Euclid, "Universal Mathematics" and the Origin of Logic

Andre Rodin

rodin@ens.fr

CUNY, 5 May 2009

Plan of the Talk:

1) Structure of Euclid's Mathematics: Postulates, Axioms, Problems, Theorems.
2) Aristotle: From Universal Mathematics to Logic and Metaphysics
3) Difficulties of Aristotle's Logic vis-à-vis Euclid's Mathematics
4) Conclusion

Euclid, *Elements*: 3 kinds of fundamentals (archai): Definitions (to skip), Postulates, Axioms

Definitions of Book 1:

D1.1. A point is that of which there is no part.

D1.2. And a line is a length without breadth.

D1.3. And the extremities of a line are points.

D1.4. A straight-line is whatever lies evenly with points upon itself.

D1.5. And a surface is that which has length and breadth alone.

D1.6. And the extremities of a surface are lines.

D1.7. A plane surface is whatever lies evenly with straight-lines upon itself.

……………………..

D1.13. A boundary is that which is the extremity of something.

D1.14. A figure is that which is contained by some boundary or boundaries.

D1.15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from a single point lying inside the figure are equal to one another.

D1.16. And the point is called the center of the circle.

D1.17. And a diameter of the circle is any straight-line being drawn through the center, which is brought to an end in each direction by the circumference of the circle. And any such (straight-line) cuts the circle in half.

Definitions of Book 7:

D7.1. A unit (or monad) is (that) according to which each existing (thing) is said (to be) one.

D7.2. And a number (is) a multitude composed of units (monads).

…………..

D7.11. A prime number is one (which is) measured by a unit alone.

D7.12. Numbers prime to one another are those (which are) measured by a unit alone as a common measure.

D7.13. A composite number is one (which is) measured by some number.

……..

D7.20. Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third (is) of the fourth.

D7.21. Similar plane and solid numbers are those having proportional sides.

D7.22. A perfect number is that which is equal to its own parts [= equal to the sum of its factors].

Postulates:

P1. Let it have been postulated to draw a straight-line from any point to any point.

P2. And to produce a finite straight-line continuously in a straight-line.

P3. And to draw a circle with any center and radius.

P4. And that all right-angles are equal to one another.

P5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angle (and do not meet on the other side).

Remarks:

1) Postulates P1-P3 are NOT propositions, they don't have truth-values. They are (descriptions of) fundamental operations (events). Infinitive form of the verb is never used in Greek as a replacement for the imperative mood. Modal or existential interpretation of Postulates is not necessary.
2) P1-P3 involve two fundamental "generic forms": circle and straight line. The idea of construction by ruler and compass is not a pure convention!
3) P4-P5 are problematic

Axioms (Common Notions)

A1. Things equal to the same thing are also equal to one another.

A2. And if equal things are added to equal things then the wholes are equal.

A3. And if equal things are subtracted from equal things then the remainders are equal.

A4. And things coinciding with one another are equal to one another.

A5. And the whole [is] greater than the part.

Remarks:

1) Unlike Postulates Axioms apply both to Geometry and Arithmetic, this is the reason why Euclid calls them "common". They provide "theory of equality". In the geometrical case equality means (roughly) equicomposability. In the arithmetical case congruence (coincidence) means one-one correspondence between units.
2) Postulates and Axioms don't comprise any *specific* proposition taken for granted. Such propositions are provided only by Definitions. Axioms provide only a *general* framework for doing mathematics.
3) The distinction between Postulates and Axioms is naturally interpreted in terms of Plato's ontology according to which the domain of mathematics is "intermediate" between the domain of pure Becoming (sensibilia) and the domain of pure Being (Ideas). Postulates are principles of "Mathematical Becoming" while

Axioms are principles of “Mathematical Being” (Proclus). A similar remark can be made about Problems and Theorems.

Problems and Theorems (no “propositions”!)

"Science as a whole has two parts: in one it occupies itself with immediate premises [= first principles], while in the other it treats systematically the things that can be demonstrated or constructed from these first principles, or in general are consequences of them. In the geometrical reasoning this second part is again divided into solving problems and finding theorems. The name "problem" is appropriate where what in a sense doesn't exist is produced, set, brought into view and arranged, while the name "theorem" is appropriate where something that is attributed or not attributed is seen, known and proved. The former [have to do with] generation, setting, application, ascription, inscription, insertion, touching and the like; the latter [have to do with] properties and essential attributes of geometrical objects, which are grasped and firmly bound by demonstration." (*Commentary*, 200.20-201.14, Morrow's translation, corrected)

Common Structure of Problems and Theorems:

1. [*proposition*]
2. [*exposition*]
3. [*specification*]
4. [*construction*]
5. [*proof*]
6. [*conclusion*]

Ex.: 1.1 (Problem):

[*proposition*]

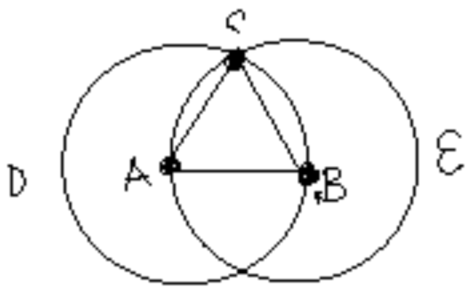
To construct an equilateral triangle on a given finite straight-line.

