Proofs and Objects in HoTT

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Cassirer's Critique of Russell 1903

"Here rises a problem that lies wholly outside the scope of "logistics" [= Formal Symbolic Logic]. All empirical judgements [..] must respect the limits of experience. What logistics develops is a system of hypothetical assumptions about which we cannot know, whether they are actually established in experience or whether they allow for some immediate or non-immediate concrete application. According to Russell even the general notion of magnitude does not belong to the domain of pure mathematics and logic but has an empirical element, which can be grasped only through a sensual perception. From the standpoint of logistics the task of thought ends when it manages to establish a strict deductive link between all its constructions and productions.

Cassirer's Critique of Russell 1903

Thus the worry about laws governing the world of objects is left wholly to the direct observation, which alone, within its proper very narrow limits, is supposed to tell us whether we find here certain rules or a pure chaos. [According to Russell] logic and mathematics deal only with the order of concepts and should not care about the order or disorder of objects. As long as one follows this line of conceptual analysis the empirical entity always escapes one's rational understanding. The more mathematical deduction demonstrates us its virtue and its power, the less we can understand the crucial role of deduction in the theoretical natural sciences. " (E. Cassirer, Kant und die moderne Mathematik, 1907)

A Criterion for the Foundations of Mathematics

FM should make *reasonable* the effectiveness of mathematics in natural sciences.

"Logical and mathematical concepts must no longer produce instruments for building a metaphysical "world of thought": their proper function and their proper application is only within the empirical science." (Cassirer, ib.)

Traditional geometry is constructive

"The book [of Nature] is written in mathematical language, and the symbols are triangles, circles and other geometrical figures."

The traditional geometry organizes "triangles, circles and other geometrical figures" into an *order* which does <u>not</u> reduce to a *logical* (viz. propositional) deductive order. It also involves a constructive order, in which these and further constructions are produced.

OBJECT = INDIVIDUAL

"I will fix the way I wish to use the term "object" and simultaneously say what I think useful in such abstract discussions [about objects in general] by saying that the usable general characterization of the notion of object comes from logic. We speak of particular objects by referring to them by singular terms [..]." (Ch. Parsons, Mathematical Thought and its Objects, 2008)

The notion of object as individual borrowed from logic does not allow for a constructive order different from a logical order.

Ian Müller on Euclid

"I know of no logic which accounts for this inference in its Euclidean formulation. One 'postulates' that a certain action is permissible and 'infers' the doing of it, he., does it. An obvious analogue of the procedure here is provided by the relation between rules of inference and a deduction. "

Euclid's Postulates 1-3

P1. To draw a straight-line from any point to any point;

P2. And to produce a finite straight-line continuously in a straight-line;

P3. And to draw a circle with any center and radius.

P1-3 are NOT (propositional) axioms but licenses to act!

Euclid's Common Notions 1-3

- Ax1. Things equal to the same thing are also equal to one another;
- Ax2. And if equal things are added to equal things then the wholes are equal;
- Ax3. And f equal things are subtracted from equal things then the remainders are equal.

Euclid's Common Notions 1-3

Ax1-3 replace logical rules: notice the central role of the equality concept!

$$\frac{A = C, B = C}{A = B}$$

Ian Müller on Euclid

"A Euclidean derivation, then, is a *thought experiment* of a certain kind; an experiment intended to show either that a certain operation can be performed [as in *problems*] or that a certain kind of object has a certain property [as in *theorems*]. Thus, Euclidean derivations are quite different from Hilbertian one."

Thought Experiment

The capacity to support the thought-experimentation is a key feature of traditional geometry, which makes this sort of geometry constitutive in the Early Modern science.
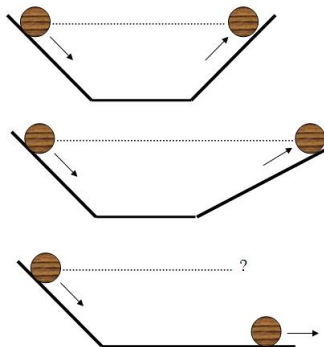
Thought Experiment

Euclid's geometry supports the thought-experimentation with
constructive rules P1-3, which apply to non-propositional objects.
Common Notions are rules, which apply to equality statements and
which are analogous to logical rules in the modern sense.

