PLURALISM IN GEOMETRY AND LOGIC: A CATEGORICAL APPROACH

Andrei Rodin

UNILOG 2010

More than an analogy...

Content:

A popular story

A popular story

- Since Antiquity people tried to prove Parallel Postulate (P5) on the basis of other principles of Euclidean geometry by the *reductio ad absurdum* but the desired contradiction didn't show up.

A popular story

- Since Antiquity people tried to prove Parallel Postulate (P5) on the basis of other principles of Euclidean geometry by the *reductio ad absurdum* but the desired contradiction didn't show up.
- Finally some of these people (Gauss, Boliay, Lobatchevsky) guessed that they are exploring a new vast territory rather than approaching the expected dead end.

A popular story

- Since Antiquity people tried to prove Parallel Postulate (P5) on the basis of other principles of Euclidean geometry by the *reductio ad absurdum* but the desired contradiction didn't show up.
- Finally some of these people (Gauss, Boliay, Lobatchevsky) guessed that they are exploring a new vast territory rather than approaching the expected dead end.
- The issue remained highly speculated until Beltrami in 1868 provided a model of Lobachevskian (Hyperbolic) geometry.

A popular story

- Since Antiquity people tried to prove Parallel Postulate (P5) on the basis of other principles of Euclidean geometry by the *reductio ad absurdum* but the desired contradiction didn't show up.
- Finally some of these people (Gauss, Boliay, Lobatchevsky) guessed that they are exploring a new vast territory rather than approaching the expected dead end.
- The issue remained highly speculated until Beltrami in 1868 provided a model of Lobachevskian (Hyperbolic) geometry.
- Finally Hilbert in his *Grundlagen der Geometrie* of 1899 put things in order by distinguishing between formal theories and their models.

A popular story

- Since Antiquity people tried to prove Parallel Postulate (P5) on the basis of other principles of Euclidean geometry by the *reductio ad absurdum* but the desired contradiction didn't show up.
- Finally some of these people (Gauss, Boliay, Lobatchevsky) guessed that they are exploring a new vast territory rather than approaching the expected dead end.
- The issue remained highly speculated until Beltrami in 1868 provided a model of Lobachevskian (Hyperbolic) geometry.
- Finally Hilbert in his *Grundlagen der Geometrie* of 1899 put things in order by distinguishing between formal theories and their models.
- Thus the geometrical pluralism has been firmly established.

Claim:

This view is VERY anachronistic: it takes Hilbert's *formalist* view of 1899 for granted, while in fact this view couldn't possibly emerge BEFORE the discovery of Non-Euclidean geometries.

Closer to Real History

Closer to Real History

- People worked on P5 because unlike other principles of Euclidean geometry this Postulate has no strong intuitive support. (The popular view according to which the "usual" geometrical intuition is Euclidean doesn't stand against this historical evidence.)

Closer to Real History

- People worked on P5 because unlike other principles of Euclidean geometry this Postulate has no strong intuitive support. (The popular view according to which the "usual" geometrical intuition is Euclidean doesn't stand against this historical evidence.)
- Boliay and Lobachevsky aimed at a general notion of geometry *independent* of P5, not at an alternative geometry. They called it *absolute* geometry (Boliay) or *pangeometry* (Lobachevsky).

Closer to Real History

- People worked on P5 because unlike other principles of Euclidean geometry this Postulate has no strong intuitive support. (The popular view according to which the "usual" geometrical intuition is Euclidean doesn't stand against this historical evidence.)
- Boliay and Lobachevsky aimed at a general notion of geometry *independent* of P5, not at an alternative geometry. They called it *absolute* geometry (Boliay) or *pangeometry* (Lobachevsky).
- This general theory split itself into parts in a rather unusual way (Unification Problem)

Closer to Real History

- People worked on P5 because unlike other principles of Euclidean geometry this Postulate has no strong intuitive support. (The popular view according to which the "usual" geometrical intuition is Euclidean doesn't stand against this historical evidence.)
- Boliay and Lobachevsky aimed at a general notion of geometry *independent* of P5, not at an alternative geometry. They called it *absolute* geometry (Boliay) or *pangeometry* (Lobachevsky).
- This general theory split itself into parts in a rather unusual way (Unification Problem)
- Beltrami in 1868 discovered a link between the problem of parallels (Lobachevsky) and the geometry of curved surfaces (Gauss) and curve spaces (Riemann).

Unification Problem: Klein's solution of 1871

Unification Problem: Klein's solution of 1871

- The appropriate notion of "absolute" geometry is that of *projective* geometry (but developed in an abstract way independently of the Euclidean background)

Unification Problem: Klein's solution of 1871

- The appropriate notion of "absolute" geometry is that of *projective* geometry (but developed in an abstract way independently of the Euclidean background)
- By specifying a "projective metric" one obtains a Riemannian manifold of constant curvature K.

