How Mathematical Concepts Get Their Bodies

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OPTOFONICA

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Kant on geometrical intuition:

Give a philosopher the concept of triangle and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on his concept as long as he wants, yet he will never produce anything new. He can analyse and make distinct the concept of straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts.

Kant on geometrical intuition:

But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle, and obtains adjacent angles that together are equal to two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that here there arises an external adjacent angle, which is equal to an internal one, etc. In such a way through a chain of inferences that is always guided by intuition, he arrives at a fully illuminated and at the same time general solution of the question. (*Critique of Pure Reason*)

Concepts without intuitions are empty and intuitions without concepts are blind.

Cassirer on symbolic forms:

In all the great systems of rationalism mathematics had been considered the pride of human reason the province of "clear and distinct" ideas. But this reputation seemed suddenly called in question. Far from being clear and distinct the fundamental mathematical concepts proved to be fraught with pitfalls and obscurities. These obscurities could not be removed until the general character of mathematical concepts had been clearly recognized until it had been acnowledged that mathematics is not a theory of things but a theory of symbols. (*An Essay on Man*)

Cassirer on symbolic forms:

Human culture derives its specific character and its intellectual and moral values, not from material of which it consists, but from its form, its architectural structure. And this form may be expressed in any sense material. .... Without this vivifying principle the human world would indeed remain deaf and mute. With this principle, even the world of a deaf, dumb, and blind child can become incomparably broader and richer than the world of the most highly developed anymal. (*An Essay on Man*)

Hilbert on Foundations of Geometry:

Let us consider three distinct systems of things. The things composing the first system, we will call points and designate them by the letters A, B, C,. . . ; those of the second, we will call straight lines and designate them by the letters a, b, c,. . . ; and those of the third system, we will call planes and designate them by the Greek letters α, β, γ . . We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as "are situated", "between"; "parallel", "congruent", "continuous", etc. The complete and exact description of these relations follows as a consequence of the axioms of geometry. (*Grundlagen der Geometrie*)

formal proof:

$$\frac{a}{\vdash a = a}$$

$$\frac{\Gamma \vdash a = b;\ \Delta \vdash b' = c}{\Gamma \cup \Delta \vdash a = c}$$

$$\frac{\Gamma \vdash f = g;\ \Delta \vdash a = b}{\Gamma \cup \Delta \vdash f\,a = g\,b}$$

$$\frac{x;\ \Gamma \vdash a = b}{\Gamma \vdash \lambda x.\ a = \lambda x.\ b} \quad \text{(if } x \text{ is not free in } \Gamma\text{)}$$

$$\frac{(\lambda x.\ a)\ x}{\vdash (\lambda x.\ a)\ x = a}$$

$$\frac{p{:}bool}{p \vdash p}$$

$$\Gamma \vdash p;\ \Delta \vdash p' = q$$

informal proof:

$$\lim_{h\to 0}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{1}{h^2}\,j\left(\frac{x}{h},\frac{y}{h}\right)f(x_0-x,y_0-y)\,dx\right)dy =$$

$$\lim_{h\to 0}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}j(u,v)\,f(x_0-hu,y_0-hv)\,du\right)dv =$$

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}j(u,v)\left[\lim_{h\to 0}f(x_0-hu,y_0-hv)\right]du\right)dv =$$

$$\left(f(x_0,y_0)\right)\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}j(u,v)\,du\right)dv\right)$$

no u,v on the left, no x_0,y_0 on the right!

Kant on symbolic intuition:

But mathematics does not merely construct magnitudes (quanta), as in geometry but also mere magnitudes (quantitatem), as in algebra, where it entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitude. In this case it chooses a certain notation for all construction of magnitudes in general .... and thereby achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction (of objects themselves), which discursive cognition could never achive by means of mere concepts. (*Critique of Pure Reason*)

Frege on geometry:

The truths of geometry govern all that is spatially intuitable, whether actual or product of our fancy. The wildest visions of delirium, the boldest inventions of legend and poetry, ... all these remain, so long as they remain intuitable, still subject to the axioms of geometry. (*Kleine Schriften*)

IS GEOMETRICAL INTUITION INDEED THAT STUBBORN?
No. Mathematics progresses through conceptualization of poorly conceptualized intuitions AND intuiting of poorly intuited concepts.