[*exposition*]

Let AB be the given finite straight-line.

[*specification*]

So it is required to construct an equilateral triangle on the straight-line AB.



[*construction*]

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center radius BA have been drawn. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another to the points A and B (respectively).

[*proof*]

And since the point A is the center of the circle CDB, AC is equal to AB. [D1.15-16] Again, since the point B is the center of the circle CAE, BC is equal to BA. [D1.15-16] But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [A1]. Thus, CA is also equal to CB. Thus, the three (straight lines) CA, AB, and BC are equal to one another.

[*conclusion*]

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.

<u>Remarks:</u>

1) Proposition of a Problem has the same grammatical form as a Postulate. *Prima facie* it is NOT a requirement to produce something. No more than a Theorem is a requirement to prove

something. In both cases requirements come into play only in *specification*. It is NOT a proposition in Frege's sense either (it lacks a truth-value).

2) Postulates alone are not sufficient for solving problems: Axioms and immediate premises (making part of Definitions) are needed too.

3) Postulates are used in *construction*; Axioms and immediate premises are used in *proof*.

Ex.: 1.5 (Theorem):

[*proposition*]

For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another.

[*exposition*]

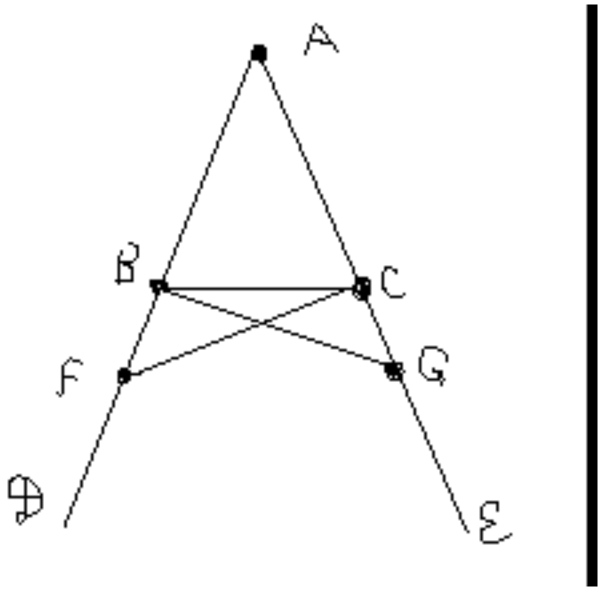
Let ABC be an isosceles triangle having the side AB equal to the side AC, and let the straight-lines BD and CE have been produced in a straight-line with AB and AC (respectively).

[*specification*]

I say that the angle ABC is equal to ACB, and (angle) CBD to BCE.

[*construction*]

For let the point F have been taken somewhere on BD, and let AG have been cut off from the greater AE equal to the lesser AF. Also, let the straight lines FC and GB have been joined.



[*proof*]

In fact, since AF is equal to AG, and AB to AC, the two (straight-lines) FA, AC are equal to the two (straight lines) GA, AB, respectively. They also encompass a common angle FAG. Thus, the base FC is equal to the base GB, and the triangle AFC will be equal to the triangle AGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. [T1.4] (That is) ACF to ABG, and AFC to AGB. And since the whole of AF is equal to the whole of AG, within which AB is equal to AC, the remainder BF is thus equal to the remainder CG. [A3] But FC was also shown (to be) equal to GB. So the two (straight lines) BF, FC are equal to the two (straight lines) CG, GB, respectively, and the angle

BFC (is) equal to the angle CGB, and the base BC is common to them. Thus the triangle BFC will be equal to the triangle CGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. [T1.4] Thus, FBC is equal to GCB, and BCF to CBG. Therefore, since the whole angle ABG was shown (to be) equal to the whole angle ACF, within which CBG is equal to BCF, the remainder ABC is thus equal to the remainder ACB. [A3] And they are at the base of triangle ABC. And FBC was also shown (to be) equal to GCB. And they are under the base.

