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Leibniz, Grassmann and Peano
Rival Interpretations of Leibniz in early 1900-ies
HoTT perspective
Martin-Löf&Voevodsky
Concluding Remarks

Geometric Characteristics as an early form of typing

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Aims:

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Aims:

- ▶ To find a proper place of HoTT in the History of Ideas about Logic, Geometry and Computing from the late 19th c. onward;

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Aims:

- ▶ To find a proper place of HoTT in the History of Ideas about Logic, Geometry and Computing from the late 19th c. onward;
- ▶ To describe a philosophical context in which epistemological implications of HoTT could be fruitfully discussed.

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Claim 1

Leibniz' idea of *Geometric Characteristics* (not to be confused with the *Characteristica Universalis* !) further elaborated in the 19th century by H. Grassmann and G. Peano is more fully realized in today's Homotopy Type theory. It supports a pattern of interplay between Logic and Geometry, which is very unlike one that has been established in the 20th century through Hilbert's *Grundlagen of 1899*. While Hilbert and those who used his conception of formal axiomatic method purported to provide purely logical foundations of Geometry and other mathematical disciplines, the *Geometric Characteristics* and HoTT makes Logic into a part of Geometry.

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Claim 2

HoTT provides a new support to Cassirer's critique of Russell in their controversy about Foundations of Geometry in 1900-ies. This controversy provides a useful historical and philosophical context for today's discussion on epistemological, metaphysical and foundational issues related to HoTT.

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Characteristica Universalis 1676

I'm thinking about a new language or a rational system of writing, which could serve for communications between different peoples. With such an instrument we could solve metaphysical and moral problems just like geometrical problems. Any disagreement between them will reduce to an error of calculation. Philosophers like mathematicians could sit down and say: let's calculate.

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Leibniz 1679: to Huygens

I believe that we must have still another properly linear geometrical analysis, which directly expresses *situm* as algebra expresses *magnitudem*.

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Characteristica Geometrica 1679

Since the [traditional] letter notation of points in figures reflect geometrical properties of these figures, I wondered if any figure could be wholly represented by symbolic means in such way that anll geometrical problems could be solved by manipulating with symbols. This cannot be done with Algebra alone since an algebraic solution is always supported with a geometrical proof.

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Leibniz 1679: to Huygens

This new characteristic ... will not fail to give at the same time the solution, construction, and geometrical demonstration, the whole in a natural manner and by an analysis.

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Hermann Grassmann (1847) Geometrische Analyse, geknüpft an die von Leibniz erfundene Geometrische Charakteristik.

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Die Formenlehre oder Mathematik. Von Robert Grassmann. Stettin, 1872.

Ersters Buch: Die Grösenlehre

Zweites Buch: Die Begriffslehre oder Logik

Drittes Buch: Die Bindelehre oder Combinationslehre

Viertes Buch: Die Zahlenlehre oder Arithmetik

Fünftes Buch: Die Ausenlehre oder Ausdehnungslehre.

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Die Begriffslehre oder Logik

In order to ground the science of concepts or Logic, we should
proceed formally and represent all proofs by equations, which are
transformed according to rules provided by the science of
Magnitude. Only this method of proof presupposes no logic and no
grammar; this is the only method making thought rigorous. [. . .]
Logic constitutes the second branch of the science of Form aka
Mathematics, so it refers to definitions and rules of science of
Magnitude.

