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Events and Intensional Sets[1]

1. Pairing and Halving

As Plato notices (*Phaed.* 96e-97b) there are two possible ways to get a pair. One (I shall call it *pairing*) is to bring two units together and the other (*halving*) is to cut a unit into two halves[2]. If the units are small objects like pebbles or coins then to use pairing is much easier, though. It would need a good deal of work to break a pebble or to cut a coin, while to bring such two things together you need nearly nothing: given two pebbles you get a pair of pebbles immediately.

Then you might wonder whether to "bring things together" or to "collect" means anything at all. Let us see. If you melt two coins and lump them together then you get a single lump but not a pair: to get a pair from the lump you would need to *halve* it. On the other hand if you halve a coin and take one half *too far* apart from the other you get two single halves; to make them a proper pair you would need to bring the halves back together - but again not *too much* together to avoid the need to apply halving again.

Such vagueness is obviously unbearable as far as a primitive concept like that of pair is concerned. Plato's decision is to rely on what he calls the eternal *form* of pair getting rid with the genetic accounts of the concept altogether. The up-to-date approach is instead to formalize the genetic accounts themselves. I take the latter approach in this paper.

Two melted and lumped coins cease to make a pair exactly when they presumably cease to exist and the lump instead comes into being. To avoid controversies about identity through time let us presume that pairing preserves identities of all the considered individuals (as well as differences between the individuals: so we do not allow that different individuals *become* identical by pairing[3]). As for the second part of the controversy, concerning the question of how far two units might stand from each other to be

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[2] This terminology is slightly confusing for I call *pairing* and *halving* two alternative ways to get a *pair*. I could use the term *coupling* instead of *pairing* but then it would not comply with the standard set-theoretic terminology which would later make a more serious confusion. Notice also that I use the term *half* loosely without assuming that two *halves* of a thing are of the same size (which is obviously irrelevant for the discussion).

[3] See Gallois, A.1998, *Occasions of Identity: A Study in the Metaphysics of Persistence, Change, and Sameness*, Clarendon Pr., Oxford

counted as collected into a pair, a popular approach is to ignore the problem by trivializing collection and allowing for pairing promiscuously:

Any two things make a pair (1)

We can strengthen (1) by supposing that

Any two things make an *unique* pair (1')

Clearly (1') amounts to how a pair is understood. By (1') all possible ways of pairing of given x,y (like putting x on the top of y or doing it the other way round) are identified: pair (x,y) is uniquely defined by its members x and y. We can say that identity of the pair is *borrowed* from that of its members. As we shall see in the part 3 of this paper this is a particular case of the *extensionality* principle.

The promiscuous pairing extrapolates what we know by our experience of manipulations with small objects like coins and pebbles over all sorts of things. Although mountains and stars unlike coins or pebbles cannot be put in a pocket we still think and talk about pairs of mountains or pairs of stars. Moreover we think and talk about pairs of sounds, colors, feelings, virtues, and what not. The only way we restrict pairing in everyday talk is by presuming that paired things are more or less of the same type. (For a pair whose members are, say, a whistle and a star seems odd.)

Such an application of a model that works for a limited domain to a wider domain by abstracting from certain constraints in many cases is well justified both epistemologically and practically. We have special devices which allow us to map habitual structures of common experience into domains where our experience is limited and not so well-structured. Particularly important device of this sort is *symbolization*: symbolizing stars by pebbles, for example, we can get a sky map. In some cases it also provides powerful means for manipulations with environments (not with stars so far though). However applying a chosen model indiscriminately we get a risk serious confusion. To avoid this we need to be sensitive to evidences showing limits of applicability of popular models, and to reserve alternative models for different cases. Indeed the intuition saying that only *close* things are to be counted as collected, however vague it might seem, makes good sense and is formalized straightforwardly. Allowing pairing and collection only for close elements we immediately get a system of opens and hence a topological structure. The standard way to do this is to put such a structure *on the top* of Set theory based on unrestricted collection. The above discussion shows a rationale behind attempts to build topology without the set-theoretic basis to make it a general framework in its own right[4].

