

# Axiomatic Method between Logic and Geometry

## Talk 1: Axiomatic Geometry according to Euclid and according to Hilbert.

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# Claim 1

The received notion of axiomatic theory stemming from Hilbert 1899 is not adequate to the recent successful practice of axiomatizing mathematical theories. In particular, the axiomatic architecture of the elementary Topos theory and (more obviously) of the Homotopy type theory (HoTT) does not fit into the standard Hilbertian pattern of formal axiomatic theory.

## Claim 2

At the same time, Topos theory and HoTT both fall under a more general and in many respects more traditional notion of axiomatic theory stemming from Euclid's *Elements*. Using some elements of HoTT I shall formulate the notion of constructive axiomatic theory in precise terms.

# Claim 3

This broader notion of axiomatic theory, which I call after Hilbert and Bernays *constructive*, is more suitable for being used as a formal framework in physics and other sciences than the received notion.

# Plan of Talk 1

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- ▶ Euclid : Doing and Showing

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- ▶ Hilbert : Form and Object. Genetic Method versus Axiomatic Method

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- ▶ Euclid : Doing and Showing
- ▶ Hilbert : Form and Object. Genetic Method versus Axiomatic Method
- ▶ Constructive Axiomatic Method after Hilbert & Bernays



# Plan of Talk 2

- ▶ Typing and Curry-Howard Correspondence : Doing is Showing ?
- ▶ Geometry and Logic in Topos theory
- ▶ Propositional and Non-Propositional types in HoTT : Curry-Howard reconsidered
- ▶ Prospects of Constructive Axiomatic Method in Physics and beyond

## Received view on theories (rough version)

A theory is a set  $T$  of statements (propositions). Axiomatic theory is a theory where one distinguishes a subset  $A \subset T$  of axioms and fixes rules of logical inference, which allow one to generate the whole of  $T$  from  $A$ .

## Received view on theories (nuanced version)

A theory comprises :

- ▶ Distinction between the syntax and the semantics ;
- ▶ Set  $WFF$  of all well-formed formulas (which under the intended interpretation are propositions).
- ▶ Semantic distinction between logical and non-logical terms
- ▶ Semantics for non-logical terms. Concept of satisfaction.
- ▶ Syntactic consequence
- ▶ Semantic consequence
- ▶ soundness, completeness (in various senses)
- ▶ ...

## Historical Claim

The theory of *Elements*, Book 1 does not fit even into the rough version of the received notion of axiomatic theory. So there is no much sense of using the nuanced version for its analysis/reconstruction.

## Müller on Euclid

I know of no logic which accounts for this inference in its Euclidean formulation. One 'postulates' that a certain action is permissible and 'infers' the doing of it, he., does it. An obvious analogue of the procedure here is provided by the relation between rules of inference and a deduction. Rules of inference permit certain moves described in a general way, e.g., the inferring of a formula of the form  $A \vee B$  from a formula of the form  $A$ . And in a deduction one may in fact carry out such a move, e.g., write ' $(P \& Q) \vee R$ ' after writing ' $P \& Q$ '. The carrying out of a deductive step on the basis of a rule of inference is certainly not itself an inference. For neither the rule nor the step is a statement capable of truth and falsehood. And if the analogy is correct, Euclid's constructions are not inferences from his constructional postulates; they are actions done in accord with them.

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- ▶ Axioms (Common Notions) :  
play the role similar to that of logical rules restricted to mathematics : cf. the use of the term by Aristotle
- ▶ Postulates :  
non-logical constructive rules

## Common Notions

- A1. Things equal to the same thing are also equal to one another.
- A2. And if equal things are added to equal things then the wholes are equal.
- A3. And if equal things are subtracted from equal things then the remainders are equal.
- A4. And things coinciding with one another are equal to one another.
- A5. And the whole [is] greater than the part.

## Common Notions (continued)

Euclid's Common Notions hold *both* for numbers and magnitudes (hence the title of “common”); they form the basis of a regional “mathematical logic” applicable throughout the mathematics.

## Aristotle on Axioms (1) :

*By first principles of proof [as distinguished from first principles in general] I mean the common opinions on which all men base their demonstrations, e.g. that one of two contradictories must be true, that it is impossible for the same thing both be and not to be, and all other propositions of this kind.” (Met. 996b27-32)*

Here Aristotle refers to a logical principle as “common opinion”.

