Renewing Foundations

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- Doneddu's Plane Euclidean Geometry

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- To develop a dialectical historically-laden notion of foundation for mathematics and science

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- To sketch a theory of persistence and evolution of mathematical notions
- To make a number of claims about relationships between mathematics and philosophy (in the Conclusion)

Three versions of the (statement of the) Pythagorean theorem: Version 1: Euclid

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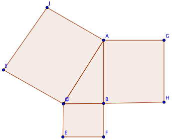
In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

(*Elements*, Proposition 1.47)



Three versions of the (statement of the) Pythagorean theorem: Version 2: Arnauld (1667)

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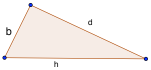
> *The square of hypothenuse is equal to (the sum of) squares of the two (other) sides (of the given rectangular triangle):* $bb + dd = hh$.

(*New Elements of Geometry, Proposition 14.26.4*)



Three versions of the (statement of the) Pythagorean theorem: Version 3: Doneddu (1965)

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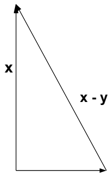
Two non-zero vectors x and y are orthogonal if and only if $(y - x)^2 = y^2 + x^2$

(Donnedu, *Euclidean plane geometry*)



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- Definition 1.8: And a plane angle is the inclination of the lines, when two lines in a plane meet one another, and are not laid down straight-on with respect to one another.

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WARNING: Equality in Euclid is NOT a binary relation.

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- Assumption 1.4: We assume that multiplication and division applies not only to number but to all magnitudes. ...

Assumption 1.5: .... When one supposes that a given magnitude is not generated by multiplication of some other magnitudes one considers this given magnitude as one-dimensional; such magnitudes are called linear. When one supposes that a given magnitude is generated by multiplication of two linear magnitudes one considers this given magnitude as two-dimensional; such magnitudes are called plane.

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- Definition: Segment is a non-ordered pair of two points called its boundaries
- Plane geometry is a theory that studies a basic set P called plane. Elements of this set are called points.
- Any subset of P is called a figure.
- There are two principle elements of geometry: (1) space viewed as a set of points and subsets of this set called figures; (2) group of transformation that determines the notion of "equality" and that is a source of geometrical properties of figures.

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This was only a VERY superficial semantical analysis of the *statement* of the theorem. As far as the structure of reasoning (proof) is taken into account the differences are even more striking!

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Question: Do they share anything in common?

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An analogy (only): 3D and 4D ontology. However dialectical links between older and newer foundations are neither causal nor contingent.

Two Metaphors of Foundations

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Contrary to what the architectural metaphor suggests foundations change while the rest remains stable (in the sense explained above) The architectural metaphor applies only locally, it doesn't work at larger historical scales. This is a serious reason for abandoning the term "foundations" as non-adequate but I shall not do this in the present discussion.

Two Metaphors of Foundations

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Neurath's Boat doesn't support the notion of *progress* (hereafter = accumulation of knowledge) The traditional metaphor supports the notion of progress locally but not globally.

Progress and Persistence of Mathematical Knowledge

The notion of progress requires that true knowledge, once it is acquired, then indefinitely continues to be available. (Think again about the Pythagorean theorem.) How is it possible? Where is it stocked?

Popper's Third World:

> *Knowledge in the objective sense consists not of thought processes but of thought contents. It consists of the content of our linguistically formulated theories; of that content which can be, at least approximately, translated from one language into another . The objective thought content is that which remains invariant in a reasonably good translation. Or more realistically put: the objective thought content is what the translator tries to keep invariant, even though he may at times find this task impossibly difficult.*

("Three worlds", 1978)

Claim: Popper's notion of thought content as an invariant of linguistic translation (may be adequate for describing the content of a religious doctrine but) is not appropriate for describing a mathematical and, more generally, scientific content.

