Identity, Equality and Equivalence in Mathematics

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Claims :

Fregean universal notion of identity does not meet needs of mathematics and natural sciences. Martin-Löf's Constructive Type theory and its homotopical interpretation by Voevodsky et al. provides a possible replacement.

Metaphysics :

The Ship of Theseus ; Identity through time : do we need a timeless ontology ?

Leibniz Law

The indiscernibility of identicals : $\forall x \forall y[(x = y) \rightarrow \forall P(Px \leftrightarrow Py)]$
The identity of indiscernibles : $\forall x \forall y[\forall P(Px \leftrightarrow Py) \rightarrow (x = y)]$

Quantum Physics (French, Saunders et al.) :

Particle states : $\mid 0 >$ and $\mid 1 >$

States of a two-particle system :

(i) distinguishable particles : $p_i = 1/4$

$(\mid 0 >\mid 0 >), (\mid 0 >\mid 1 >), (\mid 1 >\mid 0 >), (\mid 1 >\mid 1 >)$

(ii) indistinguishable ("identical") bosons : $p_i = 1/3$

$(\mid 0 >\mid 0 >), (\mid 1 >\mid 1 >), (1/\sqrt{2}(\mid 0 >\mid 1 > + \mid 1 >\mid 0 >))$

(iii) indistinguishable ("identical") fermions : $p = 1$

$(1/\sqrt{2}(\mid 0 >\mid 1 > - \mid 1 >\mid 0 >))$

Biology, Sociology :

Simondon : L'individu et sa genèse physico-biologique (l'individuation à la lumière des notions de forme et d'information) ;
Group identity, gender identity, ethnic identity...

Elementary Mathematics : Numbers

$1 + 1 = 2$

How many ones make two ? How many ones are there ?

Are all ones *equal* ? Are they *identical* (the same) ?

Elementary Mathematics : Points

Are these points discernible ?

$A\bullet \quad \bullet B$

Fig. 1

What about coinciding (congruent) points ? (This case can NOT

$$A = B$$
$$\bullet$$

Fig.2

be dispensed with !)

Modern Structural Mathematics : Groups

Definition : A group is a set G of elements a , b, ... provided with a binary operation $\oplus$ associating with every ordered pair of these elements (possibly identical) a third element of the same set (possibly identical to one of those) such that the following axioms hold :

- operation $\oplus$ is associative ;
- there exists an item 1 (called unit) such that for all a $a \oplus 1 = 1 \oplus a = a$;
- for all a there exists a^{-1} (called inverse of a) such that $a \oplus a^{-1} = a^{-1} \oplus a = 1$.

Modern Structural Mathematics : Groups

Example : Group S_2 of permutations of two items {A, B}.
Elements : *id* and *in*
Multiplication table :

- $id \circ id = id$
- $id \circ in = in$
- $in \circ id = in$
- $in \circ in = id$

Modern Structural Mathematics : Isomorphism

Definition : Groups G_1 and G_2 are called *isomorphic* iff

- elements of G_1 are in one-to-one correspondence with elements of G_2 ;
- for all elements a_1, b_1, c_1 from G_1 such that $a_1 \oplus b_1 = c_1$ the corresponding elements a_2, b_2, c_2 from G_2 satisfy $a_2 \otimes b_2 = c_2$ where $\oplus$ is the group operation in G_1 and $\otimes$ is the group operation in G_2.

Question : How many groups S_2 are there ? Answer : only one *up to isomorphism*. But. it has *many* "isomorphic copies" !

Plato : Equality as weakened Identity

The "intermediate" nature of Mathematics

"Ideal" and mathematical numbers : ideal **2** is unique but its *equal* mathematical copies 2=2=2=are many (Met. 987b) ;

Ideal numbers are not subject to arithmetical operations (Met. 1081a1082b)

Identity stricto sensu applies only to ideas but not to mathematical entities ; in mathematics identity is replaced by equality.

Frege : Universal Identity

It is not only among numbers that the relationship of identity is found. From which it seems to follow that we ought not to define it specially for the case of numbers. We should expect the concept of identity to have been fixed first, and that then from it together with the concept of number it must be possible to deduce when numbers are identical with one another, without there being need for this purpose of a special definition of numerical identity as well.