Unification Problem: Klein's solution of 1871

- The appropriate notion of "absolute" geometry is that of *projective* geometry (but developed in an abstract way independently of the Euclidean background)
- By specifying a "projective metric" one obtains a Riemannian manifold of constant curvature K.
- The case $K = 0$ gives *parabolic* (Euclidean) geometry, the case $K < 0$ gives (a family of) *hyperbolic* (Lobachevskian) geometries and the case $K > 0$ gives the new family of *elliptic* (in particular Spherical) geometires.
- Open problem: *where live Riemannian manifolds?* (Shared Space Problem)

Unification Problem: Hilbert's solution of 1899

Unification Problem: Hilbert's solution of 1899

- Ultimate Foundations of Mathematics (and of the rest of Science) is Logic

Unification Problem: Hilbert's solution of 1899

- Ultimate Foundations of Mathematics (and of the rest of Science) is Logic
- Any formal theory is OK as far as its logical properties (consistency, parsimony) are OK

Unification Problem: Hilbert's solution of 1899

- Ultimate Foundations of Mathematics (and of the rest of Science) is Logic
- Any formal theory is OK as far as its logical properties (consistency, parsimony) are OK
- A desired extra property: categoricity

Hilbert's solution: some extra features

Hilbert's solution: some extra features

- A *metaphorical* solution of Shared Space Problem: different spaces and their corresponding spaces live in *the* space of logical possibilities (??)

Hilbert's solution: some extra features

- A *metaphorical* solution of Shared Space Problem: different spaces and their corresponding spaces live in *the* space of logical possibilities (??)
- As far as logics are *many* Hilbert's solution of the Unification Problem doesn't work! The Hilbertian framework is incompatible with Logical Pluralism.

Preliminaries: Extrinsic and Intrinsic geometry

Preliminaries: Extrinsic and Intrinsic geometry

- Relativization of the objecthood and the spacehood (Gauss): every geometrical object is *intrinsically* a space; every geometrical space is *extrinsically* an object.

Preliminaries: Extrinsic and Intrinsic geometry

- Relativization of the objecthood and the spacehood (Gauss): every geometrical object is *intrinsically* a space; every geometrical space is *extrinsically* an object.
- Importance of (all) *maps* between spaces (Gauss, Lobachevsky, Klein, Grothendieck)

Preliminaries: Extrinsic and Intrinsic geometry

- Relativization of the objecthood and the spacehood (Gauss): every geometrical object is *intrinsically* a space; every geometrical space is *extrinsically* an object.
- Importance of (all) *maps* between spaces (Gauss, Lobachevsky, Klein, Grothendieck)
- INTRINSICALLY = as described in terms of *incoming* maps (unlike Riemann no "absolute" meaning of "intrinsic"; "intrinsic" $\neq$ "essential"!)

Preliminaries: Extrinsic and Intrinsic geometry

- Relativization of the objecthood and the spacehood (Gauss): every geometrical object is *intrinsically* a space; every geometrical space is *extrinsically* an object.
- Importance of (all) *maps* between spaces (Gauss, Lobachevsky, Klein, Grothendieck)
- INTRINSICALLY = as described in terms of *incoming* maps (unlike Riemann no "absolute" meaning of "intrinsic"; "intrinsic" $\neq$ "essential"!)
- EXTRINSICALLY = as described in terms of *outgoing* maps

Extrinsic and Intrinsic geometry: Examples

Distinguish between two notions of Euclidean plane:

Extrinsic and Intrinsic geometry: Examples

Distinguish between two notions of Euclidean plane:

- the universe of Euclidean planimetry (*ESPACE*) and

Extrinsic and Intrinsic geometry: Examples

Distinguish between two notions of Euclidean plane:

- the universe of Euclidean planimetry (*ESPACE*) and
- an object of Euclidean stereometry (an *eplane* in *ESPACE*)

Extrinsic and Intrinsic geometry: Examples

Distinguish between two notions of Euclidean plane:

- the universe of Euclidean planimetry ($ESPACE$) and
- an object of Euclidean stereometry (an *eplane* in $ESPACE$)
- $eplane : EPLANE \rightarrow ESPACE$

Extrinsic and Intrinsic geometry: Examples

Distinguish between two notions of Euclidean plane:

- the universe of Euclidean planimetry (*ESPACE*) and
- an object of Euclidean stereometry (an *eplane* in *ESPACE*)
- $eplane : EPLANE \to ESPACE$
- $horosphere : EPLANE \to LSPACE$ (Hyperbolic space)

Extrinsic and Intrinsic geometry: Examples

Distinguish between two notions of Euclidean plane:

- the universe of Euclidean planimetry (*ESPACE*) and
- an object of Euclidean stereometry (an *eplane* in *ESPACE*)
- $eplane : EPLANE \longrightarrow ESPACE$
- $horosphere : EPLANE \longrightarrow LSPACE$ (Hyperbolic space)
- Intrinsically horospheres and eplanes are the same but extrinsically they are very different! There is no point in saying that they are "essentially" the same (just different models of the same thing...)

Categorical geometry 1:

Geometry deals NOT ONLY with...

Categorical geometry 1:

Geometry deals NOT ONLY with...

