Translation versus Formalization: the Example of Euclid's *Elements*

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Content :

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Translation of the old content into terms of a formal axiomatic theory (or more generally, a *formal system*) built by means of modern symbolic logic.

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Example :

Jeremy Avigad, Edward Dean & John Mumma (2009). A Formal System for Euclid's Elements. Review of Symbolic Logic 2 (4) :700–768

Abstract :

> *We present a formal system, E, which provides a faithful model of the proofs in Euclid's Elements, including the use of diagrammatic reasoning.*

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- Evaluating the argument against today's standard

A By-Product of a Larger Formalization Project ?

> *The starting point of any strictly logical treatment of geometry (and indeed of any branch of mathematics) must then be a set of undefined elements and relations, and a set of unproved propositions (=axioms) involving them, and from these all other propositions (theorems) are to be derived by the methods of formal logic. Moreover, since we assumed the point of view of fromal (i.e. symbolic) logic, the undefined elements are to be regarded as mere symbols devoid of content..*

Veblen&Whitehead, The Foundations of Differential Geometry (1932)

THE SUCCESS OF THE PROJECT OF FORMALIZATION OF MATHEMATICS SHOULD NOT BE TAKEN FOR GRANTED

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- widening gap between formalist foundations and the current mathematical practice
- pragmatic compromises (Veblen&Whitehead 1932, Bourbaki, etc.)
- no criteria of the *adequacy* of formalization (to what it is supposed to be a formalization of) are explicitly given
- philosophical objections (Brouwer, Poincaré, Lautman, Gonseth, ...)

Formalization is not innocent. It doesn't amount simply to "clarification". It is a clarification of a very particular sort. Tarski's vision is specific. Remind the story of the tarskian coup of 1952 told by Wilfrid Hodges on Tuesday.

Translation

Formalization is a *conceptual translation* of sort....

Observation :

The history of Euclid's *Elements* is a (history of) continuing conceptual translation. Euclid's Letter was almost never respected. Unlike philosophers, theologians, literature critics and other *hommes de lettre* mathematicians usually did not care about older texts. They didn't try to preserve older writings in their original form. They typically tried to revise and complement older texts, sometime by re-writing these older texts wholly anew. In this respect the formalization is not an exception.

In particular, this happened when Euclid's *Elements* were translated from one natural language into another (in particular, when the *Elements* were translated into Arab and into Latin). However repeated translations of the *Elements* into new "mathematical languages" (in particular, translations of geometrical books of the *Elements* into the language of *algebra* during 16th and 17th centuries) from a mathematical point of view were even more important. Here are some examples.

Euclides Restitutus Denuo Limatus ab Omni Naevo Vindicatus

Giovanni Alfonso Borelli 1658 : *Euclides Restitutus Denuo Limatus* (Euclid Revived and Newly Polished)

Euclides Restitutus Denuo Limatus ab Omni Naevo Vindicatus

Girolamo Saccherri 1733, *Euclides ab Omni Naevo Vindicatus* (Euclid Cleared of Every Flaw)

EUCLIDES
AB OMNI NÆVO VINDICATUS:
SIVE
CONATUS GEOMETRICUS
QUO STABILIUNTUR
Prima ipsa universæ Geometriæ Principia.
AUCTORE
HIERONYMO SACCHERIO
SOCIETATIS JESU
In Ticinensi Universitate Matheseos Professore.
OPUSCULUM
EX.MO SENATUI
MEDIOLANENSI
Ab Auctore Dicatum.
MEDIOLANI, MDCCXXXIII.
Ex Typographia Pauli Antonii Montani. Superiorum permissu

Tacquet and Dechales : Identity Problem

Comparing once popular *Elements of Geometry* published by A. Tacquet in 1654 and the edition of Euclid's *Elements* (the first eight books thereof) published by M. Dechales 6 years later in 1660 it is difficult to say why the later work has Euclid's name in its title while the former doesn't. The difference between the two titles seems to be unrelated to the content of the two books although it might point to different intentions of their authors. When Tacquet's book was republished in 1725 (long after the authors death) it actually got Euclid's name on its cover !

This example shows that the question of whether or not to put Euclid's name on a geometry textbook, in 17-18th centuries was seen as a secondary issue. A more important issue was the choice between teaching geometry after older versions of Euclid's *Elements* and producing new revised versions of this book.

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Today's Elements (also outdated but having no better replacement so far..) :

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- A. Arnauld *Nouveaux Eléments de Géométrie* (1667)
- D. Hilbert, *Grundlagen der Geometrie* (1899)
- N. Bourbaki, *Éléments de mathématique* (1939 - *circa* 2000)

Example : Three versions of the (statement of the) Pythagorean theorem

Version 1 : Euclid

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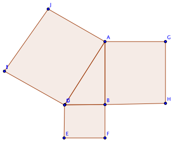
In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

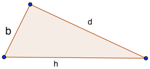
(*Elements*, Proposition 1.47)



Three versions of the (statement of the) Pythagorean theorem : Version 2 : Arnauld (1667)

> *The square of hypothenuse is equal to (the sum of) squares of the two (other) sides (of the given rectangular triangle) :* $bb + dd = hh$.

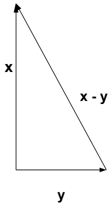
(*New Elements of Geometry, Proposition 14.26.4*)



Three versions of the (statement of the) Pythagorean theorem : Version 3 : Doneddu (1965)

> *Two non-zero vectors x and y are orthogonal if and only if* $(y - x)^2 = y^2 + x^2$

(Donnedu, *Euclidean plane geometry*)



Versions 1-3 of the Pythagorean theorem differ in their **foundations**. Still they translate the same theorem !

How translation helps to solve mathematical problems

Translation V1$\rightarrow$V2 , which translates traditional geometrical constructions into the language of algebra, allowed people in 19 th century to settle great open geometrical problems of Antiquity, including the problem of quadrature of circle. Using algebraic methods one shows that this and other similar problems are unsolvable by the required means (i.e. by compass and ruler). Such results could not be in principle obtained in the same foundational setting, in which these problems were first posed !

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Both these conditions are crucial for mathematical progress !

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Different version of the Pythagorean theorem do NOT share in common anything like an eternal *essence* or an invariant *structure* or a logical form. They share nothing in common except mutual translations !

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- It is historically naive and philosophically irresponsible to assume uncritically that the formalization to be an universal instrument of epistemic analysis.
- The translational perspective on mathematics and science links today's state of affairs with the history of a given discipline.
- The translational allows for a better planning of future research that the atemporal formal perspective.
- In particular, it provides a basis for evaluating *formalization* as a way of organizing mathematics and science.

Category Theory

Category theory is a mathematical apparatus that is helpful for modeling (non-reversible) translations.

THE END