[*conclusion*]
Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

Ex.: 7.1 (Arithmetical Theorem)

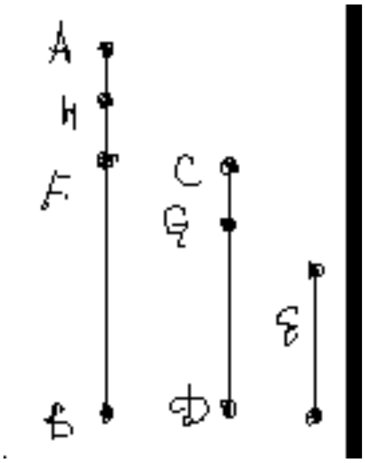
[*proposition*]
Two unequal numbers (being) laid down, and the lesser being continually subtracted, in turn, from the greater, if the remainder never measures the (number) preceding it, until a unit remains, then the original numbers will be prime to one another.

[*exposition*]

For two [unequal] numbers, AB and CD, the lesser being continually subtracted, in turn, from the greater, let the remainder never measure the (number) preceding it, until a unit remains.

[*specification*]

I say that AB and CD are prime to one another—that is to say, that a unit alone measures (both) AB and CD.



[*construction*]

For if AB and CD are not prime to one another then some number will measure them. Let (some number) measure them, and let it be E. And let CD measuring BF leave FA less than itself, and let AF measuring DG leave GC less than itself, and let GC measuring FH leave a unit, HA.

[*proof*]

In fact, since E measures CD, and CD measures BF, thus also measures BF. And (E) also measures the whole of BA. Thus, (E) will also measure the remainder and AF measures DG. Thus, E also measures DG. And (E) also measures the whole of DC. Thus, (E) will also measure the remainder CG. And CG measures FH. Thus, E also measures FH. And (E) also measures the whole of FA. Thus, (E) will also measure the remaining unit AH, (despite) being a number. The very thing is impossible. Thus, some number does not measure (both) the numbers AB and CD.

[*conclusion*]

Thus, AB and CD are prime to one another. (Which is) the very thing it was required to show.

<u>Remarks:</u>

1) *Proposition* of a Theorem IS a proposition in Frege's sense.
2) But the structure of a Theorem doesn't reduce to a *proposition* and its *proof*. Notice that Euclid concludes every Theorem with words "what was required to *show*" (deiknumi), not "what was required to *prove*" (apodeiknumi). His notion of proof is more specific. (The standard Latin translation of Euclid's "Hoper edei deixai" by "Quod erat demonstrandum" lost this important nuance.)
3) Axioms are principles of proof *similar* to logical principles.
4) Notice "arbitrary instantiation" in the *exposition*: "Let ABC be…"

5) Notice “subjective turn” or “instantiation of thinking subject” in *specification* : “I say that….”
6) *Construction* involves only instantiated concepts. It contains the core of mathematical argument.

<u>Conclusion on Euclid:</u>

Euclid’s theory is *deductive* in the sense that it is developed from first principles. However Euclid’s deduction doesn’t reduce to truth-preserving inference.

Where comes from the idea of a theory based on truth-preserving inferences from first truths?

Platonic view: Scientific knowledge (episteme) (as distinguished from dialectical knowledge, noesis) reduces to mathematical knowledge. Mathematical *Axioms* are general principles of scientific reasoning. Universal Mathematics (?) based on Axioms is a general theory of everything.

<u>Remark (on a candidate Universal Mathematics):</u> In Euclid’s *Elements* contain two independent theories of proportion: one for geometrical magnitudes (Books 5) and the other for numbers (Books 7-8). The two

theories clash in Book 10. However the idea of *general* theory of proportion has been discussed in Euclid's times:

"As for unifying bond of the mathematical sciences, we should not suppose it to be proportion, as Erathosphenes says. For though proportion is said to be, and is, one of the features common to all mathematics, there are many other characteristics that are all-pervading, so to speak, and intrinsic to the common nature of mathematics." (Proclus *Commentary* , 43.22-44.1, Murrow's translation)

In 16-17 centuries the general theory of proportion developed in a new algebraic setting was identified with Universal Mathematics (as a general theory of quantity).

Aristotle's view: Physical knowledge doesn't reduce to mathematical knowledge. Nature doesn't reduce to Form. Hence the idea of replacement of Universal Mathematics by another Universal Science (*First Philosophy* or Metaphysics), which would account for mathematical and physical objects on equal footing. The epistemic branch of this new Science Aristotle calls *Analytics*. It consists of a theory of proof (syllogistics of *Prior Analytics*) and general epistemology (*Posterior Analytics*). *Analytics* was later called *Logic* by Stoics.