Let us go back to halving. To avoid paradoxes about identity we need to move away from the brute halving by cuts and breaks. However we can try to save the method by making soft abstract cuts and breaks instead of real ones. Not really cutting a coin we can distinguish two halves of it. We need no more physical efforts to make this. Although one might doubt whether two halves of a coin make a pair in case they are not separated let us presume that they do. Then in the same vein as we did it with pairing we could suppose that

<u>Any thing has halves</u> (2)

However this still does not put halving and pairing onto equal footing. For even soft halving (2) obviously cannot be applied so promiscuously as pairing (1). There are many things which can be paired but not halved. For example you can think and talk about a pair of colors but not about halves of a color. Only things of a limited number of sorts have parts and halves in particular: material objects, texts, many believe that events also do[5]. This makes Mereology far more special a discipline than Set theory.

But cannot it be only a matter of convention? Do we not go too far applying the concept of pair approved by our experience of manipulation with small solid objects (particularly in symbolizing practices) to colors and the rest? Why not to extrapolate halving in a similar way?

Here perhaps is an answer. Let us presume that pairing is a special case of *collecting* (pairing is collecting of two things) while halving is a special case of *partitioning* (halving is partitioning into two proper parts). As far as pairs are identified by their members (see (1')) there is only one way that given x,y can be collected: the collection can be nothing but pair (x,y). Otherwise with partitioning. A given thing can generally speaking be partitioned into any number of parts but not necessarily two. While in case of collecting the result is wholly determined by what is collected in case of partitioning the result depends on how the process goes. To illustrate this point it is helpful again to use a brute physical example: if you break a pebble with a hammer you can hardly predict how much pieces you get. Halving is a precise job no matter whether you do it physically or abstractly.

Furthemore the replacement of physical halving by abstract halving does not really resolve all the controversies about identity. Consider this claim

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[4] Relevant approaches are developed in Formal Topology. For an elementary introduction see Vickers, S. 1989, *Topology Via Logic*, Cambridge University Press, Cambridge.

[5] See Varzi, A. (ed.) 2000, *Temporal Parts*: The Monist, Vol. 83, N3. In what follows I refer to events' parts informally not committing myself to the suggestion that mereology of objects is applicable to events too.

Any given thing has a *unique* (pair of) halves (2')

(2') unlike (1') does not sound plausible and we shall now see that there are good reasons to reject it. Suppose we would like to identify different ways of halving of given x by the same method as we identify different ways of pairing of given y and z. The method is that of borrowed identity. In the case of halving this would mean to identify pairs of halves by the wholes they originate from. But then we get a new problem: the borrowed identity allows us to distinguish between *pairs* of halves originated from different wholes but it is too inexact to distinguish between different halves of the *same* whole. This means that the identity of halves must be provided by halving itself: the halving must *mark* each half in a way, say, mark one half as "left" and the other as "right". Otherwise the halves are indistinguishable. It is not at all obvious that this may be done to satisfy (2'), our intuition rather suggests the opposite. This shows again a substantial difference between halving and pairing: unlike pairing halving cannot be trivialized.

2. Collection and Connection

So far we have discussed methods of getting pairs using examples of pairs of small objects like coins and pebbles. Now let us change the basic examples and speak about *events* instead of objects.

As has been already said we often think and talk about pairs of events and pairs of objects uniformly. My guess is that by doing this we think about events after the pattern given by objects (but certainly not the other way round)[6]. To prove this let us study the question of how to get a pair of events regardless of the question about a pair of objects considered above. It is important for this end to think about events as much realistically as about pebbles at hand. This is not so difficult because various events always happen around us, in our minds and our bodies. Anyone who so wishes can recognize them immediately. Just turn your attention to how you breath or blink your eyes, to what is going on in your immediate surroundings, etc. You might have nothing at hand at a moment but you can hardly ever find yourself in an event-free environment.

