (X_i) \to I$, $(u, f) \to u$.

Now suppose that we know what equality is in $\boldsymbol{Fam(C)}$ and in $\boldsymbol{B}$ but not in $\boldsymbol{C}$. This implies that we cannot think of families (of objects and morphisms of $\boldsymbol{C}$) extensionally as usual. In particular given a family $X = (X_i)$ where $i \in I$ and morphism $u : J \to I$ in $\boldsymbol{B}$ we cannot define another family $Y = (Y_j)$ by saying that $Y_j = X_{u(j)}$ because the latter equality doesn't make sense for us. Nevertheless we can achieve the same effect through requiring certain properties of $\boldsymbol{Fam(C)}$ and $\boldsymbol{p}_C$. What we need for it is to characterize morphism $\varphi_{(u,\,X)} = (u,\,(id_{Yj}))$ in $\boldsymbol{Fam(C)}$ without using equality in $\boldsymbol{C}$; (id_{Yj}) to be the family of identity morphisms of objects $Y_j = (X_{u(j)})$ in $\boldsymbol{C}$. Given $u : J \to I$ and $X = (X_i)$ φ is characterized up to unique isomorphism by the following property (i) : for any morphism $\psi = (v,\,g)$ with codomain X in $\boldsymbol{Fam(C)}$ and any v' such that $v = uv'$ in $\boldsymbol{B}$ there exist in $\boldsymbol{Fam(C)}$ a unique $\psi\prime$ such that $\psi = \phi\psi'$ and $\boldsymbol{p}_C(\psi') = v'$. In addition morphisms of the form $\varphi = (u,\,(id_{Yj}))$ satisfy the *functoriality* conditions (ii) : $\varphi_{(u,\,X)} = id_X$ (identity morphism of family X), and $\varphi_{(uv,\,X)} = \varphi_{(u,\,X)}\varphi_{(v,\,Y)}$ for each $v : K \to J$. Now we use these properties as definition of abstract functor $\boldsymbol{p : F \to B}$ called *fibration over* $\boldsymbol{B}$ (or *fibred category over* $\boldsymbol{B}$) in the case when only the property (i) is taken into account, and called *split fibration over* $\boldsymbol{B}$ in the case when in addition for each pair $(X,\,u :\ J \to \boldsymbol{p}(X))$ one makes a particular choice of $\varphi_{(u,\,X)}$ (called *splitting*) satisfying functoriality conditions (ii). Thus equality in a category can be defined as splitting of fibration over an appropriate base. Noticeably given a fibration its splitting might not exist or be not unique. I refer the reader for further details to [Bénabou 1985].

Bénabou's theory of equality in categories allows for regarding objects and morphisms of a given category as families rather than bold individuals ; these families can be occasionally split into elements through a (split) fibration in different ways dependently of the choice of base. Such splitting is reverse operation with respect to the informal identification of isomorphic objects and morphisms mentioned in the section **11**, and unlike the latter it is performed more rigorously and more "categorically". This reversal is remarkable : it shows that given the definition of equality through split fibration families are no longer thought of as extensional multiplicities, that is, as classes. Recall however that given a fibration $\boldsymbol{p : F \to B}$ categories $\mathbf{F}$, $\boldsymbol{B}$ are construed in the usual way and in particular assume the "usual" equality of morphisms and objects, so the internalized equality relates only to hypothetical category $\boldsymbol{C}$ such that $\boldsymbol{F = Fam(C)}$. This situation is quite analogous to that in the Model theory when a formal theory is interpreted in a semantical structure construed independently of this theory either informally or with the help of a meta-theory. As Bénabou stresses in the end of his paper such "meta-equality" is indispensable "unless you do something different from Category theory". In the end of the following section I shall argue that this "something different" can be a real option (however I shall not pursue it in this paper).

14 Higher Categories

Given objects A, B of category $\boldsymbol{C}$ consider class $\boldsymbol{Hom}(A, B)$ of morphisms f, g, ... of the form $A \to B$. Then turn $\boldsymbol{Hom}(A, B)$ into a new category formally introducing morphisms of the form $\alpha : f \to g$ (that is, morphisms between morphisms of $\boldsymbol{C}$). Do this for all pairs of objects of $\boldsymbol{C}$. Observe that morphisms α can be composed in two different compatible ways shown at the below diagram :

