Formal Axiomatics and Set-theoretic Construction in Bourbaki

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28 juillet 2011

Claim :

The persisting gap between the formal and the informal mathematics is due to an inadequate notion of mathematical theory behind the current formalization techniques. I mean the (informal) notion of axiomatic theory according to which a mathematical theory consists of a set of axioms and further theorems deduced from these axioms according to certain rules of logical inference. Thus I claim that the usual notion of *axiomatic method* is inadequate and needs a replacement.

Claim (continued) :

In particular, I claim that *elementary theories* like ZFC, CCAF or ETCS are not adequate as foundations (albeit they may be useful for some other purposes).

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- How formalization works : the example of Bourbaki (the published and unpublished version of Bourbaki's set theory)
- Formalization and symbolization : a comparison with the early modern symbolic algebra
- Non-restrictive constructivism in mathematics and modern science

Theorem 1.5 :

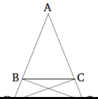
[*enunciation :*]

> *For isosceles triangles, the angles at the base are equal to one another, and if the equal straight lines are produced then the angles under the base will be equal to one another.*

Theorem 1.5 (continued) :

[*exposition*] :

> *Let ABC be an isosceles triangle having the side AB equal to the side AC ; and let the straight lines BD and CE have been produced further in a straight line with AB and AC (respectively). [Post. 2].*



Theorem 1.5 (continued) :

[*specification* :]

> *I say that the angle ABC is equal to ACB, and (angle) CBD to BCE.*

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I say that the angle ABC is equal to ACB, and (angle) CBD to BCE.

[*construction* :]

For let a point F be taken somewhere on BD, and let AG have been cut off from the greater AE, equal to the lesser AF [Prop. 1.3]. Also, let the straight lines FC, GB have been joined. [Post. 1]

Theorem 1.5 (continued) :

[*proof* :]

> *In fact, since AF is equal to AG, and AB to AC, the two (straight lines) FA, AC are equal to the two (straight lines) GA, AB, respectively. They also encompass a common angle FAG. Thus, the base FC is equal to the base GB, and the triangle AFC will be equal to the triangle AGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) ACF to ABG, and AFC to AGB. And since the whole of AF is equal to the whole of AG, within which AB is equal to AC, the remainder BF is thus equal to the remainder CG [Ax.3]. But FC was also shown (to be) equal to GB. So the two (straight lines) BF, FC are equal to the two (straight lines) CG,*

Theorem 1.5 (continued) :

[*conclusion* :]

> *Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.*

Problem 1.1 :

[*enunciation :*]

> *To construct an equilateral triangle on a given finite straight-line.*

Problem 1.1 (continued) :

[*exposition* :]

Let AB be the given finite straight-line.

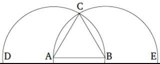
[*specification* :]

So it is required to construct an equilateral triangle on the straight-line AB.

Problem 1.1 (continued) :

[*construction* :]

> *Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another, to the points A and B [Post. 1].*



Problem 1.1 (continued) :

[*proof* :]

> *And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [Axiom 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.*

Problem 1.1 (continued) :

[*conclusion* :]

> *Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.*

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- Axioms (Common Notions) :
  play the role similar to that of logical rules restricted to mathematics : cf. the use of the term by Aristotle
- Postulates :
  non-logical constructive rules

Common Notions

A1. Things equal to the same thing are also equal to one another.
A2. And if equal things are added to equal things then the wholes are equal.
A3. And if equal things are subtracted fromequal things then the remainders are equal.
A4. And things coinciding with one another are equal to one another.
A5. And the whole [is] greater than the part.

Common Notions (continued)

Euclid's Common Notions hold *both* for numbers and magnitudes (hence the title of "common"); they form the basis of a regional "mathematical logic" applicable throughout the mathematics. Aristotle transform them into laws of logic applicable throughout the *episteme*, which in Aristotle's view does not reduce to mathematics but also includes *physics*.