Identity is a relation given to us in such a specific form that it is inconceivable that various forms of it should occur.

Frege : Universal Identity : Example

Definition : $(a, b) =_{def} \{\{a\}, \{a, b\}\}$ (Logic with Identity and an extensional Set theory are assumed)
Remark : $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$
Proposition : $(a, a) = \{\{a\}, \{a, a\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}$
Does this make sense ?

Frege : Theory of Abstraction

> *The judgment "line a is parallel to line b", or, using symbols $a \parallel b$, can be taken as identity. If we do this, we obtain the concept of direction, and say : "the direction of line a is identical with the direction of line b". Thus we replace the symbol $\parallel$ by the more generic symbol $=$, through removing what is specific in the content of the former and dividing it between a and b.*

No "isomorphic copies" but a class of equivalent individuals ? What are those individuals ? Sets ?

Geach : Relative Identity

Any equivalence relation ... can be used to specify a criterion of relative identity. The procedure is common enough in mathematics : e.g. there is a certain equivalence relation between ordered pairs of integers by virtue of which we may say that x and y though distinct ordered pairs, are one and the same rational number. The absolute identity theorist regards this procedure as unrigorous but on a relative identity view it is fully rigorous.

Geach : Relative Identity

- (1, 2) and (2, 4) are different pairs of numbers but the same rational number : $\frac{1}{2} = \frac{2}{4}$
- a and b are different straight lines but if $a \parallel b$ then a and b are the same *as* direction.

Notice that Frege and Geach use mathematical examples of the same type but interpret them differently.

Martin-Löf : Definitional and Propositional Identity

- $x : A$ - term x of type A; double interpretation of types : "sets" and propositions
- Definitional identity of terms (of the same type) and of types $x = y : A$; $A = B : type$ (substitutivity)
- Propositional identity of terms x, y of (definitionally) the same type A :
  $Id_A(x, y) : type$;
  Remark : propositional identity is a (dependent) type on its own.

Martin-Löf : Heiher Identity Types

- $x', y' : Id_A(x, y)$
- $Id_{Id_A}(x', y') : type$
- and so on

Martin-Löf : Extensional and Intensional theories

- $x = y : A$ implies $z : Id_A(x, y)$;
- Extensionality : $z : Id_A(x, y)$ implies $x = y : A$;
- Intensionality : otherwise (beware that in the given framework this is a purely negative concept !) ;
- Remark : In extensional theories the hierarchy of higher identity collapses (is trivial) ; in intensional theories it, generally, does not.

Martin-Löf versus Frege

Empirical Question : Are Morning Star (MS) and Evening Star (ES) identical ?
Problem : *MS is identical to MS* is a logical truth ; *MS is identical to ES* may be (or be not) only an empirical truth.
Frege's solution : the question is ill-posed. Expressions "MS" and "ES" *refer* to the same planet Venus but *mean* different things.
Remark : this solution requires the view *sub specie aeternitatis*.
Alternative solution : a Constructive Identity.

Categorification and Decategorification

Long ago, when shepherds wanted to see if two herds of sheep were isomorphic, they would look for an explicit isomorphism. In other words, they would line up both herds and try to match each sheep in one herd with a sheep in the other. But one day, along came a shepherd who invented decategorification. She realized one could take each herd and count it, setting up an isomorphism between it and some set of numbers, which were nonsense words like one, two, three, . . . specially designed for this purpose. ..

Categorification and Decategorification

By comparing the resulting numbers, she could show that two herds were isomorphic without explicitly establishing an isomorphism ! In short, by decategorifying the category of finite sets, the set of natural numbers was invented. According to this parable, decategorification started out as a stroke of mathematical genius. Only later did it become a matter of dumb habit, which we are now struggling to overcome by means of categorification.

(Baez and Dolan 1998)

Categorification and Decategorification

Isomorphisms between equipotent sets are *many* and they are *structured*.

Abstraction is DANGEROUS :

DECATEGORIFICATION (= ABSTRACTION OF NUMBER) KILLS THE STRUCTURE, NAMELY THE SYMMETRIC GROUP S_N ! !

CATEGORIFICATION BRINGS IT BACK !