- invariants under automorphisms (=reversible maps of a given space to itself)

Categorical geometry 1:

Geometry deals NOT ONLY with...

- invariants under automorphisms (=reversible maps of a given space to itself)
- groups of automorphisms (Klein's program)

Categorical geometry 1:

Geometry deals NOT ONLY with...

- invariants under automorphisms (=reversible maps of a given space to itself)
- groups of automorphisms (Klein's program)
- BUT with ALL maps between spaces

Categorical geometry 2:

INDEED

Categorical geometry 2:

INDEED

- an attempt to reduce Geometry to Group Theory (Klein) brings Homology and Cohomology theories, not a reduction

Categorical geometry 2:

INDEED

- an attempt to reduce Geometry to Group Theory (Klein) brings Homology and Cohomology theories, not a reduction
- the language of categories and functors turns to be the most convenient in these theories (Eilenberg and Steenrod)

Categorical geometry 3:

Categorical geometry 3:

- a solution of Shared Space Problem: spaces/objects live in a CATEGORY

Categorical geometry 3:

- a solution of Shared Space Problem: spaces/objects live in a CATEGORY
- a category of spaces has geometrical properties itself: cf. the above definitions of “extrinsic” and “intrinsic”

Categorical geometry 3:

- a solution of Shared Space Problem: spaces/objects live in a CATEGORY
- a category of spaces has geometrical properties itself: cf. the above definitions of “extrinsic” and “intrinsic”
- a more developed notion: *Grothendieck topology*

Categorical logic: methods of theory-building

Categorical logic: methods of theory-building

- a method of theory-building is NOT a matter of pure logic BUT

Categorical logic: methods of theory-building

- a method of theory-building is NOT a matter of pure logic BUT
- a matter of *presentation*

Categorical logic: methods of theory-building

- a method of theory-building is NOT a matter of pure logic BUT
- a matter of *presentation*
- generic constructions: Euclid's *Postulates* (as distinguished from *Axioms*)

Categorical logic: methods of theory-building

- a method of theory-building is NOT a matter of pure logic BUT
- a matter of *presentation*
- generic constructions: Euclid's *Postulates* (as distinguished from *Axioms*)
- Presentation of groups, Sketch theory

Categorical logic 1:

LOGIC deals NOT ONLY with...

Categorical logic 1:

LOGIC deals NOT ONLY with...

- invariants under automorphisms of universes of discurses (Tarski)

Categorical logic 1:

LOGIC deals NOT ONLY with...

- invariants under automorphisms of universes of discurses (Tarski)
- groups of such automorphisms

Categorical logic 2:

BUT

Categorical logic 2:

BUT

with categories of *translations* between various bodies of *contentual* reasoning.

Categorical logic 3:

To reason *logically* means

Categorical logic 3:

To reason *logically* means

- NOT to fit the reasoning into an adequate logical *form* BUT

Categorical logic 3:

To reason *logically* means

- NOT to fit the reasoning into an adequate logical *form* BUT
- to be *translatable* (= to fit the reasoning into an adequate translational protocol).

Categorical logic 3:

To reason *logically* means

- NOT to fit the reasoning into an adequate logical *form* BUT
- to be *translatable* (= to fit the reasoning into an adequate translational protocol).
- "Good" translations are NOT those that *preserve* something BUT

Categorical logic 3:

To reason *logically* means

- NOT to fit the reasoning into an adequate logical *form* BUT
- to be *translatable* (= to fit the reasoning into an adequate translational protocol).
- "Good" translations are NOT those that *preserve* something BUT
- those having *universal properties* (limits, colimits).

Categorical logic 4:

Categorical logic 4:

- Global features of logical categories (truth-values) arise from such local interactions.

Categorical logic 4:

- Global features of logical categories (truth-values) arise from such local interactions.
- Compare how topological features arise in Riemannian geometry.

Categorical logic 4:

- Global features of logical categories (truth-values) arise from such local interactions.
- Compare how topological features arise in Riemannian geometry.
- Example: Topos logic

Categorical logic 4:

- Global features of logical categories (truth-values) arise from such local interactions.
- Compare how topological features arise in Riemannian geometry.
- Example: Topos logic
- Other?

Logical Pluralism?

Logical Pluralism?

- Yes but only with strong *unification principles*

Logical Pluralism?

- Yes but only with strong *unification principles*
- Universal Logic is NOT a minimal logical structure BUT

Logical Pluralism?

- Yes but only with strong *unification principles*
- Universal Logic is NOT a minimal logical structure BUT
- a universal TRANSLATIONAL PROTOCOL.

Logical Pluralism?

- Yes but only with strong *unification principles*
- Universal Logic is NOT a minimal logical structure BUT
- a universal TRANSLATIONAL PROTOCOL.
- Is Category theory an adequate mathematical tool for it?

Logical Pluralism?

- Yes but only with strong *unification principles*
- Universal Logic is NOT a minimal logical structure BUT
- a universal TRANSLATIONAL PROTOCOL.
- Is Category theory an adequate mathematical tool for it?
- Is Topos logic the only interesting notion of categorical logic?

THE END