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## **Isomorphism doesn't replace identity in categories or elsewhere**

**(talk at workshop "Structure and Identity", December 8,  
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### **Plan:**

**Part I: On Category theory and Structuralism**

**Part II: On Identity in Category theory**

**Conclusions**

## **Part 1: Structure**

**"[T]he debates on whole–part relationships, stability and change, subsistency through time and the like all start implicitly or explicitly from the idea that objects precede properties, ontologically. Recent developments in mathematics (category theory) open up roads to a formal treatment of a more “structural” viewpoint. "**

**– – Karin Verelst & Wim Christiaens 2007**

### **Historical Remark:**

**The structural viewpoint in Mathematics dates back to early Hilbert (of *Grundlagen*; see Hellman 2006). Its impact on mathematical practice culminates in 1960–ies through Bourbaki's *Elements* (1939 – ?). So it can be hardly called recent. What counts as recent is a matter of convention but...**

## **Claim:**

**As far as *recent* mathematical practice is concerned (categorical mathematics as opposed to Bourbaki-style set-theoretic mathematics) the structural viewpoint (Mathematical Structuralism) is *outdated*. Category theory suggests a genuinely new view on Mathematics, which reduces neither to Mathematical Structuralism nor to a form of "Substantialism" about mathematical objects (which is usually seen as *the* only alternative to Structuralism, cf. Verelst & Christiaens). Like the structural view the categorical view extends itself from the domain of Pure Mathematics to Mathematised Science (Physics, Biology,...) and beyond.**

## **Argument:**

**1) (a) What is Structure and (b) how it makes Identity problematic?**

**a) Structure is a thing determined *up to isomorphism* ; this agrees with all the current versions of Structuralism;**

**b) Expressions (borrowed from the mathematical parlance developed in 1960-ies) "unique up to isomorphism", "equal up to isomorphism" and the like suggest a relative notion of identity (Geach) and/or the introduction of a new abstract object ("structure", cf. Fregean abstraction) without making the situation clear.**

**Basic example:**

**Structural setting of Hilbert's *Grundlagen* of 1899  
popularised in the North America by Veblen and  
other "postulate theorists" and elaborated by  
Tarski et al.:**

**Formal theory + a bunch of its *isomorphic*  
models**

**This doesn't work in mathematically interesting  
cases for 1-st order theories because of the  
*Categoricity Problem*: formal theories, generally,  
have "non-standard" (non-intended) models.**

**A controversy about Hilbert–Tarski setting (which ignores the Categoricity Problem):**

**Substantialist view (Frege): isomorphism of models supervenes over identity of models (logically and ontologically).**

**Structuralist view (early Hilbert) : models and the notion of identity they are equipped with don't matter. Isomorphism is basic.**

**Problem: How to think of isomorphism without underlying identity?**

**Rather than resolve this problem (which might have no reasonable solution at all) I argue that...**

**2) The structural approach (as described) doesn't meet needs of Mathematics**

**because**

**ALL MORPHISMS**

**but not only *isomorphisms* matter.**



## **What is a general morphism?**

**Standard structuralist explanation (Bourbaki)**

**with the ex. of groups:**

**$(G, *)$  [or  $(\_, *)$  ?] – group *structure*, where  $G$  is "underlying set" and  $*$  – group operation.**

**Remark: officially  $(G, *)$  is a set, namely a subset of  $G \times G \times G$  but in practice one usually avoids this reduction.**

**This suggests the folk structuralist metaphysics of "matter" (underlying sets) and "form" (structures).**

**Def.:** Given groups  $(G, *)$  and  $(H, +)$  map  $f: G \rightarrow H$  is morphism of groups (called in Group theory

*homomorphism*) iff for all  $x, y$  from  $G$

$$f(x*y) = f(x)+f(y). \quad (*)$$

If, in addition,  $f$  is one-to-one on sets it is called *isomorphism*.

### **Terminological Remark:**

General morphisms are told to *preserve* a corresponding *structure*. But this makes a sense (given the above definition of structure) only for isomorphisms. Consider the case of "forgetful" group homomorphism, which sends any group  $(G, *)$  into  $1 = \{1, x\}$ , where  $1x1=1$ . Here is a better (albeit less popular) terminological proposal: morphisms *respect* structures. Or using CT terminology: morphisms are *functorial* and satisfy appropriate *coherence* conditions like  $(*)$ .

### **Claim:**

The standard terminology reflects the fact in the structuralist setting a general morphisms is conceived of after the special case of isomorphism. This is NOT justified.

### **Philosophical Argument:**

Coherence is not about preservation of structure, substance or something else. Notions of structure, invariant and symmetry don't generalise to the case of general morphism.

### **Mathematical Argument:**

Sets, groups, topological spaces and many other important (so-called) structures can be *fully* conceived of in terms of their general morphisms (as categories). This doesn't work when one uses *only* isomorphisms. Remind Categoricity Problem.