Argument: A literal reproduction of written patterns never played the same role in mathematics (and science) as it did in religion and in general literature. The same holds for reproduction of written patterns up to a "reasonably good translation". Unlike religion, mathematics and science have never developed a culture of Great Books (original written sources preserving their textual identity). Mathematical texts are generally reproduced through a permanent revision including the *radical* revision, i.e., the revision of foundations. This makes a great difference (ignored by Popper) between the retention of mathematical (also scientific) contents and the retention of conceptual contents of other kinds.

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Today's systematic interest to old mathematical texts identified as written linguistic patterns (i.e. by usual philological rather than specific mathematical standards) is relatively recent (19th century). It belongs to history of mathematics but not to mathematics itself.

A false counter-example: Euclid's *Elements*

It is often claimed that until recently Euclid's *Elements* used to be a Bible of mathematics. In fact, the literature published under the title of "Euclid's Elements" since the beginning of book printing is surprisingly diverse. Revision of current versions of Euclid's book was until very recently a rule rather than an exception. The way in which Euclid's *Elements* preserved its identity through multiple re-publications differs strikingly from how poetical, philosophical and religious texts preserved their identity. The *Elements* barely existed as a fixed written literal pattern until such pattern was fixed by philologists (!) Heiberg and Menge in 1886. The alleged stickiness to Euclid's letter NEVER existed in mathematics! The history of revisions of Euclid's *Elements* still waits to be accounted for systematically.

To sum up:
Popper's notion of thought content as an invariant of linguistic translation is inadequate to mathematics and science. Mathematical and scientific knowledge cannot be stocked in libraries or in a special metaphysical world. This knowledge is continuously reproduced through research and education. Such reproduction has a character of non-trivial *renewal* rather than mechanical repetition of printed patterns or mere retention of their translational invariants.

Foundations and Education

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The need of renewal of scientific knowledge is related to the fundamental anthropological fact that the human race subsists and evolves through lives of multiple short-living individuals. There is no biological mechanism providing the accumulation of human knowledge. Every new generation, every new student should learn everything anew. It is well known that the non-trivial character of reproduction of short-living individuals (I mean the mutagenesis) is a crucial factor of biological evolution of a given biological species. I claim that the non-trivial character of reproduction of scientific knowledge is equally crucial for scientific progress.

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Example: Euclid's *Elements*

Why have Euclid's *Elements* been generally abandoned as a principal introductory mathematics textbook? Not because people found fallacies in it. Hardly because its standard of rigor was judged insufficient (an introductory textbook is not supposed to meet a high standard of rigor anyway). In principle, this happened because Euclid's *Elements* could not any longer serve its main purpose, that is, to provide a sufficient basis for further studies in any existing branch of mathematics. Euclid's *Elements* did this job perfectly well for a while but ceased to do this since mathematics significantly progressed.

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WARNING: I do NOT suggest that the change of foundations is caused by (is a reaction to) progress. I only claim that the non-trivial renewal of foundations that implies their change is a necessary condition of progress. Obviously, this condition is not sufficient.

Progress and Renewal of Foundations

The principle method of renewal of foundations is philosophical *dialectics* which should be definitely distinguished from scientific ways of reasoning appropriate in mathematics and sciences, which serve for achieving a progress in these fields. Strictly speaking, there is no possible progress in foundations. Renewal of foundations is a *condition* of scientific progress but itself does not qualify as a progress. Philosophy makes progress of mathematics and science possible but does not progress itself.

Example: The Pythagorean theorem

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Older versions of the Pythagorean theorem translate into its newer versions but, generally, not the other way round. To see this recall the "relativity of dimension" in Arnauld. "Geometrical algebra" translates into symbolic algebra but, generally, not the other way round.

Mathematical Foundations of mathematicians and Mathematical Foundations of philosophers

My proposed notion of foundations broadly agrees with what working mathematicians say about foundations. It less agrees with what many living philosophers say about foundations.