## Aristotle on Axioms (2) :

Comparison of mathematical and logical axioms :

*We have now to say whether it is up to the same science or to different sciences to inquire into what in mathematics is called axioms and into [the general issue of] essence. Clearly the inquiry into these things is up to the same science, namely, to the science of the philosopher. For axioms hold of everything that [there] is but not of some particular genus apart from others. Everyone makes use of them because they concern being qua being, and each genus is. But men use them just so far as is sufficient for their purpose, that is, within the limits of the genus relevant to their proofs.*

## Aristotle on Axioms (2), continued :

*Since axioms clearly hold for all things qua being (for being is what all things share in common) one who studies being qua being also inquires into the axioms. This is why one who observes things partly [=who inquires into a special domain] like a geometer or an arithmetician never tries to say whether the axioms are true or false. (Met. 1005a19-28)*

## Aristotle on Axioms (3) :

Reference to Ax.3 :

*Since the mathematician too uses common [axioms] only on the case-by-case basis, it must be the business of the first philosophy to investigate their fundamentals. For that, when equals are subtracted from equals, the remainders are equal is common to all quantities, but mathematics singles out and investigates some portion of its proper matter, as e.g. lines or angles or numbers, or some other sort of quantity, not however qua being, but as [...] continuous. (Met. 1061b)*

## Postulates 1-3 :

*P1 : to draw a straight-line from any point to any point.*

*P2 : to produce a finite straight-line continuously in a straight-line.*

*P3 : to draw a circle with any center and radius.*



# Postulates 1-3 (continued) :

Postulates 1-3 are NOT propositions ! They are not first truths.  
They are basic (non-logical) *operations*.

# Operational interpretation of Postulates

Postulates	input	output
P1	two points	straight segment
P2	straight segment	straight segment
P3	straight segment and its endpoint	circle

# Propositional interpretation of Postulates

P1m (modal) : Given two (different) points it is always *possible* to produce a straight segment from one given point to the other given point.

P1e (existential) : Given two (different) points there *exists* a straight segment having these given points as its endpoint.

# Claim

Under a propositional interpretation Postulates don't fit into Euclid's theory !

# Problems and Theorems

Common structure :

- ▶ *enunciation* :
- ▶ *exposition*
- ▶ *specification*
- ▶ *construction*
- ▶ *proof*
- ▶ *conclusion*

## Theorem 1.5 :

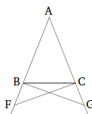
[*enunciation* :]

*For isosceles triangles, the angles at the base are equal to one another, and if the equal straight lines are produced then the angles under the base will be equal to one another.*

## Theorem 1.5 (continued) :

[*exposition*] :

*Let  $ABC$  be an isosceles triangle having the side  $AB$  equal to the side  $AC$  ; and let the straight lines  $BD$  and  $CE$  have been produced further in a straight line with  $AB$  and  $AC$  (respectively). [Post. 2].*



## Theorem 1.5 (continued) :

[*specification* :]

*I say that the angle ABC is equal to ACB, and (angle) CBD to BCE.*



## Theorem 1.5 (continued) :

[specification :]

*I say that the angle  $ABC$  is equal to  $ACB$ , and (angle)  $CBD$  to  $BCE$ .*

[construction :]

*For let a point  $F$  be taken somewhere on  $BD$ , and let  $AG$  have been cut off from the greater  $AE$ , equal to the lesser  $AF$  [Prop. 1.3]. Also, let the straight lines  $FC$ ,  $GB$  have been joined. [Post. 1]*

## Theorem 1.5 (continued) :

[proof :]

*In fact, since  $AF$  is equal to  $AG$ , and  $AB$  to  $AC$ , the two (straight lines)  $FA, AC$  are equal to the two (straight lines)  $GA, AB$ , respectively. They also encompass a common angle  $FAG$ . Thus, the base  $FC$  is equal to the base  $GB$ , and the triangle  $AFC$  will be equal to the triangle  $AGB$ , and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is)  $ACF$  to  $ABG$ , and  $AFC$  to  $AGB$ . And since the whole of  $AF$  is equal to the whole of  $AG$ , within which  $AB$  is equal to  $AC$ , the remainder  $BF$  is thus equal to the remainder  $CG$  [Ax.3]. But  $FC$  was also shown (to be) equal to  $GB$ . So the two (straight lines)  $BF, FC$  are equal to the two (straight lines)  $CG, GB$ , respectively, and the triangle  $BFC$  is equal to the triangle  $CBG$ .*

## Theorem 1.5 (continued) :

[*conclusion* :]

*Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.*

Mind the difference between monstration and demonstration (deixis kai apodeixis). *Quod erat demonstrandum* is an erroneous Latin translation of Euclid's expression. The correct one is *quod erat monstrandum*!