Groups and Categories

Think of a group as a single changing object provided with reversible self-transformations (isomorphisms). Example : groups of symmetries of regular polyhedra. The notion of *category* is a twofold generalization over the notion of group (so presented) :

- a category comprises many objects (each is provided with its own identity transformation)
- transformations are not necessarily reversible

Examples : Sets and Functions ; Topological Spaces and Continuous Transformations ; Groups and Homomorphisms of Groups.
Categories with only reversible transformations (morphisms) are called *groupoids*

Fundamental group

Fundamental group G^0_T of a topological space T :

- a base point P ;
- loops through P (loops are circular paths $l : I \rightarrow T$) ;
- composition of the loops (up to homotopy only ! - see below) ;
- identification of homotopic loops ;
- independence of the choice of the base point.

Fundamental (1-) groupoid

G^1_T :

- all points of T (no arbitrary choice) ;
- paths between the points (embeddings $s : I \rightarrow T$) ;
- composition of the *consecutive* paths (up to homotopy only ! - see below) ;
- identification of homotopic paths ;

Since not all paths are consecutive G^1_T contains more information about T than G^0_T !

Path Homotopy and Higher Homotopies

$s : I \longrightarrow T$, $p : I \longrightarrow T$ where $I = [0, 1]$: paths in T
$h : I \times I \longrightarrow T$: homotopy of paths s, t if $h(0 \times I) = s$, $h(1 \times I) = t$
$h^n : I \times I^{n-1} \longrightarrow T$: n-homotopy of $n - 1$-homotopies h_0^{n-1}, h_1^{n-1} if
$h^n(0 \times I^{n-1}) = h_0^{n-1}$, $h^n(1 \times I^{n-1}) = h_1^{n-1}$;
Remark : : Paths are zero-homotopies

Higher Groupoids and Omega-Groupoids

- all points of T (no arbitrary choice) ;
- paths between the points ;
- homotopies of paths
- homotopies of homotopies (2-homotopies)
- higher homotopies up to n-homotopies
- higher homotopies ad infinitum

G_T^n contains more information about T than G_T^{n-1} !

Composition of Paths

Concatenation of paths produces a map of the form $2I \to T$ but not of the form $I \to T$, i.e., not a path. We have the whole space of paths $I \to 2I$ to play with ! But all those paths are homotopical. Similarly for higher homotopies (but beware that n-homotopies are composed in n different ways !)
On each level when we say that $a \oplus b = c$ the sign $=$ hides an infinite-dimensional topological structure !

Grothendick Conjecture :

G_T^ω contains all relevant information about T ; an omega-groupoid is a complete algebraic presentation of a topological space.

Voevodsky

(circa 2008)
Higher (homotopical) groupoids model higher identity types in Martin-Löf's Intensional Constructive Type theory !
Disclaimer : 1-groupoids do this too but they kill higher identity types ("extensionality one level up")
Only omega-groupoids provide "fully instensional" models ;
Voevodsky's *Univalence Axiom* rules out all groupoid models except the omega-groupoid model.

Conclusions

- The concept of identity (equality or however you want to call it) as elaborated in the recent mathematical practice is non-trivial one and topological in its character.
- A primitive trivial concept of *definitional* identity seems to be dispensable along with definitions (although the identity of types of symbols is not). In any event there is no reason to consider it as fundamental.
- Category-theoretic language generalizes upon the language of group theory, which is widely used in the contemporary physics. This gives a reasonable hope that the theory of identity developed through the Homotopy Type theory can be applied in physics and then in other natural sciences.

Prospectives :

New developments in mathematics and physics in the beginning of the 20th century pushed many philosophers (Frege, Russell et al.) to give up Kantian program of Critical philosophy and to restore instead the traditional Aristotelian alliance between Logic and dogmatic Metaphysics (which is an essential part of so-called "linguistic turn").

Prospectives :

In spite of the fact that the contemporary Logic widely uses mathematical methods it remains largely detached from the mainstream research in Mathematics and Physics (and Biology) and following the Aristotelian pattern works out dummy examples like *The Ship of Theseus*. Martin-Löf's Constructive Type theory and its recent development into the Homotopy Type theory is a rare case where new research in logics and in the mainstream mathematics go tightly together. It makes a fair contrast with Frege's and Russell's dogmatic metaphysical attitude, which unfortunately still remains very influential in the international philosophical community.

THE END