Switching from Universal Mathematics to Physics and Logic:

"Just as the universal part of mathematics deals not with objects which exist separately, apart from extended magnitudes and numbers, but with magnitudes and numbers, not however *qua* such as to have magnitude or to be divisible, clearly it is possible that there should also be both propositions and demonstrations about sensible magnitudes, not however *qua* sensible but *qua* possessed of certain definite qualities." (*Met.* 1077b17-22, Ross' translation)

Remark: Here Aristotle counters Plato's argument according to which there is no possible science about sensibilia.

Switching from Mathematical Axioms to Logical Axioms (not to be confused with "immediate premises"!):

"By first principles of proof [as distinguished from first principles in general] I mean the common notions on which all men base their demonstrations, e.g. that one of two contradictories must be true, that it is impossible for the same thing both be and not to be, and all other propositions of this kind." (*Met.* 996b27-32, Heath's translation, corrected)

Remark: Aristotle extends here the notion of demonstration beyond mathematics.

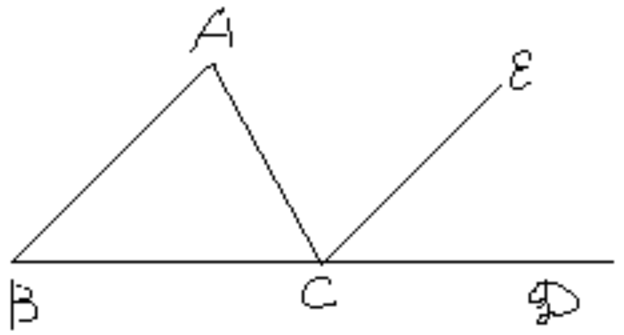
"We have now to say whether it is up to the same science or to different sciences to inquire into what in mathematics is called *axioms* and into [the general issue of] essence. Clearly the inquiry into these things is up to the same science, namely, to the science of the philosopher. For axioms hold of everything that [there] is but not of some particular genus apart from others. Everyone makes use of them because they concern being *qua* being, and each genus is. But men use them just so far as is sufficient for their purpose, that is, within the limits of the genus relevant to their proofs. Since axioms clearly hold for all things *qua* being (for being is what all things share in common) one who studies being *qua* being also inquires into the axioms. This is why one who observes things partly [=who inquires into a special domain] like a geometer or a arithmetician never tries to say whether the axioms are true or false" (*Met.* 1005a19-28, my translation)
"Since the mathematician too uses common [axioms] only on the case-by-case basis, it must be the business of the first philosophy to investigate their fundamentals. For that, when equals are subtracted from equals, the remainders are equal is common to all quantities, but mathematics singles out and investigates some portion of its proper matter, as e.g. lines or angles or numbers, or some other sort of quantity, not however *qua* being, but as [...] continuous." (*Met.* 1061b, my translation)

A difficulty acknowledged by Aristotle himself: syllogistic is incapable to account for a geometrical proof!

“Let *A* be two right angles, *B* triangle, *C* isosceles. Then *A* is an attribute of *C* because of *B*, but it is not an attribute of *B* because of any other middle term; for a triangle has [its angles equal to] two right angles by itself, so that there will be **no** middle term between *A* and *B*, **though *AB* is matter for demonstration**." (*An. Pr.* 48a33-37, Heath's translation, corrected, bold mine)

On the epistemic role of *construction* -

"Diagrams are devised by an activity, namely by dividing-up. If they had already been divided, they would have been manifest to begin with; but as it is this [clarity] presents itself [only] potentially. Why does the triangle has [the sum of its internal angles is equal to] two right angles? Because the angles about one point are equal to two right angles. If the parallel to the side had been risen [in advance], this would be seen straightforwardly" (*Met.* 1051a21-26, my translation)



Conclusion:

Aristotelian (scholastic) physics was not successful (from today's viewpoint). The era of Modern physics began when Galileo broke with the scholastic physics and re-introduced Platonic notion of mathematical physics (keeping a good deal of Aristotelian background, e.g. the understanding of importance of empirical evidence). This is just one reason to take critically Aristotle's notion of science in general and his notion of logic in particular.

The idea (dating back to Boole) of doing logic *mathematically* had the same modernising effect but this new trend has been countered by Bolzano, Frege, Russell and other modern Aristotelians. These people modernised logic in its technical aspects but at the same largely re-established the traditional Aristotelian (scholastic) understanding of logic as a technical part of Metaphysics. The study of "logical foundations of mathematics" in 20th century too often took this neo-scholastic setup for granted. As far as foundations are concerned the notion of **mathematical foundations of logic** may be more appropriate than that of logical foundations of mathematics.

The END