## Problem 1.1 :

[*enunciation* :]

*To construct an equilateral triangle on a given finite straight-line.*

## Problem 1.1 (continued) :

[*exposition* :]

*Let  $AB$  be the given finite straight-line.*

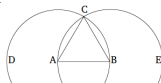
[*specification* :]

*So it is required to construct an equilateral triangle on the straight-line  $AB$ .*

## Problem 1.1 (continued) :

[*construction* :]

*Let the circle  $BCD$  with center  $A$  and radius  $AB$  have been drawn [Post. 3], and again let the circle  $ACE$  with center  $B$  and radius  $BA$  have been drawn [Post. 3]. And let the straight-lines  $CA$  and  $CB$  have been joined from the point  $C$ , where the circles cut one another, to the points  $A$  and  $B$  [Post. 1].*



## Problem 1.1 (continued) :

[*proof* :]

*And since the point  $A$  is the center of the circle  $CDB$ ,  $AC$  is equal to  $AB$  [Def. 1.15]. Again, since the point  $B$  is the center of the circle  $CAE$ ,  $BC$  is equal to  $BA$  [Def. 1.15]. But  $CA$  was also shown (to be) equal to  $AB$ . Thus,  $CA$  and  $CB$  are each equal to  $AB$ . But things equal to the same thing are also equal to one another [Axiom 1]. Thus,  $CA$  is also equal to  $CB$ . Thus, the three (straight-lines)  $CA$ ,  $AB$ , and  $BC$  are equal to one another.*



## Problem 1.1 (continued) :

[*conclusion* :]

*Thus, the triangle  $ABC$  is equilateral, and has been constructed on the given finite straight-line  $AB$ . (Which is) the very thing it was required to do.*

# Conclusion on Euclid

There is no obvious reason to assume that “behind” Euclid’s geometrical reasoning there lies a hidden system of proposition-based logic. There is no obvious way to reconstruct Euclid’s reasoning in a propositional form.

## Historical Remark

Term “proposition” as a common name for Problems and Theorems is missing in Euclid and is not recorded in Proclus Commentary and other reliable sources. This term is misleading since it suggests that Euclid’s theory is construed as an axiomatic theory in the standard (rough) sense of the word.

# Euclidean Structure in Modern Mathematics (Kolmogorov & Fomin)

[*enunciation* :]

*Any closed subset of a compact space is compact*

[*exposition* :]

*Let  $F$  be a closed subset of compact space  $T$*

[*specification* : I say that  $F$  is a compact space]

## Euclidean Structure in Modern Mathematics (continued) :

[*construction* :]

[*Let*]  $\{F_\alpha\}$  [*be*] an arbitrary centered system of closed subsets of subspace  $F \subset T$ .

[*proof* :]

[*E*]very  $F_\alpha$  is also closed in  $T$ , and hence  $\{F_\alpha\}$  is a centered system of closed sets in  $T$ . Therefore  $\bigcap F_\alpha \neq \emptyset$ .  
By Theorem 1 it follows that  $F$  is compact.

[*conclusion* : Thus any closed subset of a compact space is compact. (Which is) the very thing it was required to show.]

# Hilbert 1894

Among the appearances or facts of experience manifest to us in the observation of nature, there is a peculiar type, namely, those facts concerning the outer shape of things, Geometry deals with these facts [..]. Geometry is a science whose essentials are developed to such a degree, that all its facts can already be *logically deduced* from earlier ones.

# Hilbert 1899 : logical and non-logical terms

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- ▶ Any two distinct points of a straight line completely determine that line



## Hilbert 1899 : logical and non-logical terms

- ▶ Any two distinct points of a straight line completely determine that line
- ▶ If different points  $A, B$  belong to straight line  $a$  and to straight line  $b$  then  $a$  is identical to  $b$ .