Cassirer's critique explained

Russell's logic [=mathematics] comprises *only* rules applicable to propositions, such as *modus ponens*, but has no rules for (non-propositional) objects. For this reason it doesn't support (a proper form of) thought-experimentation and cannot be effective in natural sciences.

Cassirer's argument extends to Hilbert's axiomatic approach.

Hintikka (2011) on the Hilbert-style axiomatic method

"What is crucial in the axiomatic method [..] is that an overview on the axiomatized theory is to capture all and only the relevant structures as so many models of the axioms."

"The class of structures that the axioms are calculated to capture can be either given by intuition, freely chosen or else introduced by experience."

Hintikka (2011) on the Hilbert-style axiomatic method

[Where logic comes from?] "[N]ew logical principles are not dragged [..] by contemplating one's mathematical soul (or is it a navel?) but by active *thought-experiment* by envisaging different kinds of structures and by seeing how they can be manipulated in imagination. [..]

[M]athematical intuition does not correspond on the scientific side to sense-perception, but to *experimentation*."

Observation

In the Hilbert-style axiomatic approach the issue logical semantics
(meaning of logical constants) is separated from usual semantic
considerations: logic, including its proper semantics, is supposed to
be fixed first; the intended non-logical semantic is fixed only
afterwards.

A problem with Hilbert-style axiomatic method

Thought-experimentation without definite constructive <u>rules</u> (which need not to be fixed once and for all), i.e., rules applied to non-propositional objects, remains "a pure chaos" (Cassirer) and remains very limited. Compare playing with pieces of wood with playing chess.

Logic does not solve the problem

Hilbert-style formal mathematics works only on the basis of rich intuitive mathematics: cf. Hilbert's *anschauliche Geometrie*. But no rich intuitive mathematics can be possibly developed without using (formal) constructive rules. Mathematical intuition is not, generally, a chaotic spontaneous activity. Logical rules cannot perform the function of constructive rules.

Hilbert-style axiomatic method in science

YES-NO questioning games with Nature are important BUT the mathematical modeling of natural phenomena does not reduce to such games.

Modern science does *not* allow for a direct truth-evaluation of formal axiomatic theories with empirical data: scientific theories are essentially *model-based*. Mathematical models used in science can not be satisfactorily specified by listing some formal propositions, which these models are supposed to satisfy.

Objection

It may be objected that the above argument does not apply to the 20th century fundamental physics, which use only mathematical models of a very abstract sort.

Suppes (2002) and other proponents of the *non-statement* aka *semantic* view of scientific theory recognizes that scientific theories are model-based but he believes that Tarski-style set-theoretic models can be good for all scientific purposes. Moreover, he thinks that such models are more appropriate in today's science than traditional intuitive models.

Hilbert Problem 6

Explore "all logically possible theories" which admit a physical semantic.

Exploring merely "logical" possibilities may not provide anything interesting for Physics. This is for two independent reasons:
1) the chosen logical framework may be not appropriate;
2) it cannot provide a sufficient control over intended models: explicit constructive rules for model-building are needed.

Thought experiments in QM are just as important (and arguably even more important) as they are in the Early Modern science. Thought experiments in QM involve manipulations with objects just like Galileo's thought experiment.

FIG. 1
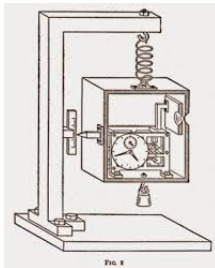
looks outdated?

New Physics needs a new constructive Geometry rather than no Geometry.

Beware the meaning of being "constructive"!

Desiderata for formal framework

- support deduction from first principles (first elements), including non-propositional ones (primitive objects, types, etc.)
- combine logical rules with constructive rules (rules for non-propositional objects)

Constructive axiomatic method

Theories satisfying the above desiderata I shall call *constructive* axiomatic theories.

This use of the term "constructive" has a historical grounding (ex. Hilbert&Bernays 1934) but is not standard. This notion of being constructive does not fix any specific set of rules.

Examples of constructive axiomatic theories

- ▶ Euclid (geometry)
- ▶ HoTT (!)

M-L (1983) Constructive Proof theory

"[P]roof and knowledge are the same. Thus, if proof theory is construed not in Hilbert's sense, as metamathematics, but simply as a study of proofs in the original sense of the word, then proof theory as the same as theory of knowledge, which, in turn, is the same as logic in the original sense of the word, as the study of reasoning, or proof, not as metamathematics." (Martin-Löf 1983)

Propositions in M-L (1983)

"Classical" notion of proposition as truth-value is rejected and replaced by the "intuitionistic" one:

"A proposition is defined by laying down what counts as a proof of the proposition."