Manin on foundations

Cantors theory of the infinite had no basis in the older mathematics. You can argue about this as you like, but this was a new mathematics, a new way to think about mathematics, a new way to produce mathematics. In the final analysis, despite the arguments, the contradictions, Cantors universe was accepted by Bourbaki without apology. They created "pragmatic foundations", adopted for many decades by all working mathematicians, as opposed to normative foundations that logicists or constructivists tried to impose upon us. (....) So Bourbaki in fact did something completely different from what these guys (= philosophers, A.R.) think. (...)

Manin on foundations

A rebuilding of what I call the "pragmatic foundations of mathematics" will continue. By this I mean simply a codification of efficient new intuitive tools, such as Feynman path integrals, higher categories, the "brave new algebra" of homotopy theorists. (...) I am pretty strongly convinced that there is an ongoing reversal in the collective consciousness of mathematicians: the right hemispherical and homotopical picture of the world becomes the basic intuition, and if you want to get a discrete set, then you pass to the set of connected components of a space defined only up to homotopy. I see in this an analogy with a rebuilding of pragmatic foundations in terms of category theory and homotopic topology

I agree with what Manin says about the rebuilding of foundations but disagree with his notion of "pragmatic foundations". I assume that Manin calls foundations "pragmatic" because he is disappointed by the notion of foundations of mathematics developed in today's mainstream philosophy. I'm disappointed by this notion of foundations of mathematics too but I still believe that foundations of mathematics deserve and even require a systematic philosophical treatment. I also believe that normative claims can be appropriate in philosophy of mathematics.

Lawvere on foundations

A foundation makes explicit the essential general features, ingredients, and operations of a science, as well as its origins and generals laws of development. The purpose of making these explicit is to provide a guide to the learning, use, and further development of the science. A "pure" foundation that forgets this purpose and pursues a speculative "foundations" for its own sake is clearly a nonfoundation.

I tend to agree with what Lawvere says in this passage but I realize that the issue deserves a more systematic discussion. I definitely stick to the *audiatur altera pars* principle. The sad reality is that philosophically-minded mathematicians and mathematically-minded philosophers today barely manage to engage themselves into a rational dialog about foundations of mathematics.

What is Next?

Today, as ever, foundations of mathematics need a renewal. Whether set-theoretic foundations will stand or fall in the future does not seem me to be a serious issue. I don't see any reason why these particular foundations unlike their numerous predecessors may turn to be eternal.

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Unless the progress of mathematics stops tomorrow the set-theoretic foundation will soon be no longer adequate to this discipline. Many working mathematicians doing cut-edge research say that set-theoretic foundations of mathematics are already not quite adequate today.

What is Next?

Since the progress of mathematics accelerates (at least in long-term) it is even reasonable to predict that the life span of set-theretic foundations will be shorter than that of any older foundations. Thinking of set-theoretic foundations *as if* they were eternal seems me not only theoretically wrong but also irresponsible with respect to younger generations of our students.

What is Next?

A more pertinent issue is what is coming next. After Manin and Lawvere, I am enthusiastic about tentative category-theoretic foundations. This ongoing foundational renewal like any other requires further systematic philosophical efforts. I'll briefly describe my approach to this issue in the end of this talk (if I'll have time for it).

Historicicity of Mathematics

The ongoing replacement of set-theoretic foundations by tentative category-theoretic foundations is similar to any other foundational renewal occured in the past except that it proceeds faster than earlier renewals. This acceleration of foundational renewal is strongly correlated with the acceleration of mathematical progress (without being causally dependent). The acceleration of renewal increases the importance of *historical thinking* in mathematics and its philosophy.

Historicicity of Mathematics

Even if a future New Math project will produce in few decades a complete set of introductory mathematical courses based on purely category-theoretic notions - just like the older New Math project did this few decades ago using the naive Set theory - students might still profit from being capable to read older Bourbaki-style mathematical publications. Such reading may turn to be NOT ONLY of purely historical interest: some people may produce new mathematics using these older sources. This is why I believe that future foundations of mathematics should explicitly involve major older foundations and provide some account of the dialectics of their renewal.

