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Categories without Structures

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Plan:

1. Renewing foundations
2. Claim
3. Mathematical Structuralism and Set theory
4. Structuralist motivations in Category theory
5. Categories versus Structures
6. Categorical foundations (conclusion)

1. Renewing foundations

Historical observation:

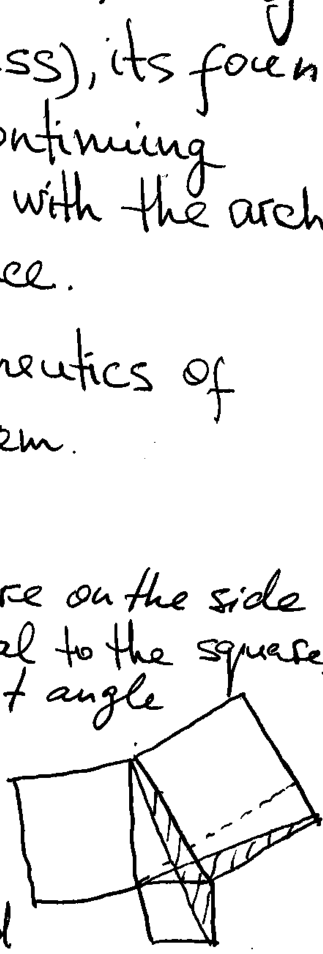
Foundations is the most dynamic part of mathematics. While the principle body of mathematical knowledge is the subject to continuing growth (progress), its foundations is a subject to continuing renewal. This is in odds with the architectural metaphor of science.

Example: a historical hermeneutics of Pythagorean theorem.

① Euclid's "Elements" 1.47

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle

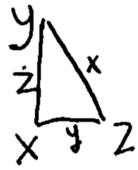
[equality ~ equicomposability]



② Long & Marrow 1997

Let XYZ be a right triangle with lengths of legs x and y, and hypothenuse of length z. Then

$$x^2 + y^2 = z^2$$



③ Donnedu 1965

Two non-zero vectors x and y are orthogonal iff $(y-x)^2 = y^2 + x^2$

While the theorem in some sense remains the same its foundations change dramatically. (In which sense? Do ①–③ share a common structure?)

Big picture:

The progress (= cummulative development) of science requires preservation of the earlier acqured knowledge. But it cannot be preserved in a frozen condition, it needs a permanent renewal. This is what foundations serve for. Renewal of foundations doesn't reduce to mechanical repetition: it is radical (pleonastically). There is no progress in foundations.

There is no eternal „absolute" foundations (albeit there are eternal mathematical truths)

2. Claim:

Structuralist foundations of mathematics supported the mainstream research throughout 20th century.
Set-theoretic foundations of mathematics is a (controversial) version of Structuralist foundations. Cf. Bourbaki.
The idea of categorical (= category-theoretic) foundations of mathematics has emerged in 1960-ies as an attempt to create a better vehicle for Mathematical Structuralism
However categorical foundations have a potential to overcome Structuralism.

My present purpose is to outline this new development.
My ambition is to make philosophy of mathematics active rather than only reactive.

Slogan: The subject-matter of mathematics is covariant transformation, not invariant form!

3. Set theory, ~~Category theory,~~ and Mathematical Structuralism

Maclane 1996 ("Structure in Mathematics")

"All infinite cyclic groups are isomorphic, but this infinite group [notice the switch to singular-A] appears over and over again - in number theory, in ornaments, in crystallography, in physics. Thus the "existence" of this group is really a many splendored matter. An ontological analysis of things simply called "mathematical objects" is likely to miss the real point of mathematical existence."

Cf. the case of natural numbers: they are even more promiscuous both internally and externally. What's new?

- ✓ Prolifiration of structures
- ✓ Free building of structures

Platonic world

partaking ↑
ideas
numbers
magnitudes
sensibilia

~~Plato contra Aristotle~~

there is no Substan[ce]
only Form!

Official definition (Hellman)

"Structuralism is a view about the subject matter of mathematics according to which what matters are structural relationships in abstraction from the intrinsic nature of the related objects. <...> The items making up any particular system [sic!] exemplifying the structure in question are of no importance; all that matters is that they satisfy certain general conditions - typically spelled out in axioms defining the structure or structures of interest"

Cf. Hilbert's "Grundlagen" (1899) and his often-quoted letter to Frege. But... he is more explicit as to „exemplification" of structures:

"One merely has to apply a univocal and reversible one-to-one transformation and stipulate that the axioms for the transformed things be correspondingly similar..."

