Univalence and Constructive Identity

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Summary :

The non-standard identity concept developed in the Homotopy Type theory allows for an alternative analysis of Frege's *Venus* example illustrating his distinction between the *sense* and the *reference* of linguistic expressions. This alternative analysis explains how empirical evidences can justify judgements about identities (like Morning Star and Evening Star are the same planet) and accounts for the constructive aspect of such judgements. This analysis suggests a way in which the Homotopy-Type-theoretic identity may apply in Physics and other Natural Sciences.

Frege on Sense and Reference

Identity in Homotopy Type theory

Morning Star = Evening Star Homotopically

To Weaken or To Construct ?

Frege 1892 (beginning)

$a = a$ and $a = b$ are obviously statements of different epistemic value ; $a = a$ holds a priori and, according to Kant, is to be labeled analytic, while statements of the form $a = b$ often contain very valuable extensions of our knowledge and cannot always be established a priori. The discovery that the rising sun is not new every morning, but always the same, was one of the most fertile astronomical discoveries. Even today the identification of a small planet or a comet is not always a matter of course. Now if we were to regard identity as a relation between that which the names a and b designate, it would seem that $a = b$ could not differ from $a = a$ (provided $a = b$ is true).

Frege 1892 (the end)

When we found $a = a$ and $a = b$ to have different epistemic values, the explanation is that for the purpose of knowledge, the sense of the sentence, viz., the thought expressed by it, is no less relevant than its reference, i.e. its truth value. If now $a = b$, then indeed the reference of b is the same as that of a, and hence the truth-value of $a = b$ is the same as that of $a = a$. In spite of this the sense of b may differ from that of a and thereby the thought expressed in $a = b$ differs from that of $a = a$. In that case the two sentences do not have the same epistemic value. If we understand by judgment the advance from the thought to its truth value, as in the above paper, we can also say that the judgments are different.

Frege 1893

Identity is a relation given to us in such a specific form that it is inconceivable that various kinds of it should occur

Venus Example

$a = Morning\ Star\,;\ b = Evening\ Star$
$Morning\ Star = Evening\ Star = Venus$

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- the obscure nature of *sense* aka *meaning* ;
- no special account for the case when in $\vdash a = b$ terms a and b have the same sense (meaning) ;
- no account of how empirical or other evidences justify judgement $\vdash a = b$;
- linguistic examples from the everyday talk and historical narrative are used for fixing the notion of identity in empirical sciences.

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- propositional identity of terms belonging to the same given type A ($Id_A(x, y)$);
- *propositions as types* : terms of type $Id_A(x, y)$ are proofs (witnesses) of proposition $Id_A(x, y)$.

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- composition of the *consecutive* paths is up to homotopy ;

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- Remark : : Paths are zero-homotopies

Omega-Groupoids : Grothendieck Conjecture

A (properly defined) ω-groupoid uniquely determines a topological space. This allows for thinking about ω-groupoids as homotopy groupoids without the loss of generality.

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- n-groupoid semantics are intensional up to nth dimension.

Univalence Axiom : What it does

Rules out LCC and all (higher) groupoid models except the ω-groupoid model.
(UA does some other important things that I do not mention.)

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- UA says that u is an equivalence in *TYPE*

Univalence Axiom : Why it rules out finite-dimensional models

In order to compare n-map in a n-groupoid on has to rise the dimension and consider $n + 1$ maps. However since $\omega + 1 = \omega$ one may compare ω-maps with ω-maps (a "geometric closure").

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- Propositional : $Id_A(MS, ES)$ where A is a type of observable celestial bodies.

Morning Star = Evening Star

Interpretation : Identities of MS and ES are fixed ; the identification of their different appearances in the given context is not questioned. The identification of MS with ES is problematic : it is established by constructing invertible maps $p_i : MS \to ES$ from the observations of MS to the observations of ES (I assume that an "observation of MS" includes an observation of other celestial bodies). Using the above notation we consider p_i as terms of type $Id_A(MS, ES)$

Venus

The object Venus ($=MS = ES$) is construed in terms of invariants of transformations p_i (like the distance to the Sun). The identification of MS and ES turns the class $\{p_i\}$ into a group of transformations.

Groups and Groupoids in Physics

Thinking of physical objects in terms of invariants of groups of (passive) transformations is well-established in physics. Example : an extended event in the flat relativistic space-time. However in many physical contexts groupoids of paths make better physical sense. Example : the fundamental groupoid of a curve relativistic space-time versus the fundamental group of the same manifold. A moral : the reduction of groupoids to groups is not, generally, justified.

Beyond Groups and Groupoids

Consider paths p_i as 1-cells and path hotomotopies $r_k^{i,j} : p_i \longrightarrow p_j$ as 2-cells of 2-groupoid G_2 and check the coherence conditions against the observations ; the resulting 2-groupoid is the identity 2-groupoid of Venus. Homotopically different paths p_i represent different "proofs" of identity $Id_A(MS, ES)$; the space of these paths makes part of the object Venus and reflects the way in which this object is built out of observations. Idem for $r_k^{i,j}$ and the higher-order homotopies.

Hypothesis : the topology of "the" physical space-time can be reconstructed in such homotopical terms.

Question : are there physical examples where the higher-order homotopical structure is made explicit ?

Manin Problem

quoted in French&Krause 2006, *Identity in Physics : A Historical, Philosophical, and Formal Analysis*, Ch.6 :

> *We should consider possibilities of developing a totally new language to speak about infinity. Classical critics of Cantor (Brouwer et al.) argued that, say, the general choice axiom is an illicit extrapolation of the finite case. I would like to point out that this is rather an extrapolation of the common-place physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behavior.*

Manin Problem (continued)

Even 'sets' of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the 'set' of grains of sand. In general, a highly probabilistic 'physical infinity' looks considerably more complicated and interesting than a plain infinity of 'things'.

(1977 : Mathematical Problems I : Foundations)

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Two non-primitive identity relations :

- Classical relation $=$ such that the full substitutivity of identicals is assured ;
- The non-Classical relation $\equiv$ called *indistinguishability* such that $x \equiv y$ and $y \in Z$ does not imply $x \equiv z$ (no substitutivity for $\in$-formulas).

Weakening Identity versus Constructing Identity

A weak point of weak identity : the idea according to which the wanted non-Classical identity concept can be obtained by taking some conceptual content out of the Classical identity concept (the "weakening")
Philosophical underpinning : Structuralism

Weakening Identity versus Constructing Identity (continued)

In Voevodsky's approach homotopy types are emphmultiplicities provided with a specific identity relation (more precisely : specific identity type) between their elements (terms). Classical emphsets are a particular homotopy type among others. One may suggest that classical and quantum particles are of different homotopy types (i.e. of different identity types).

Constructivism ?

NOT in the sense of barring infinities but rather in the (Neo-)Kantian sense of making infinities physical, i.e., re-establishing the link between mathematical concepts and experience. Constructivism in mathematics like constructivism in natural sciences should not be understood as a set of restrictions. Constructivism, generally, is the idea of science as active involvement ; the *ways* of involvement cannot be specified in advance. Constructivism works both bottom-up and top-down.

THE END