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Peano (1888)

Calcolo geometrico secondo l'Ausdehnungslehre di Hermann
Grassmann, preceduto delle operazioni della logic deductive

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Geometric Formations:

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Geometric Formations:

▶ numbers [the 0th species]

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Geometric Formations:

- ▶ numbers [the 0th species]
- ▶ linear combinations of points (the 1st species)

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Geometric Formations:

- ▶ numbers [the 0th species]
- ▶ linear combinations of points (the 1st species)
- ▶ linear combinations of lines (the 2nd species)

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Geometric Formations:

- numbers [the 0th species]
- linear combinations of points (the 1st species)
- linear combinations of lines (the 2nd species)
- linear combinations of triangles (the 3d species)

Aims and Claims
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Geometric Formations:

- ▶ numbers [the 0th species]
- ▶ linear combinations of points (the 1st species)
- ▶ linear combinations of lines (the 2nd species)
- ▶ linear combinations of triangles (the 3d species)
- ▶ linear combinations of tetrahedra (the 4th species)

Operations on Formations:

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Operations on Formations:

▶ summing formations of the same species,

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Operations on Formations:

- summing formations of the same species,
- multiplication of formations by number,

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Operations on Formations:

- summing formations of the same species,
- multiplication of formations by number,
- progressive product for formations of species n, m given that $m + n < 4$

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Propositions as formations of (-1)th species?

propositional univalence

1. The *equivalence* of two entities **a** and **b** of the system is defined, that is, a proposition, indicated by $\mathbf{a} = \mathbf{b}$, is defined, which expresses a condition between two entities of the system, satisfied by certain pairs of entities, and not by others, and which satisfies the logical equations:

$$(\mathbf{a} = \mathbf{b}) = (\mathbf{b} = \mathbf{a}), \quad (\mathbf{a} = \mathbf{b}) \cap (\mathbf{b} = \mathbf{c}) < (\mathbf{a} = \mathbf{c}).$$

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The geometric calculus is preceded by an introduction that treats of the operations of deductive logic; they present a great analogy with those of algebra and of geometric calculus. Deductive logic, which forms part of the science of mathematics, has not previously advanced very far, although it was a subject of study by Leibniz, Hamilton, Cayley, Boole, H. and R. Grassmann, Schröder, etc. The few questions treated in this introduction already constitute an organic whole, which may serve in much research. Many of the notations introduced are adopted in the geometric calculus.

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Are classes "Formations of the zeroth species?

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Russell: Principles of Mathematics 1903

[A]ll pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles.

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Hilbert 1899

Let us consider three distinct systems of things. The things composing the first system, we will call *points* . . . ; those of the second, we will call *straight lines* . . . and those of the third system, we will call *planes*. (1899)

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Pieri 1898 -1900, Hilbert&Bernays 1934, Tarski 1959

Géométrie envisagée comme un système purement logique (Pieri 1900): point-based axioms

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Tarski 1959

[E]lementary geometry is ... a theory with standard formalization
.... It is formalized within elementary logic, i.e., first-order
predicate calculus. All the variables $x, y, z, \ldots$ occurring in this
theory are assumed to range over elements of a fixed set ; the
elements are referred to as points, and the set as the space. The
logical constants of the theory are [follows the usual list]. As
non-logical constants ... we pick two ... : the ternary predicate β
used to denote the betweenness relation and the quaternary
predicate δ used to denote the equidistance relation

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Question

: What about yet another "system of things", namely the system of *propositions*? Is the fundamental distinction between *logical* and *extra-logical* (aka *non-logical*) terms is a type distinction?

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Leibniz Revival

@book
Russell:1900,
Author = B. Russell,
Publisher = London,
Title = A Critical Exposition of the Philosophy of Leibniz,
Year = 1900

@book
Couturat:1901,
Author = L. Couturat,
Publisher = Paris,
Title = La Logique de Leibniz. D'après des documents inédits,
Year = 1901

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Leibniz Revival

@book
Cassirer:1902,
Author = E. Cassirer,
Publisher = Marburg: Elwert,
Title = Leibniz' System in seinen wissenschaftlichen Grundlagen,
Year = 1902

Includes the *Kritische Nachtrag* with a critique of Russell 1900 and
Couturat 1901

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Russell: A Double Review

@article
Russell:1903,
Author = B. Russell,
Journal = Mind (New Series),
Number = 46,
Volume = 12,
Pages = 177-201,
Title = Recent Work on the Philosophy of Leibniz,
Year = 1903

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Russell 1903 on Couturat 1901

Leibniz's Geometrical Calculus, which is discussed in chapter ix., is distinctly disappointing. ... He failed to make a Geometrical Calculus, and merely introduced a new and less convenient system of co-ordinates, the system of bipolars or tripolars.