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Aristotle on Axioms (1) :

> *By first principles of proof [as distinguished from first principles in general] I mean the common opinions on which all men base their demonstrations, e.g. that one of two contradictories must be true, that it is impossible for the same thing both be and not to be, and all other propositions of this kind." (Met. 996b27-32)*

Here Aristotle refers to a logical principle as "common opinion".

Aristotle on Axioms (2) :

Comparison of mathematical and logical axioms :

> *We have now to say whether it is up to the same science or to different sciences to inquire into what in mathematics is called axioms and into [the general issue of] essence. Clearly the inquiry into these things is up to the same science, namely, to the science of the philosopher. For axioms hold of everything that [there] is but not of some particular genus apart from others. Everyone makes use of them because they concern being qua being, and each genus is. But men use them just so far as is sufficient for their purpose, that is, within the limits of the genus relevant to their proofs.*

Aristotle on Axioms (2), continued :

Since axioms clearly hold for all things qua being (for being is what all things share in common) one who studies being qua being also inquires into the axioms. This is why one who observes things partly [=who inquires into a special domain] like a geometer or a arithmetician never tries to say whether the axioms are true or false. (Met. 1005a19-28)

Aristotle on Axioms (3) :

Reference to Ax.3 :

> *Since the mathematician too uses common [axioms] only on the case-by-case basis, it must be the business of the first philosophy to investigate their fundamentals. For that, when equals are subtracted from equals, the remainders are equal is common to all quantities, but mathematics singles out and investigates some portion of its proper matter, as e.g. lines or angles or numbers, or some other sort of quantity, not however qua being, but as [...] continuous. (Met. 1061b)*

Postulates 1-3 :

P1 : to draw a straight-line from any point to any point.
P2 : to produce a finite straight-line continuously in a straight-line.
P3 : to draw a circle with any center and radius.

Postulates 1-3 (continued) :

Postulates 1-3 are NOT propositions! They are not first truths. They are basic (non-logical) *operations*.

Operational interpretation of Postulates

Postulates	input	output
P1	two points	straight segment
P2	straight segment	straight segment
P3	straight segment and its endpoint	circle

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- Problems et Theorems share a common 6-part structure (*enunciation*,*exposition*, *specification*, *construction*, *proof*, *conclusion*), which does NOT reduce to the binary structure *proposition - proof*.
- Postulates 1-3 and *enunciations* of Problems are NOT propositions but (non-logical) operations.
- Euclid's mathematics aims at *doing* AND *showing* but not only at showing and moreover not only to proving certain propositions

Conclusion on Euclid

The modern notion of axiomatic theory prima facie does not apply to the theory of Euclid's *Elements*

Modal and Existential Interpretation :

The Modal and the Existential interpretation of Postulates and Problems turns the theory of Euclid's *Elements* into an (informal) axiomatic theory in the modern sense of the term. None of these two interpretations is *historically* justified. None of these two interpretations is innocent.

Euclidean Structure in Modern Mathematics

[*enunciation :*]

Any closed subset of a compact space is compact

[*exposition :*]

Let F be a closed subset of compact space T

[*specification :* I say that F is a compact space]

Euclidean Structure in Modern Mathematics (continued) :

[*construction* :]

> *[Let]* $\{F_\alpha\}$ *[be] an arbitrary centered system of closed subsets of subspace* $F \subset T$.

[*proof* :]

> *[E]very* F_α *is also closed in* T*, and hence* $\{F_\alpha\}$ *is a centered system of closed sets in* T*. Therefore* $\cap F_\alpha \neq \emptyset$*. By Theorem 1 it follows that* F *is compact.*

[*conclusion :* Thus any closed subset of a compact space is compact. (Which is) the very thing it was required to show.]

ELEMENTS DE LA THEORIE DES ENSEMBLES, Rédaction 50

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Rédaction n° 050

Philosophical Statement

First of all let us clarify what we understand under the name of mathematical theory. A mathematical theory is a study of one or more categories of elements, of their properties and relations that unify them, and of constructions made out of them. Such a study cannot proceed without assuming a number of mutually consistent propositions concerning these elements, these properties, these relations and these constructions. The purpose of the theory is to deduce from these premises some other propositions, so that their exactness depends only on the exacness of the premises but does not require any further hypothesis.