## **Corollary:**

To conceive of groups, topological spaces, etc., as *structures* one first takes for granted a model (the "intended" one) of Set-theory. This set-theoretic "matter" (a substantialist remainder) is indispensable in the structural setting because of the Categoricity Problem. So the structural view doesn't go through. CT allows for getting rid of this remainder, and so it makes the structuralist dream true. However CT treats isomorphisms on equal footing with morphisms of different sorts. This is why categorical reconstructions of mathematical "structures" are not themselves "structural". For in structural (Hilbert- or Bourbaki-style) reconstructions one conceives of reconstructed concepts *up to isomorphism* while in categorical reconstructions one conceives of them "*up to general morphism*".

**This makes a big difference.**

### **3) How to think "up to general morphism" (rather than up to isomorphism)?**

To conceive of mathematical concepts (sets, groups, topological spaces,...) as *categories* (objects and morphisms between them). This later notions can be taken as primitives and the rest construed out of them (if fact objects are dispensable: see below).

Categories (at least "big" ones like SET, GROUP, TOP, etc. which are most useful and most common in mathematical practice) are *not* structures but things "more general" (or in any event somewhat different) than structures. Noticeably they don't have "isomorphic copies" (all sets are in SET, etc.).

**Categorical model-theoretic setting (instead of Hilbert–Tarski’s): *Functorial Semantics* (Lawvere 1963–2004).**

**a category of models + theory as "generic model"**

**Remark:**

**The requirement of *categoricity* (in the old Veblen’s sense) doesn’t make sense for Functorial Semantics.**

**For except trivial cases categories of models have more than just one object (up to isomorphism). One looks instead for "good" categorical properties of these categories (universal properties, etc.). Nothing like the old structural approach.**

## **Terminological Remark:**

**The expression "up to general morphism" is not appropriate; I suggested it only for the sake of the argument. For objects (and morphisms) in a category are, generally, *many* and except special cases they don't collapse to *one*.**

**What is going on with Identity in categories?**

## Part II: Identity

General Remark: Ambiguities about identity of mathematical objects like numbers, geometrical figures, algebraic groups, etc. are traditional and systematic. There are exactly five Platonic solids, *the* cube is one of them, but still (in a different sense) there are many cubes. *The* symmetric group  $S_3$  has as many "isomorphic copies" as one likes. The type–token distinction is not a remedy (because there is no obvious notion of mathematical token available).

Category theory makes it more difficult to hide Identity Problem behind the usual talk about "isomorphic copies" and calls for some solution. Consider *the* category of sets, groups or the like (i.e. usual working categories). As far as "all" such things are comprised by a given category all isomorphic copies must be already there. So one



**must identify and distinguish them properly. Or one may find an excuse:**

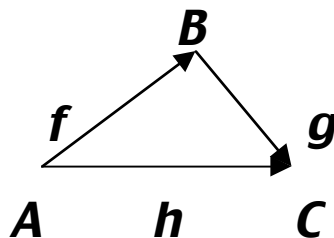
**“Strictly speaking the “canonical” isomorphisms. . . are necessary (instead of equality—A.R.) . . . Having realised this it is best, in the interests of clarity, to forget them.”**

**-- Fourman, "The Logic of Topoi" , 1977**

**Thus CT amplifies traditional worries about identity in Mathematics and makes people think about a structuralist solution ("isos instead of identities"). But CT doesn't suggest such a solution.**

## Two identity concepts in CT:

(1) God-given identity (equality) = as everywhere in Mathematics. Its principle role is in the definition of the operation of *composition* of morphisms:  $fg = h$ . The same graphically:

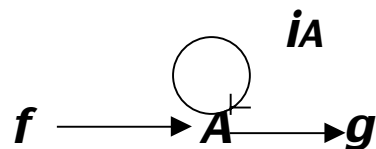


Note: Hereafter I write composition in the direct geometrical order.

Note: Not all morphisms are composable, so the operation is *partial*. Composition  $fg$  of Morphisms  $f: A \rightarrow B$  and  $g: C \rightarrow D$  exists iff  $B=C$ . Thus identity (1) gets involved when one checks whether or not given morphisms are composable (in a given order)

(2) *Identity morphism* of the form  $i_A: A \rightarrow A$  (each object is required to have one):

**Def.:** Given object  $A$  for all incoming morphisms  $f: \rightarrow A$   $f i_A = f$  and for all outgoing morphisms  $g: A \rightarrow$   $i_A g = g$ . Graphically:



**Note:** Given an object  $A$  there are, generally, other morphisms of the form  $A \rightarrow A$  than  $i_A$ .

**Note:** Uniqueness of  $i_A$  follows from its definition.

**Remark:** Objects can be formally replaced by their identity morphisms. So the notion of object is actually redundant in CT. Objects *are* identities.

**Remark:** Identity (2) is *contextual* in the sense that its definition involves other morphisms (in fact all "neighbouring", i.e. incoming and outgoing morphisms) of the same category.

**Remark:** Identity (2) is a *morphism*, not a relation. (1) is taken for granted in the definition of (2). So officially only (1) is "real" identity while (2) is a specific mathematical object.