Cf. Awodey 1996: "The subject-matter of pure mathematics is invariant form". Cf. Erlangen

There is no invariant form unless the transformation in question is reversible!

Argument: (Existence of an) isomorphism is an equivalence relation $A \leftrightarrow B$ $A \approx B$

[Mind the ambiguity of the term.]

Then the „invariant form" is given by Fregean abstraction: $C/_{\approx}$

Existence of general morphism is not an equivalence (unless the direction is ignored)

$A \to B$ (relation) doesn't imply $B \to A$

Frege's abstraction doesn't work

$$\begin{tikzcd} & \text{Structure} & \\ \text{EX1} \arrow[ur] \arrow[rr, leftrightarrow, "\sim"'] & & \text{EX2} \arrow[ul, "\text{abstraction}"'] \end{tikzcd}$$

„particular systems exemplifying the structure are of no importance"

$$\begin{tikzcd} & A & \\ B \arrow[ur] \arrow[rr] & & C \arrow[ul] \end{tikzcd}$$

commutative? Every object and morphism is important!

4. Structuralist motivations in Category theory.

Bourbaki (Dieudonné) „Architecture" 1950

„The [set-theoretic] difficulties did not disappear until the notion of set itself disappeared <...> in the light of the recent work on logical formalism. From this new point of view mathematical structures become, properly speaking, the only „objects" of mathematics."

Every structure (in Bourbaki's sense) allows for a notion of „structure-preserving" map (morphism). Every type of structure with corresponding morphisms form a category. Hence the idea that categories reflect structures (further abstraction)

Indeed structure-preserving?

Ex: „forgetful group homomorphism"

$(G, \otimes) \to (1, \circ) \quad 1 \cdot 1 = 1$

In a structuralist setting the notion of isomorphism is basic and that of general morphism is derived.

The „elementary correspondence" (unordered pair) is always „reversible"

Cf. in ZF $\{a, b\}$ (pairing)

$\langle a, b \rangle = \{a, \{a, b\}\}$ (pairing twice)

5. Categories versus structures

Are categories structures?

Pro: (1) Obviously! And quite abstract ones

A set of morphisms + incidence relation (domain/codomain), + operation of composition + few axioms

(2) Functors are maps preserving this categorical structure

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u, "f"] \arrow[r] & A' \arrow[u, "f'"'] \end{tikzcd}$$

in C $\qquad$ in C'

Contra: (1) Big categories have no isomorphic copies (there is the categorical equivalence but...)

(2) There non-structure-preserving morphism:

If we think of morphism as functors functors are not structure-preserving..

An alternative view:

Structures are categories of a particular kind; not the other way round.
Categories instead of "types" of structure.
When the notion of morphism (functor) is taken as primitive there remain nothing structural about it.

Hilbert's (version of) axiomatic method and Set-theoretic foundations are two principal pillars of Structuralism.
Categorical foundations need a different axiomatic method!

Lawvere 1965 "Category of Categories as a Foundation of Mathematics"

"In the mathematical development of recent decades one sees clearly the rise of conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were ~~naturally~~

thought to be made of. The question then naturally arises whether one can give a foundation of mathematics which expresses wholeheartedly this conviction concerning what mathematics is about, and in particular, in which classes and membership in classes do not play any role:

<u>My claim:</u> The structuralist motivation doesn't fully correspond to the content of this paper.

Double thinking about the membership
1) "Official" – abstract relation
sets are "black boxes"
2) Usual: elements as constituents
(Substance strikes Back!)

The idea to describe sets in terms of morphisms (functions) is no more and no less structuralist than the "official" ZF. Only the choice of primitive is different.

Two layers in Lawveres categorical foundation of 1965

1) Standard Hilbert-style axiomatic method
(the structuralist layer)

Axioms for the "elementary theory":

$A \xrightarrow{x} B$

$\Delta_0(x) = A$

$\Delta_1(x) = B$

$$\begin{tikzcd} \cdot \arrow[r, "x"] \arrow[rr, bend right, "u"'] & \cdot \arrow[r, "y"] & \cdot \end{tikzcd}$$

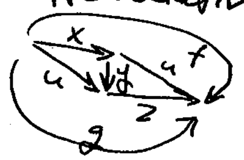
$\Gamma(x, y; u)$

(Objects are identified with the identity morphism

3 groups of axioms: Bookkeeping

Identity: $\Gamma(\Delta_0(x), x; x)$ $\boxed{\Gamma(x, \Delta_1(x); x)}$

Associativity



Definitions

$f = g$

"By a category we, of course, understand (intuitively) any structure, which is an interpretation of the elementary theory of categories and by a functor we understand (intuitively) any triple consisting of two categories and a rule T which assignes to each morphism x of the first category a unique morphism xT of the second category in such a way that ...."