The distinction between logical and non-logical concepts plays a fundamental role in Hilbert's 1899 axiomatics because it provides a sense of being formal for his axiomatic theories. The form in point is a logical form. That means that logical semantics (which is not explicitly construed in this framework!) is rigidly fixed and the non-logical semantics is left variable. A formal mathematical theory is grounded in logic and logic alone. This is disregarding non-logical epistemic sources of specific axioms (experience, pragmatic value, intuition, etc.) Logic in this context is thought of as a system of rules for handling propositions (with all their constituents such as individuals of various types).

## Hintikka 1997 on Hilbert-style axiomatic method

The basic clarified form of mathematical theorizing is a purely logical axiom system.

# What are possible “fillings” for non-logical elements of a formal theory?

- ▶ usual intuitions
- ▶ interpretations of the given formal theory in other informal theories (ex. : arithmetical models of geometric theories or one geometric theory in another one)
- ▶ “thought-things” (*Gedankendinge*)

All of these contribute to the standard notion of *model* due to Tarski. One may wonder how a combination of these three very different concepts can be coherent.

## Hilbert 1918 : Axiomatizing Logic

[I]t appears necessary to axiomatize logic itself and to prove that number theory and set theory are only parts of logic. This method was prepared long ago (not least by Frege's profound investigations); it has been most successfully explained by the acute mathematician and logician Russell. One could regard the completion of this magnificent Russellian enterprise of the axiomatization of logic as the crowning achievement of the work of axiomatization as a whole.

Notice a conceptual gap (Hintikka). Whatever the axiomatization of logic may be it can not be an axiomatization in 1899 sense !

## Hilbert 1927 : Intuition Strikes Back !

No more than any other science can mathematics be founded by logic alone ; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought.



## Hilbert 1927 : Intuition Strikes Back !

If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction. This is the basic philosophical position that I regard as requisite for mathematics and, in general, for all scientific thinking, understanding, and communication.

# Hilbert 1927 : Intuition Strikes Back !

And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable.

## Hilbert 1927 : Intuition Strikes Back !

[I]n my theory contentual inference is replaced by manipulation of signs [ausseres Handeln] according to rules; in this way the axiomatic method attains that reliability and perfection that it can and must reach if it is to become *the basic instrument of all research* (italics mine - A.R.).

A “logical” proof comes back to a symbolic (albeit not genuinely geometrical) construction. So *showing* comes back to *doing* here. Cf. Curry-Howard.

Beware that the intended logical semantics of these symbolic construction plays a crucial role anyway. If one drops this semantics then, in particular, Hilbert's big idea of formal consistency proof makes no sense. Why on earth the possibility or impossibility to construct a symbolic expression of form  $p \& \neg p$  is so important? Only because the intended logical semantics makes it important.

## Performance of this method in the 20th century mathematics and science

is VERY poor - as far as one takes seriously Hilbert's intention to make this method into the basic instrument of all research. This fact is not directly related to theoretical limits put on Hilbert Program by Gödel's incompleteness results and other metatheoretical results. I disagree with the popular view according to which this problem is only practical and pragmatic. I claim that it is genuinely theoretic.

## “Mathematical Practice”

Mathematical research is a theoretical activity par excellence. The *philosophy of mathematical practice* is an intellectual compromise designed for avoiding a substantial argument with those people who stick to early 20-th century views and research programs and don't want to revise these views under a pressure of new theoretical evidences. The distinction between theory and practice is badly construed or badly applied in this case.

# Metamathematics

In fact, the Hilbert-style formal axiomatic method proved to be very efficient for proving meta-theorems about formal systems - cf. the independence results for CH and AC. Noticeably all such results are proved by “usual” informal mathematical methods - and not only finitary constructive methods as Hilbert hoped. The foundational impact of such results is unclear since it is unclear whether the formal counterparts of mathematical theories represent these theories in any reasonable sense adequately. Beware that such formal counterparts are typically abstract combinatorial constructions but not finite strings of symbols as Hilbert conceived of them. Generally, such constructions are not finitary generated. The question of epistemic significance of idealization in mathematics is a genuine theoretical and philosophical problem, not a practical one.