Can events be *collected* and particularly paired? Well, you might suggest that memory is a device for making such collections. But let us speak about events themselves first before we discuss memory. Apparently we have no control over past events but have some control over present and future events, moreover it is likely that we are able to make some events happen by our will. Particularly I can smoke a cigarette or wink my eye. By doing the two things simultaneously I produce an event *pair* which is my

[6] Cf. H. Bergson's critique of what he calls the *spatialization of time* which is the representation of times, events, and processes by spatial entities such as points and lines used in any sort of graphs showing how certain values vary with time.

smoking *and* winking[7]. But unlike the case of pebbles the applicability of the method is clearly very limited. During my travels I collect different types of coins used in different parts of the world. The coins originated from different places at different times are then all collected in my purse. I cannot do anything similar with events occurring in different places and at different times. Of course I cannot do it with rocks or buildings either but this looks more like a technical difficulty. With events the difficulty is apparently much harder. This makes me believe that in the case of events pairing and collection generally speaking fail: unlike objects events cannot be collected.

What about halving then? Following the strategy of "thinking realistically" let us choose an example of an event "at hand", or more precisely an event, which is actually going on near (or around, or better still *with*) you at the moment. For every reader's convenience I suggest the event of your present reading of this paper. Whenever you started and whenever you will finish (I mean the reading, not necessarily the paper) there is a part of the event, which has already passed, and there is another part of the event which is going to happen. Each of the two parts may be extremely short (you could just have started to read the paper from the middle and be about to drop it), but if you are actually now reading these lines both parts have some duration anyway. The two parts, namely the past part and the future part, look like two *halves* of the event, the present being the *boundary* between the two[8]. Surprisingly the two halves come already marked (one as past and the other as future) which resolves the difficulty about the identity of halves mentioned in the end of the previous section. Thus an event, which is "given" in the most immediate way, that is an event, which one immediately observes or participates in, is always halved in a sense into its past and future parts.

However the analogy between partitioning of an event of this or another sort and breaking a pebble or another case of partitioning of *object* is obviously loose. We cannot break an event into pieces or reconstruct it from given pieces although it is easily possible to do so with a spatial entity *representing* an event like a film or written text. In particular the past part of any ongoing event cannot be physically separated from its future part. It is arguable of course that the impossibility of separating parts physically does not prevent things from having parts and does not prevent us from recognizing the parts. I assume however that the separation must be at least in a weak (not necessarily physical) sense *possible*. I cannot see how the latter condition in the case of events can be met. This makes me doubt that we can apply the

[7] A famous example of an event pair given by D. Davidson is that of a rotating and warming up ball (Davidson, D. 1969, "The Individuation of Events" in *Essays in Honor of Carl G. Hempel*, ed. Rescher, N., Dordrecht, pp. 216-34)

[8] Alternatively one might say that the present is not one but two boundaries which are not identical but *coincide*: one of them bounds the past and the other bounds the future. See Smith, B. 1997, "Boundaries: An Essay in Mereotopology" in *The Philosophy of Roderick Chisholm* (Library of Living Philosophers), Open Court, LaSalle, pp. 534–561 for the discussion.

same concept of parthood for events and objects and have one mereological account for both. I conclude that halving fails in the case of events just as pairing does[9].

Now I'm going to suggest what can be considered as a third way to get a pair (or more precisely to get a *dual* of pair), which Plato has missed. Think again about the past (P) and the future (F) of your present reading (hereafter I avoid speaking about its past and future *parts*). P and F are two things, but they do not make a pair (collection) because they occur at different times. As I have argued above there is no way to "bring them together". The *symbols* P and F are together here on the paper (or the computer screen) but the things they symbolize by the above convention are not and cannot be. But if *events* P and F are not collected how can we think about those *two things* at all? How can we consider two or more things without collecting them? In fact there is another way of doing this. It is just as important for our common conceptual scheme as the collection. I shall call it *connection*. P and F are connected by the *present* (of) reading. The connection can be loosely thought of as a shared boundary between the two. The present reading enables us to think about its past and future without taking the two things *en bloc*.

