Locality and Category Theory

Andrei Rodin[*]

[*] Kaliningrad State University

1 Introduction

Cartesian coordinates are ubiquitous in Modern mathematics and science. Particularly this is a usual tool to account for space and time, spacetime, and movement in space and time. It might even seem that to use Cartesian coordinates is the only way to give a strict mathematical account of those issues. In what follows I am trying to show that this is not the case suggesting an alternative tool, namely (abstract) categories. I argue that the categorial account of space and time better complies with both common everyday spatio-temporal thinking and principles of General Relativity (GR) and Quantum Mechanics (QM). Although I discuss only the case of space and time (spacetime) in the proper sense of the words the suggestion can be apparently generalized to cover different cases of abstract mathematical spaces and physical "parameter spaces".

2 Cartesian metaphysics, its epistemological implications, and contemporary science.

A closer look shows that the Cartesian coordinates are not a "neutral" mathematical tool as they seem to be but a tool which presupposes at least two strong metaphysical hypotheses. Although it is possible to weaken significance of the hypotheses distinguishing between mathematical apparatus used to account for physical reality and the physical reality itself, this strategy doesn't look promising and hardly can be followed too far. In any event the compatibility of principles on which a mathematical apparatus applied to reality is built with what we think about the reality itself should be considered as a priority. Particularly this concerns the case of a geometrical space used to represent the physical space or spacetime.

The two hypotheses about a geometrical space supplied by Cartesian coordinates are the following:

1) Space (as well as any object in it) consists of points;

2) Space is infinite.

The first hypothesis means that Cartesian coordinates provide a *pointwise* description of space and objects in it. Attempts to weaken the hypothesis in Aristotelian vein introducing a modal distinction between actual and possible points and/or the actual and the potential infinity became unpopular in mathematics since the Set Theory appeared on the scene. Nevertheless a similar trick which calls for the idea of potentially infinite series of increasingly precise geometrical representations of a physical object (which has the *exact* pointwise representation of the object as its limit case) still remains popular. Notice that this geometrical idea comes together with much more general epistemological idea of approximation to reality with increasingly precise theoretical models.

The problem is that the idea of infinite geometrical approximation openly contradicts to what physics says about the microstructure of physical objects. For QM justifiably shows that a pointwise (and even an arbitrarily close to pointwise) description of a trajectory in spacetime is generally speaking impossible. This result sounds paradoxically only if we believe that any precise account of space-time should be ultimately poinwise. The fact that GR supposedly gives a pointwise description of the physical spacetime creates a deep tension between GR and QM which makes problems for any attempt to build a quantum theory of gravitation[1].

The reasons to rule out the idea of infinite geometrical approximation suggested by QM should make us also rather skeptical about its epistemological counterpart, that is the idea of infinite approximation to reality by models. (Remember also how the former idea was witty challenged by B. Mandelbrot with his "fractal" examples[2].)

Provided that the physical space or spacetime is represented by a geometrical space the second hypothesis means that all the existing physical objects or events can be located with one and the same Cartesian coordinate system. The original idea was that any triple of Cartesian coordinates will do to locate any object in space (at any given moment of time) and any quadruple of coordinates will do to locate any event in spacetime. In this sense the Cartesian account of space, time or spacetime might be called universal. Apparently this second hypothesis also comes with an epistemological generalization which says that all physical objects and events are to be treated in some important sense uniformly. (This was a point of divergence of new Modern science from the old Aristotelian paradigm which supposed substantial difference between sublunary and celestial things.)

However GR allows only local frames. Cartesian coordinates in GR are applied locally, that is within a neighborhood of point of its origin. The gap between the universal character of Cartesian coordinates and their local use in GR at least causes technical inconveniences. But it is also plausible to suggest that this gap makes a serious conceptual difficulty for the whole theory.

Neither the Cartesian understanding of universality of scientific method exactly complies with methodological trends of contemporary science. Particularly Bohr's Correspondence Principle according to which microscopic and macroscopic accounts should comply with each other without being reducible to each other (or to a third theory), certainly goes beyond the Cartesian epistemology. In this case epistemological and geometrical ideas also go together.

[1]See Tian Yu Cao (forthcoming)
[2]See Benoit Mandelbrot (1980)

3 Is the Cartesian account of space intuitively clear?

Although the Cartesian coordinates are commonly known and widely used, this does not mean that the Cartesian account of space and time perfectly complies with common intuitions and everyday talks about those issues. In everyday life it is enough to say, for example, that *this book is on that shelf* - not trying to locate the book pointwise by point nor trying to make the shelf a part of an universal address system, that is identifying the book's place only *locally*. This allows to suggest that the categorial account of space and time presented below better complies not only with the today's physics but also with the today's common intuition.

4 Categories instead of coordinates

Formally speaking the suggested account is nothing but an *interpretation* in terms of space, time, and movement of basic concepts of the Category Theory[3]. Below I give only the interpreted version of the theory mentioning corresponding categorial concepts in brackets.

Suppose *things* $A, B, ...$ and mutual *placements* $f, g, ...$ of things $f : A \to B$ (to be read A is *placed* at B). Intuitively the latter are to be thought of after examples like this of a book on a shelf (but also of a coat on a peg – to include cases when a "place" is not a container). Notice that thing A can be placed at thing B generally speaking in many different ways $f, g, ...$ Then assume the following two axioms:

1) Placements are composable. That is $f : A \to B$ and $g : B \to C$ give an unique $gf : A \to C$.

2) Every thing is placed at itself $1_A : A \to A$ such a way that for any placement *at* $A f : X \to A$ $1_A f = f$ and for every placement *of* A $g : A \to Y$ $g1_A = g$.

