# "Universal Mathematics" in Euclid and the Origin of Logic in Aristotle

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# Ambiguity of term "axiom" in Aristotle

"An axiom (axiôma) is a statement worthy of acceptance and is needed prior to learning anything. Aristotle's list [of examples] here includes the most general principles such as non-contradiction and excluded middle, and principles more specific to mathematicals, e.g., when equals taken from equals the remainders are equal. It is not clear why Aristotle thinks one needs to learn mathematical axioms to learn anything, unless he means that one needs to learn them to learn anything in a mathematical subject or that axioms are so basic that they should form the first part of one s learning." (Henry Mendell, Aristotle and Mathematics, Stanford Encyclopedia of Philosophy, first published in 2004, underlining mine - A.R.)

# Analogy with mathematics

"Ancillary to his discussions of being qua being and theology (Metaphysics vi.1, xi.7), Aristotle suggests an <u>analogy</u> with mathematics. If the analogy is that there is a super-science of mathematics coverying all continuous magnitudes and discrete quantities, such as numbers, then we should expect that Greek mathematicians conceived of a general mathematical subject as a precursor of algebra, Descartes' mathesis universalis (universal learning/mathematics), and mathematical logic." (Henry Mendell, Aristotle and Mathematics, Stanford Encyclopedia of Philosophy, first published in 2004, underlining mine - A.R.)

# Claim 1 :

Aristotle refers to the established mathematical meaning of the term "common notion" (or simply "common") using interchangeably the term "axiom". This mathematical meaning of the term can be reasonably reconstructed through the text of Euclid's *Elements* (notwithstanding the chronological distance between Euclid's and Aristotle's akmes). Aristotle suggests a significant change of the meaning of the term by using it to denote (what we would call today) a logical principle ("law of logic"). Beware that this meaning of the term "axiom" is not identical to the current logical meaning of the term.

# Claim 1 (continued) :

The idea of logical principle and, more generally, the very idea of logic, is Aristotle's original invention. Aristotle's logical principles (like the principles of non-contradiction and excluded middle) are also ontological principles. Aristotle's notion of logico-ontological principle aka axiom allows him to consider axioms as common principles of all sciences (including mathematics and physics) without making physics into a branch of mathematics after Plato. Thus Aristotle's novel claim can be formulated as follows : principles common for all sciences are logical (and ontological) principles rather than universal mathematical principles, i.e., axioms aka common notions in Euclid's sense of the term. Aristotle changes the meaning of the terms "axiom" and "common notion" accordingly.

# Claim 2 :

Aristotle's logic is <u>not</u> adequate to Euclid's geometry AND Aristotle knows about this. The principle reason of this inadequacy is the fact that Aristotle's logic leaves out the constructive side of Euclid's geometry (which involves 5 Postulates in addition to 5 Axioms). Aristotle observes that the assumption about pre-existence of geometrical objects suppresses the explanatory force of the traditional geometrical reasoning. This problem remains pertinent in logic and mathematics of the 20th century.

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- Axioms (Common Notions) :
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- Postulates :
  non-logical constructive rules

# Common Notions

A1. Things equal to the same thing are also equal to one another.
A2. And if equal things are added to equal things then the wholes are equal.
A3. And if equal things are subtracted from equal things then the remainders are equal.
A4. And things coinciding with one another are equal to one another.
A5. And the whole [is] greater than the part.

# Axioms

Κοιναὶ ἔννοιαι.

α΄. Τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.
β΄. Καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἴσα.
γ΄. Καὶ ἐὰν ἀπὸ ἴσων ἴσα ἀφαιρεθῇ, τὰ καταλειπόμενά ἐστιν ἴσα.
δ΄. Καὶ τὰ ἐφαρμόζοντα ἐπ᾽ ἀλλήλα ἴσα ἀλλήλοις ἐστίν.
ε΄. Καὶ τὸ ὅλον τοῦ μέρους μεῖζόν [ἐστιν].

Common Notions

1. Things equal to the same thing are also equal to one another.
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3. And if equal things are subtracted from equal things then the remainders are equal.†
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# Common Notions (continued)

Euclid's Common Notions hold *both* for numbers and magnitudes (hence the title of "common"); they form the basis of a regional "mathematical logic" applicable throughout the mathematics. Aristotle transform them into laws of logic applicable throughout the *episteme*, which in Aristotle's view does not reduce to mathematics but also includes *physics*.