Axiomatic theory ?

Fundamental Sets

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- one should be able to distinguish an element of the set
- one should be able to establish between the element and the set the relation of membership

The Subset Axiom

> *Any predicate of type A defines a subset of A; any subset of A can be defined through a predicate of type A.*

(Predicate of type A is a predicate P such that for every element a of set A $P(a)$ has a definite truth-value. The subset S of set A defined by P consists of such and only of such a for which $P(a)$ is true.)

Next Bourbaki introduces the concept of *complement* of a given subset, of *powerset* $P(A)$ of a given set A (i.e., the set of all subsets of A) ; of *union*, *intersection* and *cartesian product* of sets (described as *operations* on sets), of *relation* and *function* between sets. Having these basic concepts in his disposal the author says :

> *In any mathematical theory one begins with a number of fundamental sets, each of which consists of elements of a certain type that needs to be considered. Then on the basis of types that are already known one introduces new types of elements (for example, the subsets of a set of elements, pairs of elements) and for each of those new types of elements one introduces sets of elements of those types.*

(continued)

So one forms a family of sets constructed from the fundamental sets. Those constructions are the following :
1) given set A, which is already constructed, take the set $P(A)$ of the subsets of A ;
2) given sets A, B, which are already constructed, take the cartesian product AxB of these sets.
The sets of objects, which are constructed in this way, are introduced into a theory step by step when it is needed. Each proof involves only a finite number of sets. We call such sets types of the given theory ; their infinite hierarchy constitutes a scale of types.

Structure

On this basis Bourbaki describes the concept of *structure* as follows :

> *We begin with a number of fundamental sets :* $A, B, C, ..., L$ *that we call base sets. To be given a structure on this base amounts to this :*
> *1) be given properties of elements of these sets ; 2) be given relations between elements of these sets ; 3) be given a number of types making part of the scale of types constructed on this base ; 4) be given relations between elements of certain types constructed on this base ; 5) assume as true a number of mutually consistent propositions about these properties and these relations.*

(Informal) Bourbaki and Euclid

Principles of building mathematical theory described in the Bourbaki's draft are not so different from Euclid's (in spite of the above statement). These principles adequately describe what is done in a large part of research mathematics of the 20th century. Like Euclid Bourbaki begins his exposition with principles of building mathematical objects but not with certain propositions about some abstract entities assumed as axioms. Propositions appear only in the very end (the 5th item of the above quote), and even it is usual to call them "axioms" (like in the case of axioms of group theory') it is clear that they are rather analogous to Euclid's *definitions*.

(Informal) Bourbaki and Euclid (continued)

While for Euclid the basic data is a finite family of *points* (everything else is constructed through Postulates) for Bourbaki the basic data is a finite family of *sets* and everything else is constructed as just described. While for Euclid the basic type of geometrical object is a *figure* for Bourbaki the basic type of mathematical object is a *structure*. In both cases the constructed objects come with certain propositions that can be asserted about these objects without proofs because they immediately follow from corresponding definitions. In both cases the construction of objects is a subject of certain *rules* but not the matter of a mere stipulation.

HOWEVER ALL OF THAT REMAINS “UNOFFICIAL” !

The Published "Official" Version

The first chapter of the treatise, which has the title *Description of Formal Mathematics*, begins with an account of *signs* and *assemblies* (strings) of signs provided with a definition of mathematical theory according to which such a theory

> *... contains rules which allow us to assert that certain assemblies of signs are terms or relations of the theory, and other rules which allow us to assert that certain assemblies are theorems of the theory.*

The Published "Official" Version (continued)

In the published formalized version the set-theoretic constructions are replaced by syntactic constructions that formally *prove the existence* of certain sets. Basic objects of the formalized set theory are no longer sets and and structures but symbolic expressions that can be interpreted as propositions *about* sets and structures.