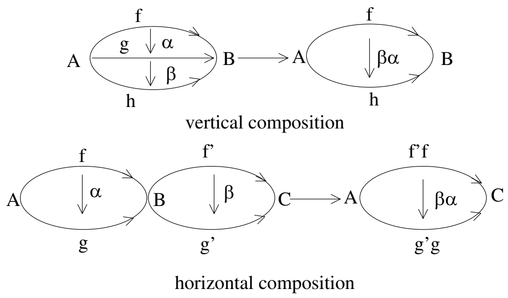

Fig 7

Requiring now natural equational condition to the effect that all morphisms "work properly" (which I shall not list here) we obtain a *2-category*. It comprises objects A, $B\ldots$, morphisms f, g, ... between objects (the same as in $\boldsymbol{C}$) called in this context *1-morphisms*, and "morphisms between morphisms" α, β, ... called *2-morphisms*. An example of 2-category which has been around from the very beginning of Category theory (that is, some 20 years earlier than the abstract notion of 2-category has been introduced in [Ehresmann 1965]) is 2-category ***2-Cat*** having (some or all) categories as objects, functors between these categories as 1-morphisms and natural transformations between the functors as 2-morphisms. Let me now explain what 2-categories have to do with the internalization of identity (equality).

Remark that in a 2-category we have not only the usual composition of 1-morphisms $(f : A \to B)(g : B \to C) = gf : A \to C$ but also functor $\boldsymbol{F} : \boldsymbol{Hom}(A, B) \times \boldsymbol{Hom}(B, C) \to \boldsymbol{Hom}(A, C)$ (provided that in category $\boldsymbol{Hom_C}$ having Hom-categories of $\boldsymbol{C}$ as objects Cartesian product

$\times$ is available [31]). On 2-morphisms this functor acts as their *horizontal* composition (while in Hom-categories 2-morphisms are composed *vertically*). If functors of the form $\boldsymbol{F}$ preserve identities in Hom-categories (2-identities) then equalities in $\boldsymbol{C}$ may be omitted without any lost. This means that we don't even need to define $\boldsymbol{C}$ as a category but may think of it as a class of "objects" and "morphisms" between these objects, and then define composition of these morphisms "from above" through functors like $\boldsymbol{F}$. In this case one may speak indeed about "replacement of relations by morphisms" : 2-identites from Hom-categories make in $\boldsymbol{C}$ the job of equalities. The situation here is quite analogous to one we've seen in fibred categories : at the top "meta-" level of construction (namely in Hom-categories and in the category $\boldsymbol{Hom_C}$ of the Hom-categories) one uses the "god-given" equality but at the bottom level equalities are got rid of.

An apparent difference between the two approaches is this : in higher categories the notion of class is used at all levels including the lowest one while in fibred categories this notion is used only at the "meta-" level and at the lower "internalized" level classes are replaced by non-extensional *families*. But is the assumption that objects of 2-category form a class necessary ? Prima facie it is the case. For in order to compose 2-morphisms α, β in a Hom-category (that is, vertically) we need to check that domain of β equals codomain of α. So 1-morphisms (objects

[31]Given objects A, B in a category their product $A \times B$ is defined up to unique isomorphism by the following (universal) property : given any object X with morphisms $X \to A$, $X \to B$ there exist unique morphism $X \to A \times B$ such that the following diagram commutes :

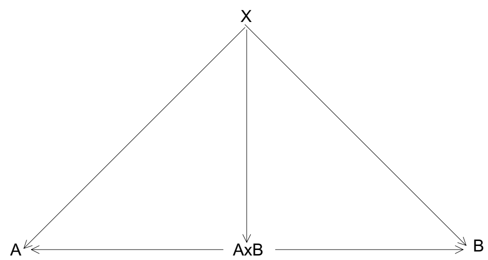

Fig. 8

Check that this works as usual when A, B, X, $A \times B$ are sets.

of Hom-categories) should form a class of well-distinguishable elements (provided with a notion of equality allowing for distinguishing them). Notice that if we take this view then the internalization of equality in $\boldsymbol{C}$ just described will be only partial : it will apply to equalities of the form $gf = h$ but not to equalities of the form $f = f$. However it is easy to get around this point through identification of 1-morphisms with 2-identities, so all needed equational conditions could be written in terms of 2-morphisms. Then one may claim that in $\boldsymbol{C}$ "there is no equality", and hence its elements don't need to form a class.

It should be noted that the common interest of people working in higher Category theory is not internalization of equality as such but *weakening* of equality, that is, finding a rigorous way of "replacing" equalities with certain isomorphisms. This approach is quite natural in the given context since the requirement that functors of the form $\boldsymbol{F}$ preserve all 2-identities is unreasonably strong (see section **11**). Since we are no longer obliged to think of $\boldsymbol{C}$ as a category in the usual sense we get a room for playing. Instead of imposing on Hom-categories and on $\boldsymbol{Hom_C}$ equational conditions implied by the assumption that $\boldsymbol{C}$ is a category we can use weakened conditions which don't imply that 2-isomorphisms replacing equalities in $\boldsymbol{C}$ are identities. Such weak 2-categories have been first introduced in [Bénabou 1967] under the name of *bicategories*. In bicategories (i) the usual associativity law $h(gf) = (hg)f$ in $\boldsymbol{C}$ is replaced by the requirement of existence of *associativity (2-)isomorphism* $a : h(gf) \rightarrow (hg)f$ (eventually called *associator* by other authors), and (ii) the usual axiom of identity $f1_A = f$ and $1_B f = f$ for $f : A \rightarrow B$ is replaced by the requirement of existence of *unit (2-) isomorphisms* $l : f1_A \rightarrow f$ and $r : 1_B f \rightarrow f$. These isomorphisms are subjects of equational conditions called *coherence laws*, which I shall not list here[32].