(1) and 2) are standard axioms of an abstract category.)

So we have a number of *things* which are somehow *placed* with respect to each other. It is not supposed that *every* considered thing is somehow placed with respect to any other. Notice also that the whole considered category is to be thought of as representing a particular limited "situation" or "locus". This means that unlike Cartesian coordinates categories work only *locally*. How to bring a number of such "situations" together is a further question which I leave here aside.

<u>Def.1:</u> A **rests with respect to** B iff there is only one placement of A at B; A **moves with respect to** B iff there is more than one placement of A at B.

[3]For standard definitions of those concepts see for example R. Goldblatt (1979)

Intuitively this means that if A happens to be placed *differently* at B then it moves with respect to B; if A is placed at B uniquely then it rests with respect to B. The idea is that when A moves with respect to B it is differently placed at B *in different times*. But time is not defined yet, and I define it after movement, not the other way round. The fact that the categorial account allows to immediately define movement substantially differs it from the Cartesian account. In this sense the categorial account might be called *dynamic*.

Def.2: **Space** is a thing S of a given category such as any thing A of the category is placed at S, i.e. there is a placement $f : A \to S$.

The idea is simple: place is where *everything* is placed. The next definition shows however that Def.2 defines rather *spacetime* while *space* proper is its special case:

Def.3: A space S of a category is called **static** iff every thing *rests* at S, i.e. for any thing A there is one and only one arrow $A \to S$; a space which is not static is called *dynamic*.
(Static space is *terminal* object of given category.)

Lemma 1: If a category has a static space then it is unique (up to isomorphism). (This is a fact about *the* terminal object of a category.)

Static space might be naturally thought of as a "momentary photo" of the whole category. This gives a hint how to define time and a moment of time (Def.6).

Def.4: T is called **trajectory** of A iff (1) A is placed at T and (2) if any thing B is placed at T then B is placed at A.

The idea is that any trajectory T is exclusive in a sense that unlike just a "path" it can be traced by *only one* thing A plus things which are placed at A. Think about a 4D trajectory of a body in the Classical spacetime.

Def.5: **Time** is a trajectory of the static space (S) of the category.

The idea is that the static space (think about the Classical space) can move only "in time". Helpfully the following is true:

Lemma 2: If the static space S is placed at any thing T of given category then T is trajectory and hence is a time.

Notice that time as defined above is a dynamical space (by composition of placements).

Def.6: A placement $S \to T$ of the static space is called **moment**.
(This is the standard categorial definition of point.)

Given the definition of moment it is easy to define simultaneity of placements at given dynamical space. There is no natural way however to define simultaneity for the whole category at once; such an attempt unavoidably involves conditional "synchronizations" between different "times" (Def.5). Still given a chosen time and synchronizations we can define spatial *points* as simultaneous moments of different dynamical spaces. The fact that the notion of point is well-defined but not primitive in Category Theory seems to be an important advantage of this account. The categorial account suggests a view that points are not what a geometrical object consists of but what it may have under certain conditions.

Notice that no *order* of moments nor "arrow" of time is presupposed by Deff.5, 6. Nevertheless it can be shown how both things arise in most cases.

5 Ontological issues

A further question is about *things* supposed by this account. Are they objects or events or somewhat else? Consider the following definitions (which are mutually *dual* in Category-theoretical sense, that is transformable into each other by the reversal of arrows):

Def.7: **Extension** is a meet of placements (arrows)

Def.8: **Intension** is a split of placements (arrows)

A thing might have both extension and intension. As having an extension a thing is a "place" for other things, or rather a "situation" or *event* in which other things happen to be. As having an intension a thing A looks like an enduring and movable *object* (particle). Thus to be an event and to be an object might be thought of as *dual* aspects of the same thing (where the sense of being dual is defined as above).

6 Perspectives

Hopefully the aforementioned dual ontology can clarify the phenomenon of Quantum duality. It is helpful for this end that Deff.7, 8 are indeed more general than Classical conceptions of particle and event. The same is true about Def.4 of trajectory which does not imply that a trajectory is a linearly ordered set of points. Thus this categorial account can be hopefully applied to QM to clarify its metaphysical "mysteries" and probably to simplify its apparatus.

The fact that this account apparently fits principles of GR and later theories of gravitational field, and particularly complies with the idea that there is only one sort of extension (but not spatial extension and temporal extension), as well as the local character of this account, allow to suggest that the account might help to link conceptually GR and QM and thus to hint some principles for a future quantum theory of gravitation.

However the account as it is presented here seems to be too abstract to be applied to physical theories directly. To make such an application possible it seems necessary to develop the account with a certain model which is more familiar in physics, for example for the category of differentiable manifolds.

On the other hand given account in its abstract form allows to approach issues of space, time, movement and identity in a general metaphysical setting. Questions which might be treated with the suggested apparatus include among others the endurance/perdurance controversy, identity through time, and ontology of events[4].

References

[Goldblatt, 1979] Goldblatt, R. (1979): *Topoi: the Categorial Analysis of Logic.* Amsterdam—New York—Oxford.

[Mandelbrot, 1980] Mandelbrot, B. (1980): *The Fractal Geometry of Nature.* New York.

[Rodin,] Rodin, A. *Endurance, Perdurance and Quantum Duality.* submitted to International Journal of Philosophy of Science.

[Yu Cao, ming] Yu Cao, T. (forthcoming): "Prerequisites for a Consistent Framework of Quantum Gravity." *Studies in History and Philosophy of Modern Physics* .

[4] See Andrei Rodin (forthcoming)