Conclusion on Bourbaki

What makes the major difference between the formal and the informal versions of Bourbaki's set theory is the character of its *objects*; otherwise the two theories proceed similarly. Both follow Euclid's pattern. What remains unclear is this : In which sense if any the formal set theory tells us something about *sets* and further set-theoretic constructions. Prima facie it only tells us something about ways of *talking* about sets and set-theoretic constructions. It may only work if the St. John's Dogma is true. I shall now show that as far as mathematics is concerned this Dogma is false.

A comparison with the Symbolic Algebra

MacLaurin, *A Treatise of fluxions* :

> *The improvement that have been made by it [the doctrine of fluxions] ... are in a great measure owing to a facility, conciseness, and great extend of the method of computation, or algebraic part. It is for the sake of these advantages that so many symbols are employed in algebra. ... It [algebra] may have been employed to cover, under a complication of symbols, obstruse doctrines, that could not bear the light so well in a plain geometrical form ; but, without doubt, obscurity may be avoided in this art as well as in geometry, by defining clearly the import and use of the symbols, and proceeding with care afterwards.*

Formalization (in the modern sense of the term) is supposed to play the same epistemic role as the symbolic algebraization as MacLaurin describes it : to make clear some "obstruse doctrines" through determining the "use of the symbols". There is however an essential difference between the two approaches. While *algebraic* symbolic constructions mimic the constructions of mathematical *objects* referred to by the corresponding symbols the symbolic constructions of modern formal theories mimic informal *descriptions* of certain objects but not the construction of these objects themselves ! While the symbolic algebra represents forms of human constructive activities, and for this reason may guide human actions in the real material world, the formal mathematics reflects only *logical* forms of the pure speculative thought.

Speculative Thought and Modern (Galilean) Science

But the modern Galilean science requires an active intervention of humans into the nature rather than a passive observation of and speculation on natural phenomena. This is why the modern science turns to be so helpful for technology. This is why the formal mathematics is useless in the modern experimental science and technology (with the possible exception of the software engineering that does not deal with the hardware development).

St. John's Dogma

The problem of the formal mathematics is that it takes the *logical* form to be fundamental. This dogma is incompatible with the Galilean science.

Constructivism

The constructivist thinking in mathematics from the very beginning of the 20th century took a conservative bend and began a fight against then-new ways of mathematical thinking including the set-theoretic thinking. This tendency can be traced back to Kronecker who required every well-formed mathematical object to be constructible from natural numbers. More recently Bishop was inspired by similar ideas (and in particular by Kronecker's works).

Constructivism (continued)

Brouwer's *intuitionism* (which qualifies as a form of constructivism) also put rather severe restrictions on his contemporary mathematics as well as on essential parts of earlier established mathematical results. Even those constructivists who like Markov tried to develop constructive mathematics as a special part of mathematics rather than reform mathematics as a whole understood the notion of mathematical construction very restrictively and almost exclusively in computational terms.

Axiomatic Method

I claim that the general issue of the constructive thinking in mathematics concerns the very method of theory-building (the axiomatic method) rather than some particular principles like the principle of the excluded middle, etc. The modern axiomatic method is non-constructive by its very design because it doesn't require to provide rules of construction of mathematical objects (except formulae). What makes a system of postulates coherent ?

Axiomatic Method (continued)

A genuinely constructive method of theory-building needs non-propositional principles similar to Euclid's Postulates and algebraic rules, which allow for building of and operating with objects and not only with formal expressions telling us something *about* these objects. Such a general method must not specify the postulates just like the standard axiomatic method does not specify axioms. Following Hilbert one should rather focus on questions concerning the mutual compatibility of postulates and the like.

Open Problems

What makes a system of postulates coherent ? What is an appropriate analogue of the logical consistency for postulates ? What is an appropriate notion of dependency for postulates ? (Cf. the constructive type theory with dependent types.)

THE END