The notion of 2-category allows for a straightforward geometrical analogy : think of objects as points, of 1-morphisms as oriented lines, and of 2-morphisms as oriented surfaces bounded by the lines. Whether we use this analogy (which is more profound than it might appear at the first glance) or not the notion of 2-category calls for the inductive generalization to the notion of n-category for arbitrary n and further to ω-category (leaving bigger ordinals apart). The *strict* (meaning *non-weakened*) versions of the notions of n- and ω-category look unproblematic : the *enrichment* of a given category $\boldsymbol{C}$ bringing the notion of 2-morphism explained in the beginning of this section can be easily reformulated as an inductive step bringing the notion of k-morphism provided with k different kinds of composition ([Leinster 2001], p.8). However the

[32] See [Bénabou 1967] or [Leinster 2001], p.9

notions of *weak* n- and ω-categories which are more interesting (both purely mathematically and for applications) are much less obvious. There are many alternative definitions of weak n- and ω-categories around ; ten of them are presented in [Leinster 2001]. A somewhat different approach based on introduction of a new kind of morphism rather than relaxing axioms is presented in [Kock 2005]. A specific obstacle for putting these things into order, which is not irrelevant to the issue discussed in this paper, is this : it is not clear which notion of equivalence one should apply to answer the question whether two given definitions of weak n-category are equivalent or not. There are obvious reasons to think that the suitable notion of equivalence should be n-categorical itself but this makes the reasoning circular. It seems that new conceptual inventions and not only technical developments are still wanted in this field.

The construction of strict n- (ω) category is transparent because it is built in the strict upward order. However as we have seen in the case of weak n-categories the opposite downward order of construction gets involved. In fact the idea of the downward construction is basic for Category theory but not specific for its higher-dimensional branch[33]. At the same time the core upward inductive construction of n-category (strict or weak) starting with a class of objects, morphisms between the objects, morphisms between these morphisms, etc., remains present in all alternative definitions of this notion. Although the co-presence of these two opposite orders is ubiquitous in mathematics (think about Euclidean triangles for example[34]) it seems that in the case of n-categories the two orders fail to match each other. As far as we start to build a n-category from *classes* of objects and morphisms we tacitly assume a lot about their identities, so the sense of further "weakening of identity" through higher-dimensional conditions becomes unclear. For this and other reasons it seems interesting to figure out a categorical (or categorical-like) construction which would avoid the classical upward geometrical concept-building leaving more room for the downward approach.

From an epistemological viewpoint the case of n-categories with $n > 2$ is interesting because it no longer allows for thinking of different levels of the construction along the distinction between the "object level" and the "meta-level". When n becomes big the reiteration of "meta-" becomes

[33]The talk about dimensions refers to the geometrical aspect of the notion of n-category modestly mentioned above in the main text as an "analogy".

[34]I mean the following. To *construct* a triangle one proceeds in the bottom-up order : one takes three points and connects them by straight lines. But to *define* the notion of triangle, say, as a 3-angled polygon one proceeds in the opposite sense assuming the notion of polygon first.

pointless and with $n = \omega$ it becomes senseless. So we cannot take refuge at the "meta-level" but must revise our understanding of identity from the outset. Let me demostrate this with the following simple example.

Remind the "partial categorification" of an isomorphism class of sets which we have achieved in section **9** : given such a class we considered all isomorphisms between its member-sets, then identified the sets and some (but not all) isomorphisms and got a symmetric group. Let it be finite symmetric group S_N for simplicity. Now we can see that a more profound alternative to Fregean approach requires turning all sets into a category. Let us however for the sake of the example improve on S_N in a different way. Namely, let us categorify it further taking into account isomorphisms of S_N, that is, group $Aut(S_N)$ of *automorphisms* of S_N. Here we can remark something interesting. Except the trivial cases $N = 1$ and $N = 2$ when there exist only the identity authomorphisms, and except the "pathological" case $N = 6$ we have $Aut(S_N) = S_N$[35]. (This latter equality sign can be read as the isomorphism relation. Considering isomorphisms in question explicitly we get S_N back !) So taking into consideration automorphisms of higher order $(Aut(Aut(S_N)) = Aut^2(S_N)$ and so on) brings nothing new : we have $Aut^n(S_N) = S_N$ for all n and all $N \neq 1, 2, 6$. Remark that S_N equipped with $Aut^k(S_N), k = 1, 2, \ldots n$ is a very simple albeit not completely trivial example of strict n-category. The property of symmetric groups just mentioned is a case of what Baez and Dolan [Baez & Dolan 1998] call *stabilization* in n-categories (p.13).

