PLURALISM IN GEOMETRY AND LOGIC: A CATEGORICAL APPROACH

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More than an analogy...

Content:

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- Finally Hilbert in his *Grundlagen der Geometrie* of 1899 put things in order by distinguishing between formal theories and their models.
- Thus the geometrical pluralism has been firmly established.

Claim:

This view is VERY anachronistic: it takes Hilbert's *formalist* view of 1899 for granted, while in fact this view couldn't possibly emerge BEFORE the discovery of Non-Euclidean geometries.

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- This general theory split itself into parts in a rather unusual way (Unification Problem)
- Beltrami in 1868 discovered a link between the problem of parallels (Lobachevsky) and the geometry of curved surfaces (Gauss) and curve spaces (Riemann).

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- The appropriate notion of "absolute" geometry is that of *projective* geometry (but developed in an abstract way independently of the Euclidean background)
- By specifying a "projective metric" one obtains a Riemannian manifold of constant curvature K.
- The case $K = 0$ gives *parabolic* (Euclidean) geometry, the case $K < 0$ gives (a family of) *hyperbolic* (Lobachevskian) geometries and the case $K > 0$ gives the new family of *elliptic* (in particular Spherical) geometires.
- Open problem: *where live Riemannian manifolds?* (Shared Space Problem)

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- A desired extra property: categoricity

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- A *metaphorical* solution of Shared Space Problem: different spaces and their corresponding spaces live in *the* space of logical possibilities (??)
- As far as logics are *many* Hilbert's solution of the Unification Problem doesn't work! The Hilbertian framework is incompatible with Logical Pluralism.

Preliminaries: Extrinsic and Intrinsic geometry

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- $eplane : EPLANE \longrightarrow ESPACE$
- $horosphere : EPLANE \longrightarrow LSPACE$ (Hyperbolic space)
- Intrinsically horospheres and eplanes are the same but extrinsically they are very different! There is no point in saying that they are "essentially" the same (just different models of the same thing...)

Categorical geometry 1:

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- groups of automorphisms (Klein's program)
- BUT with ALL maps between spaces

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- an attempt to reduce Geometry to Group Theory (Klein) brings Homology and Cohomology theories, not a reduction
- the language of categories and functors turns to be the most convenient in these theories (Eilenberg and Steenrod)

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- a category of spaces has geometrical properties itself: cf. the above definitions of "extrinsic" and "intrinsic"
- a more developed notion: *Grothendieck topology*

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- generic constructions: Euclid's *Postulates* (as distinguished from *Axioms*)
- Presentation of groups, Sketch theory

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- groups of such automorphisms

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with categories of *translations* between various bodies of *contentual* reasoning.

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- to be *translatable* (= to fit the reasoning into an adequate translational protocol).
- "Good" translations are NOT those that *preserve* something BUT
- those having *universal properties* (limits, colimits).

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- Other?

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- a universal TRANSLATIONAL PROTOCOL.
- Is Category theory an adequate mathematical tool for it?
- Is Topos logic the only interesting notion of categorical logic?

THE END