# Common Notions (continued)

Euclid's Common Notions hold *both* for numbers and magnitudes (hence the title of "common"); they form the basis of a regional "mathematical logic" applicable throughout the mathematics. Aristotle transform them into laws of logic applicable throughout the *episteme*, which in Aristotle's view does not reduce to mathematics but also includes *physics*. Aristotle describes and criticizes a view according to which Common Notions constitute a basis for *Universal Mathematics*, which is a part of mathematics shared by all other mathematical disciplines. In 16-17th centuries the Universal Mathematics is often identified with Algebra and for this reason Euclid's Common Notions are viewed as *algebraic* principles.

# Postulates 1-3 :

*P1 : to draw a straight-line from any point to any point.*
*P2 : to produce a finite straight-line continuously in a straight-line.*
*P3 : to draw a circle with any center and radius.*

# Postulates

## Αἰτήματα.

α΄. Ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

β΄. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ᾽ εὐθείας ἐκβαλεῖν.

γ΄. Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

δ΄. Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

ε΄. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ᾽ ἄπειρον συμπίπτειν, ἐφ᾽ ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

## Postulates

1. Let it have been postulated to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).†

† This postulate effectively specifies that we are dealing with the geometry of *flat*, rather than curved, space.

# Postulates 1-3 (continued) :

Postulates 1-3 are NOT propositions! They are not first truths. They are basic (non-logical) *operations*.

# Operational interpretation of Postulates

| Postulates | input | output |
|---|---|---|
| P1 | two points | straight segment |
| P2 | straight segment | straight segment |
| P3 | straight segment and its endpoint | circle |

# Aristotle on Universal Mathematics 1

*There are certain mathematical theorems that are universal, extending beyond these substances. Here then we shall have another intermediate substance separate both from the Ideas and from the intermediates,-a substance which is neither number nor points nor spatial magnitude nor time. (Met. 1077a)*

# Aristotle on Universal Mathematics 1

ἔτι γράφεται ἔνια καθόλου ὑπὸ τῶν μαθηματικῶν παρὰ ταύτας τὰς οὐσίας. ἔσται οὖν καὶ αὕτη τις ἄλλη οὐσία μεταξὺ κεχωρισμένη τῶν τ' ἰδεῶν καὶ τῶν μεταξύ, ἣ οὔτε ἀριθμός ἐστιν οὔτε στιγμαὶ οὔτε μέγεθος οὔτε χρόνος.

# Aristotle on Universal Mathematics 2

> *Alternation used to be demonstrated separately of numbers, lines, solids, and durations, though it could have been proved of them all by a single demonstration. Because there was no single name to denote that in which numbers, lengths, durations, and solids are identical, and because they differed specifically from one another, this property was proved of each of them separately. Today, however, the proof is universal, for they do not possess this attribute qua lines or qua numbers, but qua manifesting this generic character which they are postulated as possessing universally. (An. Post. 74a)*

alternation : $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{c} = \frac{b}{d}$

# Aristotle on Universal Mathematics 2

καὶ τὸ ἀνάλογον ὅτι καὶ ἐναλλάξ, ᾗι ἀριθμοὶ καὶ ᾗι γραμμαὶ καὶ ᾗι στερεὰ καὶ ᾗι χρόνοι, ὥσπερ ἐδείκνυτό ποτε χωρίς, ἐνδεχόμενόν γε κατὰ πάντων μιᾶι ἀποδείξει δειχθῆναι· ἀλλὰ διὰ τὸ μὴ εἶναι ὠνομασμένον τι ταῦτα πάντα ἕν, ἀριθμοί μήκη χρόνοι στερεά, καὶ εἴδει διαφέρειν ἀλλήλων, χωρὶς ἐλαμβάνετο. νῦν δὲ καθόλου δείκνυται· οὐ γὰρ ᾗι γραμμαὶ ἢ ᾗι ἀριθμοὶ ὑπῆρχεν, ἀλλ᾽ ᾗι τοδί, ὃ καθόλου ὑποτίθενται ὑπάρχειν.