## Conclusion on Hilbert

Hilbert's architecture of axiomatic mathematics involves "doing and showing" as well as Euclid's architecture. But unlike Euclid Hilbert places these two layers into different theories. A formal theory  $T$  with all its intended and unintended interpretations (models) *shows that* certain propositions (i.e., theorems) hold. A metatheory  $M_T$  of theory  $T$  accounts for syntactic constructions, which express propositions and proofs in  $T$ . In Hilbert's original conception  $M_T$  is not formal but rather Euclid-style. The epistemic gain is that the constructions in question are combinatorial and don't involve higher infinities and arguably no infinity at all.

## Open Question

I leave now aside well-known metatheoretical issues such as Incompleteness, which make Hilbert's Program unfeasible in its original form. Still there remains an important logical and epistemological issue, which in my view is even more important :

Whether or not the proposition-based concept of theory assumed for  $T$  (but not for  $M_T$ !) is a good one? I shall argue that the answer should be in negative.

## Genetic method : Hilbert 1900

The idea : Mathematical objects are built from other such objects.  
More complex objects are built from simpler ones.

## Hilbert's example

Dedekind Cuts and Cochy sequences. Both are “built from” natural numbers.

Notice that neither of the two “constructions” is constructive in any the usual senses of the word (Turing, Bishop, Markov, et. al)!

## Genetic method : Hilbert 1900

Despite the high pedagogic and heuristic value of the genetic method, for the final presentation and the complete logical grounding of our knowledge the axiomatic method deserves the first rank.

## Hilbert & Bernays 1934

The term axiomatic will be used partly in a broader and partly in a narrower sense. We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics. [..].

## Hilbert & Bernays 1934

[F]or axiomatics in the narrowest sense, the *existential form* comes in as an additional factor. This marks the difference between the *axiomatic method* [in the narrow sense?] and the *constructive* or *genetic* method of grounding a theory. While the constructive method introduces the objects of a theory only as a *genus* of things, an axiomatic theory refers to a fixed system of things [...] given as a whole. Except for the trivial cases where the theory deals only with a finite and fixed set of things, this is an idealizing assumption that properly augments the assumptions formulated in the axioms.

## Hilbert & Bernays 1934

Euclid does not presuppose that points or lines constitute any fixed domain of individuals. Therefore, he does not state any existence axioms either, but only construction postulates. (op. cit. p. 20a)



## Gabbay & Gabbay et. al on the above passage

Unless we see construction as a state-changing process, this [Hilbert and Bernays' ] remark is actually not justified : From a static logical point of view, existence sentences are always implied by construction postulates.

A charitable reading of this passage implies that Hilbert&Bernays do see Euclid's constructions as state-changing processes, i.e., as genetic procedures in the sense of Hilbert 1900. If I am right here then a popular reading of the above passage according to which Hilbert&Bernays refer here to a contentful version of the same axiomatic method is erroneous.

Given that

- ▶ Hilbert&Bernays qualify Euclid's theory as axiomatic in the "broadest" sense and that
- ▶ the identification of the "broadest" axiomatic method with the received notion of axiomatic method is erroneous, one concludes that
- ▶ the "broadest" axiomatic method is broader than the received axiomatic method and includes the genetic method in the sense of Hilbert's 1900 paper.

## Rules versus Axioms

The common ground that allows for combining the axiomatic method in the narrow (i.e. the received) sense and the genetic method in the sense of Hilbert 1900 is the concept of *rule*. The received method applies only logical rules, i.e., rules of operating with propositions. The constructive method applies rules to propositional and non-propositional objects.

# Objectual Constructivity

A constructive axiomatic theory comprises non-propositional principles similar to Euclid's Postulates, which allow for building of and operating with non-propositional objects and not only with formal expressions telling us something *about* these objects. While usual notions of constructivity in logic and mathematics specify such rules in one way or another I leave it wholly open *what* such rules may or should be.

# Constructive theories

In constructive theories propositions are objects of special type on equal footing with objects of non-propositional types. Objects of these different types belong in this case to one and the same theory!

The theory of *Elements*, Book 1 qualifies as constructive in *that* sense. As we shall see HoTT does so too. HoTT also provides a further insight on how the propositional and non-propositional layers of theory relate to each other.

THE END