2) Categorical layer

(basic theory + stronger theory)

Smooth passage:

« The axioms of the basic theory are those of elementary of abstract categories plus several more axioms »

But an abrupt change of the viewpoint:

« Of course, now that we are in the category of categories, the things denoted by capitals will be called categories rather than objects, and we shall speak of functors rather than morphisms »

A linguistic convention ??

The above structuralist definition of functor is no longer used; « Rule T » disappears.

Functor (= morphism) becomes a primitive!

1 - terminal

0 - initial (coterminal)

Axioms

2: $0 \to 1$

Object in A: $1 \xrightarrow{x} A$
(= identity morphism)

3:
$$\begin{tikzcd} 0 \arrow[r, "\alpha"] \arrow[dr, "\gamma"'] & 1 \arrow[d, "\beta"] \\ & 2 \end{tikzcd}$$

Morphism in A: $2 \xrightarrow{m} A$

$m \in A$

Composition in A: $3 \xrightarrow{t} A$

$\alpha t = f$

$\beta t = g$

$\gamma t = h$

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "h"'] & & \cdot \end{tikzcd}$$

4:
$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] \arrow[d] & 3 \\ 1 \arrow[r] \arrow[ur] & 2 \arrow[u] \end{tikzcd}$$

Associativity of composition in A

$4 \xrightarrow{a} A$

Theorem Schema:

$\forall A [A \models \Phi]$ where Φ is a theorem of the elementary theory

A is a category

\+ Axioms ensuring existence of functors

(in particular ensuring that the category of discrete categories (sets) is a model of the elementary theory.)

Claim: There is nothing structural in the second layer

Guess: The first layer is redundant:
An equivalent of the "elementary theory" can be assumed to begin with.
Circularity? No more than in axiomatic theories of sets: they also require some notion of set (class, collection) to begin with.
What about the language. Some version of internal language, diagrammatic language?
New notion of theory:
1) No categoricity in the old sense:
a category of models
2) "The theory appears itself as a generic model
Commentary of 2003 to Thesis of 1963
The distinction btw. theory and model is given up? Theory is no longer a "scheme" (cf. Hilbert)

Categorification (= taking morphisms – and higher morphisms – into account) and structural abstraction point to the opposite direction. The structural abstraction can be described as decategorification

Baez & Dolan 1998 („Categorification")

"The category FinSet, whose objects are finite sets and whose morphisms are functions is a categorification of the set N of natural numbers.
<...>
Long ago when shepherds wanted to see if the two herds of sheep were isomorphic they would look for an explicit isomorphism. In other words they would line up both herds and try to match each sheep in one herd with a sheep in another. But one day along came a shephard decategorification [= structural abstraction – AR]. She realised one could take each herd and "count" it setting up an isomorphism between it and some set of „numbers" which were nonsense words like "one, two, three,..." specially designed for this purpose. By comparing the resulting numbers, she could show that two herds were isomorphic without explicitely establishing an isomorphism!

In short, by decategorification of the category of finite sets, the set of natural numbers was invented. According to this parable, decategorification started out as a stroke of mathematical genius. Only later did it become a matter of dumb habbit, which we are now struggling to overcome by means of categorification." (1

Structural abstraction = de categorification!

The subject-matter of mathematics is a covariant transformation (= functor), not invariant form!

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6. Conclusion

Back to Pythagorean theorem. Development of mathematics is ultimately non-reversible. Instead of trying to extract the eternal ~~invari~~ invariant structure behind older mathematical results we should rather think how to translate them from the past to the present and further to future generations. Category theory can serve as a translation protocol.

Categorical foundations translate ~~the~~ mathematics of the past into mathematics of the future.

Lawvere & Rosebrugh 2003

"A foundation makes explicit the essential general features, ingredients, and operations of a science as well as its origins and general laws of development. The purpose of making these explicit is to provide a guide to the learning, use, and further development of the science. A „pure" foundation that forgets this purpose and pursues a speculative „foundations" for its own sake is clearly a nonfoundation."