A connection between events is certainly an event itself[10] and in many cases is naturally interpreted as a *change*. To see this think again about breaking a pebble. This event connects what happens before the pebble is broken with what happens afterwards.

Considering breaking a pebble we think both of the unbroken pebble and of a set of pebble's fragments resulted from the break. However we certainly do not think of the unbroken pebble *and* the fragments as of different items of the same collection (as we do it with the fragments alone).

The concept of connection is a temporal counterpart to that of collection, the latter being supported by our spatial intuition. A connection of *two* events is a temporal version of pairing. The formal account of connection developed in the next part of the paper justifies the duality between the two concepts. I suppose that it might be possible to develop a temporal version of the standard spatial mereology too but I leave this for a future study.

One might object that connection of events, unlike pairing of objects also works only in a very special case of subsequent events immediately following one after another. While collection allows us to pair any object with nearly any other the applicability of connection is apparently very restricted.

To extend the applicability of the method let us take *memory* into consideration. The basic fact about memory is not that it *collects* reminiscences but that it allows us to *recollect* past events. It is of course plausible to believe that recollecting past events we pick up reminiscences collected in our memory beforehand but this is an explanatory account. What I want to stress is how the recollection goes: we

[9] For mereotopology as applied also to events see Varzi, A.1997, "Boundaries, Continuity, and Contact", *Nous*, 31, pp. 26-58.

recollect events *one after another* but not simultaneously (except the special case when we recollect simultaneous events). We recollect past events, as well as plan or predict future events or just imagine fictitious events, under the same form under which we perceive present actual events: events go by series but not collections. There is an important difference though. Tracing the memory we are free to skip from one past event to another ignoring their real time order. Memory and temporal imagination connects events which are not connected in real time just as vision and spatial imagination collects objects which in no physical sense make collections. Normally we restrict this freedom by various principles including real time order, causality, associations, etc. However taking the same line as in case of pairing we can allow for connecting events promiscuously:

<u>Any two events are connected</u> (3)

We also can identify different ways to connect events by supposing that

<u>Any two events have a *unique* connection</u> (3')

Notice that the borrowed identity works with connection just as with pairing: a connection of two events is identified by the events it connects.

As well as in the case of unrestricted collection (Set theory) to make the picture realistic we must restrict connection with a topological structure. (Particularly because (3') ignores the events' order: breaking a pebble is not the same thing as putting its fragments together.) The relevance of topology is even more obvious in this case because the concept of connection is normally considered as topological. What we get with the unrestricted connection is a sort of trivialized topology open for further specification. Extensional set theories presuming unrestricted *collection* can be regarded as the same sort of trivialized topologies making no distinction between what is close and what is far[11].

3. Intensional Sets

By Cantor's famous word

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[10] This feature differs connections of events from boundaries of objects for in the latter case a boundary is not a thing of the same sort as what it bounds (say, a surface of a solid body is not a solid body).

[11] This is of course not a strict claim as far as the standard set-theoretical topological setting is concerned. For to get a trivial topology called *discrete* which counts every subset of given set X as open we need to fix X first.

By a "set" we understand every collection to a whole M of definite, well-differentiated objects m of our intuition or our thought. We call these objects the "elements" of M.[12]

Although "set" and "collection" sound like interchangeable terms this is not a tautology because it says that a set is a set *of* things called its elements. What are the elements? Cantor's explanation does not make it clear. What Cantor does make clear throughout development of his theory is the fact that sets *of* sets are allowed, i.e. that sets can be elements of other sets. This fact suggests the idea of applying the Occam razor, supposing that *sets* is the only sort of entities considered by the theory while the term "element" is relational: there is binary relation _ held between sets and read "belongs to", or "is an element of". This is the standard interpretation used in order to formalize Set theory, particularly with ZF system.