Now we can fix some n, assume the "usual" equality only in $Aut^n(S_N)$ and for $k < n$ write the group operation as $ab \to c$ (instead of $ab = c$) claiming that this operation is determined up to isomorphism by the fact that $Aut^k(S_N)$ (in particular, $Aut^0(S_N) = S_N$) is isomorphic to $Aut^n(S_N)$. The elementary character of this construction makes its downward determination just as easy as the upward one. However it also makes the whole idea of the reiteration of "levels" plainly redundant : given $Aut(S_N) = S_N)$ our "n-group" is just S_N "up to itself" ! This suggests a different move : instead of describing the group operation through equational conditions reverse the optic and reconstruct a notion of equality (identity) on the basis of the operation.

What we can expect to get in this way is "identity up to S_N" rather than a universal identity concept suitable for all mathematical and logical needs. Remark however that identity "up to symmetric group of isomorphisms" applies to such a fundamental mathematical object as a class. Thus this particular kind of identity has a very general significance

[35] [Kurosh 1960, 92]

in mathematics and logic[36].

15 Conclusion

It might be argued that before a new account of identity is well established in mathematics it is premature to start any philosophical discussion about it. I don't think so. I dare to think that not only philosophers can find a lot of interesting stuff relevant to their subject in the contemporary mathematics but that mathematicians too can be motivated by new philosophy in their work, moreover if it concerns such a traditional philosophical issue as identity. This is how things worked for centuries (including the heroic time of debates on foundations of mathematics at the edge of 19th and 20th centuries), and I cannot see any reason why in 21st t century they should be different. Although philosophical motivations could and in many contexts certainly *should* be swept out of ready-made mathematical theories philosophical reasons often play an important role in bringing new mathematical theories about. That is why the fact that the issue of identity in categorical mathematics is not yet well settled in mathematics gives me reason to discuss it right now rather than to the opposite.

Paraphrasing [Quine 1966] we can say : one man's paradox is another man's definition. Considering the invariance through change as a basic feature (if not a definition) of identity we may avoid the *Paradox of Change* but the price will be the lost of primitive and universal character of the identity concept. In my view this price must be paid anyway. People use the word "same" in many different context-dependent senses in everyday talks as well as in scientific discourses. What physicists exactly mean talking of the "same experiment", "same observation", "same effect", "same model", "same theory", "same event" or "same particle" ? The type/token distinction doesn't give us all needed answers. In biology and social sciences things become even more complicated. A philosophical approach to the issue requires first of all distinguishing, specification and theoretic systematization of different senses of the "same" rather than picking one of them, stipulating it as basic and explaining away others. I don't think that Frege is right assuming that the notion of identity is unique and simple. I think that Plato was more to the point

[36] I realize the risk of drawing any general conclusion concerning n-categories on the basis of the example of "symmetric n-group" and suggest the reader to consider this example on its own rights. Anyway a n-categorical view on symmetric groups seems me interesting.

noticing that nothing like the "absolute" identity applies to physical and mathematical matters, so he had to stipulate a special realm of eternal Ideas where it might work. But unlike Plato I am rather interested in identities, which might work in mathematics, physics and other sciences.

We have seen that the issue of identity has been crucial in the development of programs of unification of mathematics since the end of 19th century. Frege's attempts to "fix the identity" of natural numbers as continued by Russell shaped the mainstream philosophy of mathematics (although hardly the mainstream mathematics) in the XX-th century. The issue of identity remains central in the current program of categorification of mathematics. The fact that the working concept of identity in mathematics is weak and diversified noticed by Plato has been interpreted by Frege, Russell, and their followers as an evidence of the lack of rigor in this discipline, and they tried to fix the problem through introduction of a universal logical notion of identity. Categorification, in contrast, purports to further weakening and diversification of identity revealing genuinely new mathematics in doing so. Interestingly categorification revives certain philosophical ideas, which during the XX-th century remained marginal, like Geach's idea of relative identity. It also leads to a repudiation of the idea shared by the majority of Analytic philosophers since Frege that the issue of identity must be firmly fixed from the outset in any serious theoretical enterprise. In the category-theoretic framework the issue of identity is an issue to be studied (both from a general point of view and in every particular case) but not one to be rigidly fixed in advance[37].

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[37]An extended version of the paper can be found at : http ://arxiv.org/pdf/math.CT/0509596.

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