Historicicity of Mathematics

Stressing the importance of history for mathematics and its philosophy I assume that the projected future is more important than the reconstructed past. So I do not suggest here to confuse doing new mathematics and new philosophy of mathematics with doing history of these subjects. I rather suggest that the future cannot be reasonably projected and theoretical efforts cannot be properly directed unless the past is well understood and taken into account in our today's actions. This is why I think that the old habit of thinking about mathematics and its foundations *sub species aeternitatis* is misfortunate and must be definitely abandoned.

Historicicity of Mathematics

Philosophy of mathematics, which uses mathematical texts published a century ago for making up timeless metaphysical theories about the essence of mathematics and mathematical objects, cannot be appealing for working mathematicians, mathematical teachers or any other people interested in mathematics. What philosophy of mathematics can and, in my view, should produce (and permanently reproduce) is a foundation capable to support and facilitate further progress of mathematics preserving its systematic unity. Making outdated mathematical approaches into refined a-historical metaphysical theories about mathematics does not help philosophy to fulfill this task.

Structuralism and set-theoretic foundations

Set-theoretic foundations of mathematics allow for building mathematical objects as *structured sets*. A structured set consists of (1) a base set and (2) a system of relations between elements of the base set, which are specified by appropriate axioms. Crucial for set-theretic foundation is the notion of *isomorphism* between structured sets, i.e. one-to-one correspondence between elements of the given sets that preserves relations between these elements.

Structuralism and set-theoretic foundations

In set-theoretic mathematics the notion of isomorphism plays, roughly, the same role as the notion of equality (not to be confused with logical identity) plays in traditional mathematics. In set-theoretic mathematics structured sets are thought of "up to isomorphism"; a structured set thought of up to isomorphism is colloquially called a *structure*. A philosophy of mathematics supporting this way of thinking about mathematical object is called Mathematical Structuralism.

Structuralism and set-theoretic foundations

Controversies about Structuralism in set-based mathematics stressed by Benacerraf and other authors are related to controversial attempts to apply the same structural method for construing sets themselves, i.e., attempts to construe sets as structured sets. This is exactly what is going on in axiomatic theories of sets like ZF where sets are construed as sets holding a single primitive relation called "membership". (I use here the term "set" informally as synonymous to "collection". In order to avoid the obvious circularity, people distinguish between these things.) However important these controversies can be they have never suggested a new way of doing mathematics that could compete with the structural set-theoretic way of doing mathematics briefly outlined above.

Categorical foundations

An isomorphism can be described as a *reversible transformation*: one-to-one correspondence is a symmetric construction; it has no preferred direction. Thinking about isomorphism as reversible transformation one can describe a structure as an invariant of such transformations of an appropriate kind. In this setting, one can consider transformations of a more general sort called *morphisms* or *functors*, which are, generally, not reversible. Think about group homomorphisms, for example. The project of category-theoretic foundations of mathematics amounts to the attempt of rebuilding the core of older mathematics, and developing new mathematics, on the basis of the notion of morphism (functor) taken as primitive.

Structuralism and category-theoretic foundations

Many enthusiasts of category-theoretic foundations of mathematics consider such foundations as a better realization of principles of Mathematical Structuralism. I disagree. One cannot think "up to homomorphism" in anything like the same way in which people think up to isomorphism doing structuralist mathematics. Non-reversible transformations unlike reversible ones, generally, have no invariants. So in the new setting one cannot, generally, describe mathematical objects as structures. Instead of studying invariants of reversible transformations categorical mathematics studies transformations themselves.

About dialectics

Finally, I would like to stress that the replacement of standard set-theoretic foundations by categorical foundations is not a matter of taste. Unlike Manin, I don't think about this replacement as a pragmatic solution either. The notion of non-reversible transformation (morphism) is a straightforward generalization of that of isomorphism. There is an objective order of ideas, that leads one from thinking of sets and structures to thinking of categories and functors. It goes without saying that this order exists only in human collective thinking, that is in human intellectual history. This order of ideas is not a deductive order. It is a *dialectical* order.