AN AESTHETIC CHALLENGE: TO PROVIDE MODERN MATHS WITH APPROPRIATE INTUITIONS

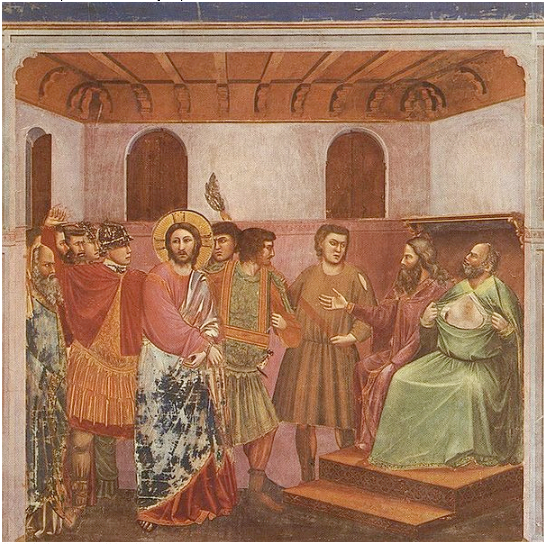
Historical hints: Projective duality

220px-Railroad-Tracks-Perspective.jpg (JPEG Image, 220×2... http://wpcontent.answc

Did Bertrand Russell see this?

File:Giotto - Scrovegni - -32- - Christ before Caiaphas.jpg

From Wikipedia, the free encyclopedia



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CIAMĒTIS·
FODERVNT·MANVS
MEAS·ET·PEDES·ME
OS·ET·DINVMERA
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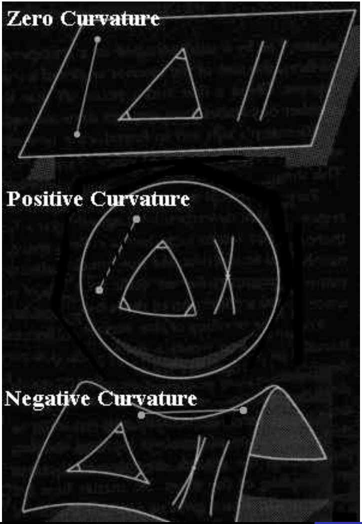
Historical hints: Riemann manifolds

curvature.jpg (JPEG Image, 298×469 pixels) http://library.thinkquest.org/27930/media/curvatur



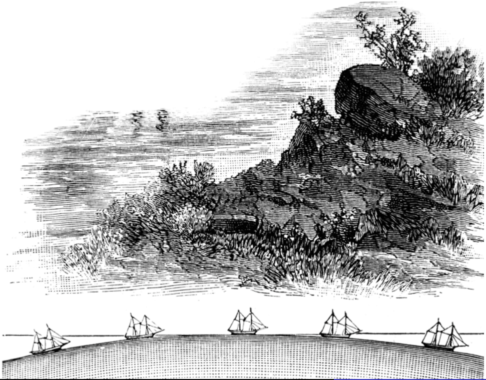
Projective plane as a riemannian manifold

DSCF0017.jpg (JPEG Image, 470×363 pixels) http://www.math.binghamton.edu/alex/PICTURES/KnotC

Smoothness

curved_surface_printing-1.jpg (JPEG Image, 690×519 pixels) http://www.allproducts.com/traffic/tceo/curved_surface_pri

Scaling

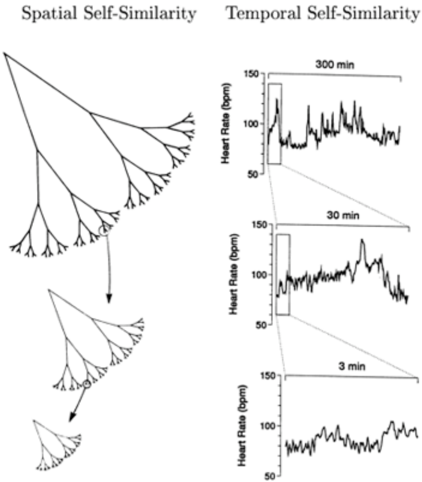


Fractal Monsters

hokusai_wave_1.jpg (JPEG Image, 500×335 pixels) http://oceanworld.tamu.edu/students/waves/images/hokus

Non-smooth Self-similarity

fractal4.gif (GIF Image, 353×400 pixels) http://mcdb.colorado.edu/courses/3280/images/networks/fr



Fractals

2364129390_4fe946df4a.jpg (JPEG Image, 500×500 pixels) http://farm3.static.flickr.com/2156/2364129390_4fe946df4

A GREAT AESTHETIC CHALLENGE: QUANTUM SPACE-TIME

Quantum Space-Time

Black Hole?