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Identity and Categorification

A principle aim of Frege's logical reform of mathematics is "fixing the sense of an identity". Frege assumes that identity is a general logical concept which is not specific to mathematics but is stable across all possible domains and contexts. His method of "definition through abstraction" aims at reduction of multiple colloquial meanings of "same" in mathematics to the universal logical identity concept through introduction of new abstract objects. As presented informally in his *Grundlagen* Frege's abstraction amounts to "replacement" of given equivalence by identity, where the latter relates to a newly introduced abstract object .

The project of Categorification of mathematics proposed by Baez&Dolan (1998), on the contrary, aims to a diversification of the identity concept in mathematics in a way similar to that earlier proposed by Geach (relative identity). The main principle of Categorification announced by Baez&Dolan is "never mistake equivalence for equality" (and moreover for identity). Unlike Frege's reform the Categorification results from recent developments in pure mathematics and theoretical physics rather than philosophy.

In my paper I purport to find a common ground for comparison of these two opposite approaches.

Replacement of given equivalence xEy by identity x=y proposed by Frege allows for a stronger interpretation than "abstraction". Namely, E can be interpreted as *reversible transformation* (isomorphism), which turns x to y and the other way round, and the identity = - as identity *through* this transformation. Surprisingly this ontological shift matters mathematically. For the language of transformations is not formally equivalent to that of relations as one might expect but is richer in a sense. Given relation xRy there might exist, generally speaking, many different well-distinguishable transformations turning x into y, while xRy only says that there exists one. Moreover reversible transformations (of same object) form a *group* structure. In particular isomorphisms between sets form symmetric groups. This fact remains completely hidden when one reduces transformations to relations. Among other things we get here a new (group-theoretic) identity concept (as unit of group). The problem is that a group of transformations is not a logical concept but a particular mathematical object which arguably needs certain identity conditions of its own.

To "identify" a group (and its identity) by *internal* means one may repeat the trick and use another group-theoretic identity for it. Consider symmetric group S_N . S_N "identifies" all sets of N elements by collapsing them into one. This collapse is not trivial because S_N has distinct elements, and in particular its identity 1. To identify (different copies of) S_N consider a group $Aut(S_N)$ of automorphisms of S_N. Elements of $Aut(S_N)$ in their turn are identified through group of automorphisms $Aut^2(S_N)$) of $Aut(S_N)$, and so on. This looks like standard infinite regress but in fact the above construction is surprisingly well-behaved. In the case N=2 all $Aut^n(S_N)$ are identities. In the case N>2 with a peculiar exception N=6 all $Aut^n(S_N)$ are isomorphic to S_N , so the infinite series gets stabilized immediately, and we have a sort of fix point here rather than regress.

The Categorification can be viewed as a generalization of the above elementary example through the following two steps:
(i) The concept of group is generalized up to that of *category*. For this end one considers multiple objects and non-reversible transformations (morphisms) between the objects.
(ii) Logical notions are reconstructed by *internal* categorical means through category-theoretic construction of *topos*.

I conclude by a comparison of identity concepts in Frege and in the Topos theory.

Literature:
Baez J.C., Dolan J. (1998), *Categorification* arXiv:math. QA 19802029 v1. (Internet document)
Frege, G. (1964) *The basic laws of Arithmetics*. Translated and edited, with an introduction by M. Furth; NY