# Proclus on Universal Mathematics and Proportion

"As for unifying bond of the mathematical sciences, we should not suppose it to be proportion, as Erathosphenes says. For though proportion is said to be, and is, one of the features common to all mathematics, there are many other characteristics that are allpervading, so to speak, and intrinsic to the common nature of mathematics." (Commentary on Euclid, 43.22-44.1, Murrow's translation)

# Eudoxus' theory of proportion in Euclid

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- Arithmetical proportion and its applications : Books 7-9
- Unexpected mix ( ?) : Book 10

Proposition 10.5 : Commensurable magnitudes have to one another the ratio which (some) number (has) to (some) number.
Proposition 10.6 : If two magnitudes have to one another the ratio which (some) number (has) to (some) number, then the magnitudes will be commensurable.

# Aristotle on Axioms (1) :

> *By first principles of proof [as distinguished from first principles in general] I mean the common opinions on which all men base their demonstrations, e.g. that one of two contradictories must be true, that it is impossible for the same thing both be and not to be, and all other propositions of this kind." (Met. 996b27-32)*

Here Aristotle refers to a logical principle as "common opinion".

# Aristotle on Axioms (1) :

ἀλλὰ μὴν καὶ περὶ τῶν ἀποδεικτικῶν ἀρχῶν, πότερον μιᾶς ἐστὶν ἐπιστήμης ἢ πλειόνων, ἀμφισβητήσιμόν ἐστιν (λέγω δὲ ἀποδεικτικὰς τὰς κοινὰς δόξας ἐξ ὧν ἅπαντες δεικνύουσιν) οἷον ὅτι πᾶν ἀναγκαῖον ἢ φάναι ἢ ἀποφάναι, καὶ ἀδύνατον ἅμα εἶναι καὶ μὴ εἶναι, καὶ ὅσαι ἄλλαι τοιαῦται προτάσεις, πότερον μία τούτων ἐπιστήμη καὶ τῆς οὐσίας ἢ ἑτέρα, κἂν εἰ μὴ μία, ποτέραν χρὴ προσαγορεύειν τὴν ζητουμένην νῦν. μιᾶς μὲν οὖν οὐκ εὔλογον εἶναι: τί γὰρ μᾶλλον γεωμετρίας ἢ ὁποιασοῦν περὶ τούτων ἐστὶν ἴδιον τὸ ἐπαΐειν;

# Aristotle on Axioms (2) :

Comparison of mathematical and logical axioms :

> *We have now to say whether it is up to the same science or to different sciences to inquire into what in mathematics is called axioms and into [the general issue of] essence. Clearly the inquiry into these things is up to the same science, namely, to the science of the philosopher. For axioms hold of everything that [there] is but not of some particular genus apart from others. Everyone makes use of them because they concern being qua being, and each genus is. But men use them just so far as is sufficient for their purpose, that is, within the limits of the genus relevant to their proofs. (Met. 1005a19-28, my translation)*

# Aristotle on Axioms (2), continued :

*Since axioms clearly hold for all things qua being (for being is what all things share in common) one who studies being qua being also inquires into the axioms. This is why one who observes things partly [=who inquires into a special domain] like a geometer or a arithmetician never tries to say whether the axioms are true or false. (Met. 1005a19-28)*

# Aristotle on Axioms (2) :

λεκτέον δὲ πότερον μιᾶς ἢ ἑτέρας ἐπιστήμης περί τε τῶν ἐν τοῖς μαθήμασι καλουμένων ἀξιωμάτων καὶ περὶ τῆς οὐσίας. φανερὸν δὴ ὅτι μιᾶς τε καὶ τῆς τοῦ φιλοσόφου καὶ ἡ περὶ τούτων ἐστὶ σκέψις: ἅπασι γὰρ ὑπάρχει τοῖς οὖσιν ἀλλ' οὐ γένει τινὶ χωρὶς ἰδίᾳ τῶν ἄλλων. καὶ χρῶνται μὲν πάντες, ὅτι τοῦ ὄντος ἐστὶν ᾗ ὄν, ἕκαστον δὲ τὸ γένος ὄν: ἐπὶ τοσοῦτον δὲ χρῶνται ἐφ' ὅσον αὐτοῖς ἱκανόν, τοῦτο δ' ἔστιν ὅσον ἐπέχει τὸ γένος περὶ οὗ φέρουσι τὰς ἀποδείξεις: ὥστ' ἐπεὶ δῆλον ὅτι ᾗ ὄντα ὑπάρχει πᾶσι (τοῦτο γὰρ αὐτοῖς τὸ κοινόν), τοῦ περὶ τὸ ὂν ᾗ ὂν γνωρίζοντος καὶ περὶ τούτων ἐστὶν ἡ θεωρία.