"A proposition is true if it has a proof, that is , if a proof of it can be given."

$t : T$ in Martin-Löf 1983

- t is an element of set T
- t is a proof (construction) of proposition T
- t is a method of fulfilling (realizing) the intention (expectation) T
- t is a method of solving the problem (doing the task) T

HoTT

"The central new idea in homotopy type theory is that types can be regarded as spaces in homotopy theory, or higher-dimensional groupoids in category theory." (HoTT Book 2013).

One more item to the above list of interpretations? NOT just that.

h-stratification in MLTT

- ▶ (i) Given space A is called *contractible* (aka space of h-level -2) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- ▶ (ii) We say that A is a space of h-level $n + 1$ if for all its points x, y path spaces $paths_A(x, y)$ are of h-level n.

h-hierarchy

- (-2) single point *pt*;
- (-1) the empty space $\emptyset$ and the point *pt* : truth values aka *classical* or "mere" propositions
- (0) sets aka *intuitionistic* propositions aka theorems
- (1) (flat) groupoids
- (2) 2-groupoids
  - ▶
  - ▶
- (*n*) *n*-groupoids
  - ▶ . . .
- (ω) ω-groupoids

The above stratification of types is a robust mathematical structure in MLTT discovered via the homotopic interpretation of MLTT syntax. MLTT intended semantic does <u>not</u> take this structure into account. HoTT semantics does.

HoTT semantics (or the version thereof that I defend) does <u>not</u> license the idea that *every* type is a proposition.

It recovers within the MLTT syntax the <u>classical</u> notion of proposition as well as the intuitionistic notion of proposition-as-set (under a different name) and determines the precise place of both in the hierarchy of types. These semantic decisions are not arbitrary but based on the robust mathematical structure of *h*-stratification of types. *h*-stratification should be reflected semantically!

HoTT semantics for $t : T$ for (-1)-types

propositions and truth-values

HoTT semantics for $t : T$ for (0)-types

theorems and their proofs / sets and their elements

HoTT semantics for $t : T$ for higher -types

(also valid for lower types):

spaces and points, which support higher-order structures from elements of some other spaces (viz. map spaces);

objects are points;
constructions are points provided with additional higher-order structures: paths, surfaces (homotopies), etc.

Spaces in Euclid and in the modern geometry

In Euclid the *space* (if any) is usual though of as a universe of all geometrical objects and constructions. Compare however the notion of *plane* in the stereometrical books of the *Elements*: here a universe becomes an object (and the other way round).

In the modern geometry since Lobachevsky and Riemann this latter situation is common. Spaces are construed from other spaces; the space/object distinction is relational. In the HoTT semantics one has to think of spaces in the modern way. Space is not only a scene for performing constructions but also a subject to constructive rules.

Ladyman's objection

Objection (paraphrased): HoTT lacks sufficient polymorphism features needed for supporting the conventional notion of object.

Replies:

▶ in many relevant contexts the conventional notion is too strong and not really needed.
  Ex.: Euclid, the concept of *arbitrary object* (Kit Fine);

▶ (recognizing the problem) More polymorphism features may be possibly introduced in a future version of HoTT/UF.

A critique of Ladyman's interpretation

This interpretation does not take the *h*-stratification into account.

A way to reconsiliation

Spaces may serve, generally, as mathematical representations of concepts.

Spaces or h-level > 0 carry *intensional* higher order structures, which can be extensionally represented in the form of path- and homotopy-sets.

How to build and manipulate objects with HoTT

- ▶ Identification of Classical particles: Morning Star = Evening Star;
- ▶ Double-split experiment

(BLACKBOARD)

Conclusion

HoTT is a strong candidate for the role of constructive mathematical framework supporting an object-oriented experimental reasoning in today's science.

There are strong epistemic reason (leaving other reasons aside) for developing a HoTT-based approach in Knowledge Representation.

On Constructive Axiomatic Method :
http://arxiv.org/abs/1210.1478

Venus Homotopically:
http://philsci-archive.pitt.edu/12116/

philomatica.org

THANK YOU!