What Is Logic and What It May Be:

Categorical Logic versus Formal Logic

Introduction

Logic is usually described as a discipline concerned with codification, systematisation of and theorising on general forms of reasoning. This is, of course, a very imprecise description. Nevertheless I'm going to argue that it is too restrictive and suggest a more general and, in my view, more appropriate notion of logic. For reasons, which will become clear I shall call the traditional notion of logic implied by the above description "formal logic" and the enlarged notion of logic proposed here "categorical logic".

The plan of this article is the following. First, I elaborate on the notion of form in general and the notion of logical form in particular. Second, I consider what in my view is the principle purpose of doing formal logic in the traditional Aristotelian sense. Then I argue that since the possibility of alternative systems of formal logic is approved none of these systems serves this purpose any longer. Finally I propose a concept of categorical logic and argue that it serves the above purpose.

Forms

The concept of form is of a double metaphysical nature: people tend to conceive of forms as self-standing objects and simultaneously as properties of other objects, which are told to be of such-and-such form. Taking the latter approach one notices that given a class of objects (individuals) having the same form one can always figure out a notion of *isomorphism*, i.e. a transformation between the given objects preserving their form. This helps to provide a general notion of form because isomorphisms can be defined in this context independently as *reversible* transformations. Provide objects *A, B* with identity transformations idA and idB, which "don't change anything". Then assume that transformations of objects are composable in the usual intuitive way. Finally call transformation *f: A –>B* reversible iff there exist transformation *g: B –>A* such that *fg*=idA and *gf*=idB (I write composition in its natural geometrical order here: *fg* stands for transformation resulting from application of *f* first and *g* second.) It is straightforward to see that the existence of an isomorphism between a given pair of object is an equivalence relation.

The idea of form as object is obtained through identification of objects of the same form. Think about (Euclidean) circles. All circles are isomorphic in the sense that any given circle can be transformed into any other given circle by a composition of appropriate motion and scaling; all transformations of this sort are clearly reversible. Think about a unique Circle moving around the plane and changing its size. Forgetting about transformations think about the Circle as a common form of all individual circles.

Logical Forms

Circle belongs to an important class of form called geometrical forms. Another basic class of forms is that of algebraic forms. Logical forms belong to this latter class. Reversible transformations associated with algebraic forms are substitutions. Take any algebraic expression (a formula) like $a+b$. Variables a,b take their values in certain domain of individuals; let it be natural numbers for simplicity. Then $a+b$ can be seen as a common name of expressions 1+2, 1+3, 2+3, etc. But let's look more precisely what is involved here. Given, for example, expression 1+2 one may replace 1 by 2 and 2 (in the original expression!) by 3 and so get expression 2+3 from the same class. Obviously such replacements are reversible. This allows for thinking about both summands "up to replacement" and about the sum - as having form $a+b$. To see that the reversibility of replacements is necessary for building this form-concept think about the operation of summing of some given natural numbers. Unless the number of the given numbers is fixed this operation doesn't have any "arity", and so doesn't have an algebraic form in the usual sense albeit it is perfectly well defined.

Logical forms work similarly. Writing the scheme (form) of Aristotle's "perfect syllogism" as

A is B, all B's are C, therefore A is C

one is supposed to put at places of variables *A, B, C* some English words and obtain a meaningful sentence. What underlies this construction is the possibility of obtaining a meaningful sentence from another meaningful sentence by keeping terms "is", "all", "are" and "therefore" untouched (invariant) but replacing other terms by new ones. Ranges of variables *A, B, C* are not limited here by any particular semantic domain. It is further required that expressions of the form *A is B* and *all B's are C* (called premises) have certain truth-values (if they are both true then *A is C* is also true). As we can see this latter requirement concerns specific features of the system of logic in question. Which forms count as logical, generally speaking, depends on a given system of formal logic. In the traditional context this makes no difficulty because, as we shall now see, the traditional concept of formal logic hardly allows for alternative systems of logic.

Purpose of Formal Logic and Logical Pluralism

Here is an argument in favour of the traditional Aristotelian idea of context-independent formal logic. Unless a system of context-independent rules of reasoning is assumed and respected by all members of a given community this community cannot support a rational discussion and so cannot develop sciences and philosophy. Arguably it cannot support democracy, independent juridical system and any other social institution based on rational dialog either. As far as people share logical forms and stick to the same logical rules they can discuss and revise their different beliefs rationally. But if they disagree about logic like about anything else they fail to be rational. In older times when there was basically one system of formal logic on the market the consensus about logic was easy to reach. But since the possibility of alternative systems of logic has been clearly shown in 20th century such consensus became problematic just like any other consensus.

Categories

The notion of category can be viewed as a generalisation of the notion of form obtained through treating non-reversible transformations on equal footing with reversible ones. A category comprises a class of objects provided with transformations (called in Category theory morphisms) to themselves and to other objects. Each object is provided with unique identity morphism; all morphisms are associatively composable. Categories are ubiquitous in mathematics and science just like forms. An algebraic group (thought of "up to isomorphism as usual) is a kind of form (structure). However since not all groups are isomorphic there is no form shared by all groups. Nevertheless all groups make a category where morphisms are group homomorphisms.

Categorical Logic

I claim that a rational discussion doesn't necessarily require a system of formal logic shared by all participants but may proceed in the situation when interactions (communications) between the participants make categories with certain properties, which can be appropriately called logical. As far as logical properties are identified in the new setting one can define the notion of logical morphism (functor), which in a sense replaces that of logical form.

Traditionally logic is closely linked to linguistic practice and particularly to language. Categorical logic sticks to the traditional link with linguistic practice but deals with speech rather than ready-made language. Notice that an elementary speech act is directed from the speaker to the listener. This makes it natural to think about speech acts as morphisms. Think about a community of speakers (where every speaker is also a listener) and speech acts between these peoples. The notion of shared language is obtained from the assumption according to which all speech acts are reversible: anything one can say to anybody else he or she can also hear from anybody else. This allows for collapsing all the speakers into one ideal subject engaged into a linguistic game with itself. But Category theory points to a vast area between a wholly transparent communication based on an ideal language and the pseudo-linguistic chaos making anything like rational discussion impossible. Let's assume that our speech acts are composable in the following sense: if A says f to B and B says g to C A is in a position to say to C such fg that it will provide on *C* exactly the same effect. This is trivially the case when all speech acts are reversible but composability of speech acts doesn't imply their reversibility. The composability of speech acts makes a given speaking community into a category (provided the composition is associative and identities are well defined). Appropriately chosen further conditions imposed on speech acts turn this category into a topos provided with its "internal logic" and a notion of logical functor. But these conditions don't imply the universal reversibility of speech acts. So Category theory shows that at least in this case the project works (provided all involved category-theoretic notions are reasonably interpreted in terms of speech acts).

Conclusion

Formal logic is a very particular case of categorical logic. Categorical logic provides a flexible framework for a rational discussion, which doesn't assume any particular category-theoretic construction to be context-independent.