Thus when x_ y we say that x is an *element* of y. Let us introduce the complimentary relational term and call y a *host* of x under the same condition (cf. the case of *husband* and *wife*). This helps to formulate the question: <u>why think about sets in terms of their elements (as Cantor suggests) but not in terms of their hosts?</u>

To make the discussion more precise let me start with ZF. For simplicity I suppose the *equality* of sets to be a part of the underlying logic (logical identity); thus in what follows I speak about equality and identity of sets interchangeably. The first Extensionality axiom of ZF allows us to identify sets by their elements. Let us now replace it by Intensionality axiom which allows to identify sets by their *hosts* in a similar way. Formally it amounts to reversal of _ everywhere it occurs; this gives a reason to call the Extensionality and Intensionality axioms mutually *dual*:

ZF	**ZF***
_x_y(_z(z_ x_z_ y)_ x=y) **Extensionality**	_x_y(_z(x_ z_y_ z)_ x=y) **Intensionality**

The Extensionality axiom reduces the question of equality of sets to that of their elements: if sets have the same elements they are equal (the borrowed identity). What makes the axiom intuitively "obvious" and helpful is the sort of examples like the one about collecting coins: since particular coins normally survive collections of coins there is a reason to identify the collections by collected coins rather than the other way round.

However we actually do things the other round when we identify things by descriptions. For example by saying that a person is male we count him as an element of the set of (all) men, etc. This allows us to

[12] Cantor, G. 1895, "Beitraege zur Begruendung der transfiniten Mengenlehre", *Math. Ann.*, Bd.46, S.481-

interpret the Intensionality axiom as Leibniz' Law of Identity of Indiscernibles: if sets have (all) the same hosts (and hence the same properties) then they are equal. The fact that the axiom is naturally interpreted in terms of property talk explains why I call it Intensionality.

Consider now the following two definitions, which are also mutually dual in the same sense:

x is **atom** $=_{Def}$ __y(y_ x)	x is **world** $=_{Def}$ __y(x_ y)

or in words: atom is a set without elements and world is a set without hosts (i.e. a set which is not an element).

The fact that Extensionality allows for no more than one atom (called *the* empty set) shows that the idea to rely on intuition about collecting coins or pebbles to justify Extensionality is rather misleading. (For intuitively we treat such things as atoms but to have only one thing of the sort is certainly not enough to support the intuition.) Dually Intensionality allows for no more than one world; the fact that the hypothesis about the uniqueness of world unlike that about the uniqueness of atom does not seem counter-intuitive shows that we make different uses of Extensionality and Intensionality in our common reasoning.

It would be quite wrong to think that Extensionality is the only axiom of ZF, which amounts to the principle "think about sets in terms of their elements". I do not know a good term for the principle; it is tempting to call it extensionality in a generalized sense of the term but then we risk confusing it with the Extensionality axiom as formulated above. So let me leave it here as it stands. The most important axiom relying on the same principle is perhaps Pairing, which allows us to *construct* a set from two given elements. By dualizing Pairing in the same way we get Connection, which allows us to *extract* a certain element from two given sets:

_ a_ b(a_b_ _p_ x(x_ p_ (x=a_x=b))) **Pairing**	_ a_ b(a_b_ _p_ x(p_ x_ (x=a_x=b))) **Connection**

In the previous section we have already explained what intuitive sense we make of Connection: we must think about sets as events in time but not objects in space. This also explains why Pairing may be not guaranteed. The same applies to the dualized version of Union:

_ a(_b(b_ a)_ _y_ x(x_ y _ _z (x_ z _ z_	_ a(_b(a_ b)_ _y_ x(y_ x _ _z (z_ x _ a_

a))) **Union**	z))) **Intersection**

(While Union allows for a set with elements of elements of given set Intersection allows for a set with hosts of hosts of given set.)