(1) and (2) allow for the following definition of *isomorphism*:

**Def.:** Morphism  $f: A \rightarrow B$  is called *reversible* or *isomorphism* iff there exist  $g: B \rightarrow A$  such that  $f * g = id_A$  and  $g * f = id_B$ . (Both conditions are essential.)

**Note:** Any identity morphism is isomorphism (in the sense of this definition) but the converse doesn't hold.

**Remark:** This definition doesn't work without (1) and (2).

There is apparently no way to take the notion of iso in CT as primitive.

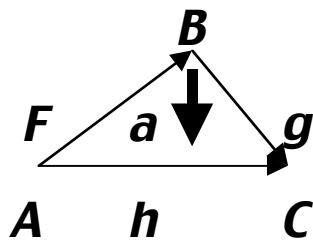
**Alternative Project:** dispense with (1) in favour of (2) (or reverse their relationships)

**Purpose:** Not take identity for granted (moreover so since it doesn't work as it should).

**Strategy:** internalisation of identity (along with other logical notions)

**Problem:** the usual notion of the internal logic (internal language) of category (topos) doesn't internalise identity (1) but only "carries it through" to the corresponding formal language. The obvious reason for it is that (1) unlike (2) is not a "properly categorical" notion.

**Idea:** Replace equality  $f g = h$  by identity (or other) morphism  $a : f g \dashrightarrow h$ . Graphically:

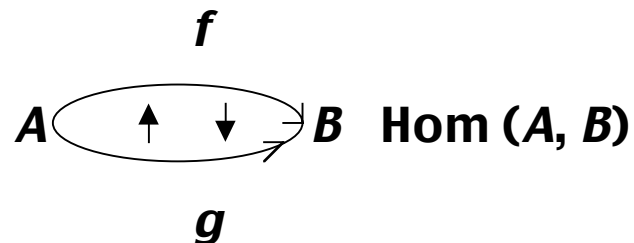


Compare the structuralist idea of "replacement of identity by isomorphism". However  $a$  is not necessary reversible (not iso). And this makes a big difference.

More precisely (and using a different “shape”):

Enrichment of categories:

Given category  $D$  consider Hom-categories instead of Hom-sets (or Hom-classes)



and the category of Hom-categories (of a given category) with product  $\times$ . This gives a new way of thinking about composition in  $D$ :

$$\text{COM: Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$$

Now identity (1) is still necessary in Hom-categories and in the ambient category of Hom-categories but no longer in  $D$ !

Instead in  $D$  we have:

– associativity coherence (2-) morphisms of form

$$a : (f g) h \dashrightarrow f(g h)$$

– identity coherence (2-) morphisms of forms

$$r : f i_A \dashrightarrow f \quad \text{and} \quad l : i_A g \dashrightarrow g.$$

Def.: Given these data  $D$  is called *bi-category*.

When  $a, r, l$  are identities the bi-category is called *strict*,

when these morphisms are isos (reversible) the bi-category is called *weak*;

otherwise it is called *lax*.

In any event  $a, r, l$  are supposed to satisfy indispensable *coherence conditions*.



**Example of weak bi-category: fundamental bi-groupoid of topological space  $T$**

**objects (0-morphisms): points •**



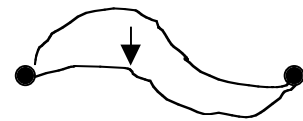
**morphisms (1-morphisms): paths**



**$[0, 1] \dashrightarrow T$**

**morphisms of morphisms (2-morphisms):**

**homotopies of paths (relative to endpoints)**



**Paths are composed (only) up to homothopy  
(because of the need of re-parameterisation).**

**The above example has a structuralist flavour because homotopies are reversible. Similar examples can be found in any “static” space where “every motion is reversible”. But more general *lax* bi-categories, which involve an irreversible dynamics, **fall apart** of the structural view.**

**How to avoid (=) also at the “meta-level”?**

**Toy example of stabilisation in an omega-category: symmetric omega-group:**

**...Aut( ... Aut(SN) ... )... [ = SN ] N =/ 1, 2, 6**

**SN is determined "up to itself"; = is not needed; identity (unit) of the groups is *canonical*:**

**Unlike other elements of the group its *Identity* (unit) always maps to itself but not permutes with other elements. It is a fixpoint.**

**Question: Is symmetry (and hence reversibility) essential for stabilisation?**

**Answer: Probably not. Look at examples outside pure logic and mathematics: biological individuals are not "reversible".**

## **Conclusions:**

**1) Structuralism inherits Platonic bias toward "Mathematics of eternal Forms" while Category theory rather supports a wide Kantian view according to which spatio-temporal intuitions are fundamental in maths. Categories are, generally, not structures. The categorical view doesn't reduce to the structural one. Isomorphism doesn't replace identity.**

**2) “Identity is a relation given to us in such a specific form that it is inconceivable that various forms of it should occur”**

**-- Frege 1903**

**This attitude must be changed. Compare the case of bosons and fermions (French&Krause 2006). Identity must be "internalised" with the rest of logic. It needs to be (re)introduced in a given theoretical context but not rigidly fixed in advance. Although CT in its existing form doesn't allow for this CT-methods may be useful for the future theory of identity.**