# Aristotle on Axioms (2) :

διόπερ οὐθεὶς τῶν κατὰ μέρος ἐπισκοπούντων ὄν: ἐπὶ τοσοῦτον δὲ χρῶνται ἐφ' ὅσον αὐτοῖς ἱκανόν, τοῦτο δ' ἔστιν ὅσον ἐπέχει τὸ γένος περὶ οὗ φέρουσι τὰς ἀποδείξεις: ὥστ' ἐπεὶ δῆλον ὅτι ᾗ ὄντα ὑπάρχει πᾶσι (τοῦτο γὰρ αὐτοῖς τὸ κοινόν) , τοῦ περὶ τὸ ὂν ᾗ ὂν γνωρίζοντος καὶ

περὶ τούτων ἐστὶν ἡ θεωρία. διόπερ οὐθεὶς τῶν κατὰ μέρος ἐπισκοπούντων ἐγχειρεῖ λέγειν τι περὶ αὐτῶν, εἰ ἀληθῆ ἢ μή, οὔτε γεωμέτρης οὔτ' ἀριθμητικός , ἀλλὰ τῶν φυσικῶν ἔνιοι, εἰκότως τοῦτο δρῶντες: μόνοι γὰρ ᾤοντο περί τε τῆς ὅλης φύσεως σκοπεῖν καὶ περὶ τοῦ ὄντος

# Aristotle on Axioms (3) :

Reference to Ax.3 :

> *Since the mathematician too uses common [axioms] only on the case-by-case basis, it must be the business of the first philosophy to investigate their fundamentals. For that, when equals are subtracted from equals, the remainders are equal is common to all quantities, but mathematics singles out and investigates some portion of its proper matter, as e.g. lines or angles or numbers, or some other sort of quantity, not however qua being, but as [...] continuous. (Met. 1061b)*

# Aristotle on Axioms (3) :

ἐπεὶ δὲ καὶ ὁ μαθηματικὸς χρῆται τοῖς κοινοῖς ἰδίως, καὶ τὰς τούτων ἀρχὰς ἂν εἴη θεωρῆσαι τῆς πρώτης φιλοσοφίας. ὅτι γὰρ ἀπὸ τῶν ἴσων ἴσων ἀφαιρεθέντων ἴσα τὰ λειπόμενα, κοινὸν μέν ἐστιν ἐπὶ πάντων τῶν ποσῶν, ἡ μαθηματικὴ δ' ἀπολαβοῦσα περί τι μέρος τῆς οἰκείας ὕλης ποιεῖται τὴν θεωρίαν, οἷον περὶ γραμμὰς ἢ γωνίας ἢ ἀριθμοὺς ἢ τῶν λοιπῶν τι ποσῶν, οὐχ ᾗ δ' ὄντα ἀλλ' ᾗ συνεχὲς αὐτῶν ἕκαστον ἐφ' ἓν ἢ δύο ἢ τρία

# Aristotle's mathematical example

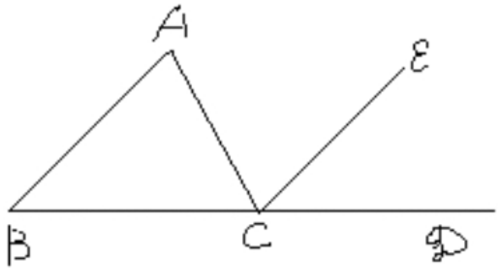
> *Let A be two right angles, B triangle, C isosceles. Then A is an attribute of C because of B, but it is not an attribute of B because of any other middle term; for a triangle has [its angles equal to] two right angles by itself, so that there will be no middle term between A and B, though AB is matter for demonstration." (An. Pr. 48a33-37)*

inadequacy of the syllogistic to geometrical proofs?