Another important axiom without which sets cannot generally speaking be thought of as constructed of their elements is Foundation. It forbids cycles x_ ..._ x and chains ....x_ ... _ y infinite to the left. Intuitively the axiom says that every set is "ultimately made of" atoms (indeed the only atom). Its dualized version (which I call Upside Down Foundation because the term Anti-Foundation is used by Peter Aczel for a different purpose[13]) also forbids cycles and chains infinite to the right. It allows us to think of all sets as extracted from their hosts, and ultimately - from the unique world:

_y(y_ x)_ _y(y_ x _ _z_(z_ x_ z_ y) **Foundation**	_y(x_ y)_ _y(x_ y _ _z_(x_ z_ y_ z) **Upside Down Foundation**

To make Foundation work we need an atom and to make Upside Down Foundation work we need a world. ZF has *the* atom (empty set) provided by the Subsets axiom and has no worlds forbidden by the Power-set axiom (any set is an element of its power-set). Since worlds and atoms are mutually dual by dualizing the rest of the axioms of ZF we get a system with one world and no atoms[14]. I leave it as an exercise for the reader (more for imagination than technical skills) to figure it out what concepts dual to those of subset, power-set, and predicate look like (I term them *superelement*, *root-set*, and *abstractor* respectively). I only note that the fact that system ZF* we get is strictly atomless in my view gives an additional reason to use it to formalize our reasoning about events. For it allows us to avoid the concept of atomic event, which goes against the intuition that every event involves a change.

4. Perspectives and Conjectures

The system ZF* has all the formal properties of ZF. Formally speaking it is just the same system provided with a non-standard interpretation of _ . This reinterpretation is obtained by reading x_ y just as y_ x is normally read. The fact that by applying such an reinterpretation twice we come back to the starting point gives a reason to call the two different interpretations mutually dual. In another sense we can say that ZF* and ZF are *incompatible* meaning that by doubling of each axiom of ZF with its dual obtained by the replacement of x_ y by y_ x at every occurrence we get an *inconsistent* system ZF+ZF*. The

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[13] Aczel, P. 1988, *Non-Well-Founded Sets* (CSLI Lecture Notes, N14), CSLI Publications, Stanford.

[14] This explains why I use the term *world* instead of convenient *proper class*: worlds are sets but proper classes are not.

inconsistency of ZF+ZF* is proved for example by the fact that ZF has an atom (the empty set) but ZF* does not. (The fact that the definition of atom works equally for *both* ZF and ZF*, and hence for ZF+ZF* as a whole, is essential for this argument.) However some axioms and even some groups of axioms of ZF are apparently compatible with their duals as for example Extensionality and Intensionality or Extensionality+Pairing+Union and Intensionality+Connection+Intersection (think about standard extensional sets forming a "bundle" for the latter case). Systems obtained by taking some axioms of ZF together with their duals from ZF* are self-dual as ZF+ZF*: reinterpreting such a system by reading of x_ y as y_ x we does not get anything new.

Such self-dual systems might have interesting applications provided pairing/connection is restricted with an underlying topological structure as discussed in the second part of this paper. By elementary means we might get structures resembling differential manifolds (provided with local Cartesian frames in tangent spaces): suppose that pairing works only locally while *connection* between the *loci* (opens) is allowed but restricted in a certain way with a global topological structure. Generally if I am right that the standard extensional reading of ZF relies on spatial concepts and intuitions while its suggested intensional reading (ZF*) relies on temporal ones then the combined systems might allow us to build various *spatiotemporal* models.

Another possible development of the suggested approach is the following. Notice that dualizing ZF, i.e. exchanging its extensional reading for an intensional one, we did not touch the underlying logic, which was the standard first-order *extensional* calculus. Could not we do the same trick with the logic turning it into intensional one by a mere reinterpretation? Think about atomic propositions not as about short phrases like *Socrates is mortal* as it is normally done but as about extensive narratives or theories such that no two or more such things can be considered (or written down) simultaneously (say, because of limits of memory). Then a logical connective tying together two propositions so understood brings an "interface" or "translator" between the two. It is not clear however what would be exact meanings of standard logic connectives under this interpretation (although it is safe to fantasize about it as long as the formal machinery remains untouched), and it is likely that a deeper reform is needed to make the project viable.