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Russell 1903 on Couturat 1901

The general conclusion, that Leibniz's logic was the true foundation
of his whole system, seems thus to be once for all demon-strated

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Russell 1903 on Cassirer 1902

Unlike M. Couturat, the present author [= Cassirer] has not yet
grasped the very modern discovery of the importance of Symbolic
Logic.

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Russell 1903 on Cassirer 1902

The criticisms which have been made in the above review are almost all of them criticisms of the Kantian philosophy itself, and those who accept that philosophy will find in Dr. Cassirer's book exactly what they desire.

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Russell - Cassirer Controversy

@book
Russell:1903 (!),
Author = B. Russell,
Publisher = London: Allen and Unwin,
Title = Principles of Mathematics,
Year = 1903

@article
Cassirer:1907,
Author = E. Cassirer,
Journal = Kant-Studien,
Volume = 12,
Pages = 1-40,
Title = Kant und die moderne Mathematik,

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Cassirer 1907 on Russell 1903

From the standpoint of logistics [= formal mathematics]rthe task
of thought ends when it manages to establish a strict deductive link
between all its constructions and productions. Thus the worry
about laws governing the world of objects is left wholly to the
direct observation, which alone, within its proper very narrow limits,
is supposed to tell us whether we find here certain rules or a pure
chaos. [According to Russell] logic and mathematics deal only with
the order of concepts and should not care about the order or
disorder of objects. As long as one follows this line of conceptual
analysis the empirical entity always escapes one's rational
understanding. The more mathematical deduction demonstrates us
its virtue and its power, the less we can understand the crucial role
of deduction in the theoretical natural sciences.

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Lawvere 1970

[E]xperience with sheaves, [..], etc., shows that a "set theory" for geometry should apply not only to *abstract* sets divorced from time, space, ring of definition, etc., but also to more general sets which do in fact develop along such parameters.

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Non-Statement View (P. Suppes et al.)

Theories are classes of models, not sets of propositions.

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Homotopy type theory: Univalent foundations of mathematics 2013

[W]hen types are viewed as propositions, they can contain more information than mere truth or falsity, and all "logical" constructions on them must respect this additional information. This suggests that we could obtain a more conventional logic by restricting attention to types that do not contain any more information than a truth value, and only regarding these as logical propositions. Such a type A will be "true" if it is inhabited, and "false" if its inhabitation yields a contradiction.

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Carry-Howard Restricted

Propositions-as-SOME-Types!

However

Every type is "reduced to proposition" through *truncation* of all its higher-order structure, i.e., through identification ("collapsing into one") of all its terms.

Conclusions

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Conclusions

▶ Leibniz' idea of Geometric Characteristicx, its transformations
  in the 19th century, and the related philosophical controversies
  of the beginning of the 20th century provide an appropriate
  context in which HoTT can be assessed from an
  epistemological viewpoint.

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Conclusions

▶ Leibniz' idea of Geometric Characteristicx, its transformations in the 19th century, and the related philosophical controversies of the beginning of the 20th century provide an appropriate context in which HoTT can be assessed from an epistemological viewpoint.

▶ HoTT supports the *Non-Statement VIew of theories* (P. Suppes et al.) by providing a precise sense in which a theory, generally does not reduce to the set of its propositions.

Conclusions

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Conclusions

▶ HoTT appears to be a more appropriate mathematical tool for representing the model-based reasoning in science and technology than more traditional tools such as Classical FOL. A reason for it is that HoTT is not only a formal symbolic calculus but also a constructive geometrical theory, which provides means for building complex models from simple elements.

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Conclusions

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- ▶ (Open Problem) The concept of *model* of HoTT needs a refinement.