# Aristotle's mathematical example

ἔστω τὸ Α δύο ὀρθαί, τὸ ἐφ' ᾧ Β τρίγωνον, ἐφ' ᾧ δὲ Γ ἰσοσκελές. τῷ μὲν οὖν Γ ὑπάρχει τὸ Α διὰ τὸ Β, τῷ δὲ Β οὐκέτι δι' ἄλλο (καθ' αὑτὸ γὰρ τὸ τρίγωνον ἔχει δύο ὀρθάς), ὥστ' οὐκ ἔσται μέσον τοῦ Α Β, ἀποδεικτοῦ ὄντος.

# Angle Sum theorem



# Aristotle's mathematical example

*Diagrams are devised by an activity, namely by dividing-up. If they had already been divided, they would have been manifest to begin with ; but as it is this [clarity] presents itself [only] potentially. Why does the triangle has [the sum of its internal angles is equal to] two right angles ? Because the angles about one point are equal to two right angles. If the parallel to the side had been risen [in advance], this would be seen straightforwardly. (Met. 1051a21-26, my translation)*

## Aristotle's mathematical example

εὑρίσκεται δὲ καὶ τὰ διαγράμματα ἐνεργείᾳ: διαιροῦντες γὰρ εὑρίσκουσιν. εἰ δ᾽ ἦν διῃρημένα, φανερὰ ἂν ἦν: νῦν δ᾽ ἐνυπάρχει δυνάμει. διὰ τί δύο ὀρθαὶ τὸ τρίγωνον; ὅτι αἱ περὶ μίαν στιγμὴν γωνίαι ἴσαι δύο ὀρθαῖς. εἰ οὖν ἀνῆκτο ἡ παρὰ τὴν πλευράν, ἰδόντι ἂν ἦν εὐθὺς δῆλον διὰ τί.

This remark hardly solves the problem.

The idea of logical foundations of geometry (and the rest of mathematics) plays a major role in the 20th century mathematics at least since Hilbert's *Foundations of Geometry* first published in 1899. In particular, this approach allowed to make sense of the proliferation of geometry that occurred in the 19th century (the invention of non-Euclidean geometries).

Application of mathematical methods in logic in the 20th century led to great advances in this old discipline (ex., Gödel's Incompleteness theorems). However the influence in the opposite direction is so far very modest : the mainstream mathematics of the 20th and the 21st century uses very little of existing logical methods (Set theory uses such methods but it hardly makes part of the mainstream mathematics in spite of its status of "official foundation".)

Thus the question of whether or not one needs anything like logical generality over and above the mathematical generality remains largely open.

# Lawvere on logic and geometry

"The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. At the same time [..] there are important influences in the other direction : a Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions [..]." Lawvere, Quantifiers and Sheaves, 1970

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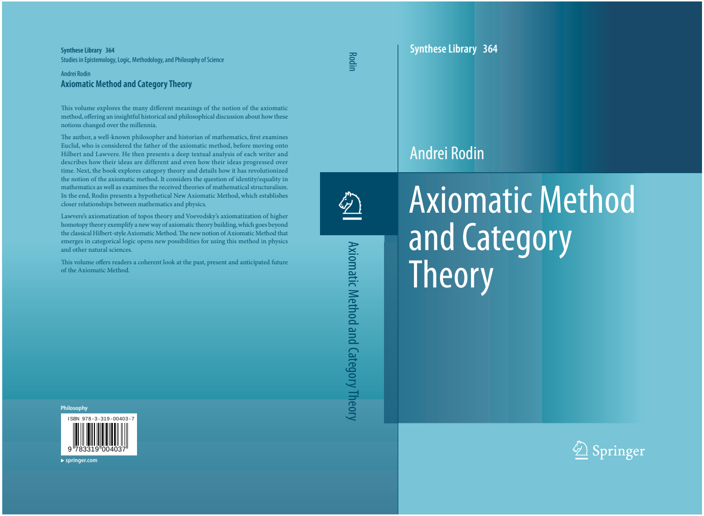
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