Chapter 1
Introduction

> *Logical and mathematical concepts must no longer produce instruments for building a metaphysical "world of thought": their proper function and their proper application is only within the empirical science.*
>
> Ernest Cassirer

> *Mathematics is a part of physics. It is a part of physics where experiments are cheap. [..] In the middle of the 20th century there were attempts to separate mathematics from physics. The results turned to be catastrophic.*
>
> Vladimir Arnold

The main motivation of writing this book is to develop the view on mathematics described in the above epigraphs. Some 200 years ago this view used to be by far more common and easier to justify than today. It is sufficient to say that it made part of Kant's view on mathematics, and that Kant's view on mathematics remained extremely influential until the very end of the nineteenth century. When Cassirer defended this Kantian view in the beginning of the twentieth century it was already polemical. When Arnold defended it in the end of the twentieth century and in the beginning of this current century it already sounded as an intellectual provocation, and so his words sound today. Kant, Cassirer and Arnold do not speak about the same mathematics: each speaks about mathematics of his own time. So the growing polemical attitude to their shared view reflects not only a change of the common opinion about the subject but a change of this subject itself. It is a common place that the modern mathematics is more abstract and more detached from physical experience than it used to be in Euclid's times and in Kant's times. When I say that I nevertheless want to defend the view on mathematics as a part of physics this means that I also want to contribute to changing the character of current mathematics, but not only to changing the common views about it.

A. Rodin, *Axiomatic Method and Category Theory*, Synthese Library 364,
DOI 10.1007/978-3-319-00404-4_1,

The above is a motivation behind this book but not its purpose. The purpose is much more limited. In order to justify the view on mathematics as part of physics I would need to write at least as much about physics as about mathematics. But this book is mainly about mathematics and about logic; physics is mentioned in it only occasionally. Yet more specifically I shall focus on the Axiomatic Method and Category Theory (including the categorical logic, which is a part of modern logic using category-theoretic methods). Let me explain why.

When Arnold talks about recent attempts to separate mathematics from physics he has in mind *Elements of Mathematics* by Nicolas Bourbaki (1939–1988) that aims at developing the whole of mathematics systematically from the first principles, i.e., on an axiomatic basis. Bourbaki's *Elements* continue the long tradition of presenting renewed foundations of mathematics in the form of *Elements*: this tradition begins with Euclid's *Elements* (and earlier versions of Greek *Elements* that have been lost) and continues through the whole history of mathematics until today. (I say a bit more about this tradition in the introductory part of Part **I**.) Arnold sees the key to the problem in Bourbaki's Axiomatic Method, and takes a notoriously hostile attitude towards the Axiomatic Method in general. I observe on my part that the problem of separating mathematics from physics concerns the specific form of the Axiomatic Method used by Bourbaki rather the Axiomatic Method in general. It is clear, in particular, that Euclid's method does not produce the same effect. And I further observe that Bourbaki's Axiomatic Method is a version of Hilbert's Axiomatic Method presented in Hilbert's *Foundations of Geometry* of 1899, which is another example of renewed mathematical *Elements* playing a more special but perhaps even more important role in the twentieth century mathematics than Bourbaki's *Elements*. So I conclude that the origin of Arnold's problem should be traced back at least to the beginning rather than only to the middle of the twentieth century. This explains my focus on Axiomatic Method and its history.

Why Category Theory? The mathematical notion of category (which has no immediate relation to the philosophical notion widely known under this name) was invented in 1945 by Eilenberg and MacLane for general purposes, some of which I explain in Chap. **9**, see also Kromer (2007) for details. In his thesis defended in 1963 (Lawvere 1963) and a series of papers based on this thesis (Lawvere 1964, 1966a,b, 1967). Lawvere put forward a program of categorical (i.e., category-theoretic) foundations of mathematics and opened a new research field known today under the name of categorical logic, see Marquis and Reyes (2012) for the most recent historical account. Although Lawvere and other people who pursued the program of categorical foundations have never explicitly challenged Hilbert's Axiomatic Method (albeit they did and do challenge some special applications of this method, most importantly its applications in the standard axiomatic set theories) I shall try to show in this book that some recent works in categorical logic and new foundations of mathematics effectively modify Hilbert's Axiomatic Method and develop it in a wholly new direction. As it always happens in the intellectual history this new development continue some earlier developments, which I shall also take into account. In the last Chapter of this book I generalize upon these tendencies and describe a hypothetical New Axiomatic Method, which admittedly does not yet exist in the form of precise logical and mathematical procedure. I hope that my proposed general philosophical vision of this new method will contribute to its future technical development and also help to use it outside the pure mathematics and its philosophy.

As the reader shall see the New Axiomatic Method establishes closer relationships between mathematics and physics and so suggests a solution of Arnold's problem. Although I cannot fully justify this claim in this book (because I am not going to discuss physics systematically) I do prepare a philosophical background for such a justification. The issue of relationships between mathematics and physics is a hardcore philosophical issue, and I believe that Arnold's problem cannot be solved without taking this philosophical issue seriously. Another hardcore philosophical issue that comes into the play as soon as one discusses the use of Axiomatic Method in mathematics is the relationships between mathematics and logic. This latter philosophical issue unlike the former is in the focus of this book. The main philosophical dilemma that I consider is, roughly, this: either (i) logic is fundamental in the sense that it gives us an independent access to an ideal space of logical possibilities where the actual world exists side-by-side with plenty of other possible worlds, which can be explored only mathematically, or as Cassirer insists in the above epigraph, (ii) logic and mathematics must stick to the actual world as we know it through empirical sciences, and by all means must avoid producing possible "metaphysical worlds of thought" even if these appear more logically coherent and more mathematical beautiful than our actual world. With many important reservations that this rough formulation requires I shall defend the latter view. The former view (which also obviously needs a more precise formulation) I call *logicism*, and when it is applied to mathematics I call it *mathematical logicism*. Beware that this meaning of "mathematical logicism" is broader than Russell's radical version of mathematical logicism according to which mathematics *is* logic (Russell 1903). So a central purpose of this book is to refute mathematical logicism and defend an alternative way of thinking about logic and mathematics.

Talking about these philosophical issues I would like to stress that I study primarily their implementation in mathematics. When in the beginning of the twentieth century Cassirer, Russell and other people discussed hot philosophical issues concerning mathematics and logic they not only made general philosophical arguments but also referred to the actual state of affairs in their contemporary science and to the history of these subjects. They also often contributed themselves to the ongoing research in mathematics and logic. In this book I follow the same pattern of philosophical discussion paying a lot of attention to some recent mathematical works and to the history of the subject but without trying to make any mathematical contribution.

Before I summarize the content of this book chapter by chapter let me say a few more words about its style and its methodology. I stick to the traditional idea according to which philosophy and its history naturally combine together. When this view is applied to the philosophy of science and mathematics the result is sometimes called the *historical epistemology* (Rheinberger 2010). So what I am doing in this book can be described as a historical epistemology of logic and mathematics.

However one important reservation is here in order. In my understanding the past history, the present state of affairs and the anticipated future of a given discipline are parts of the same whole. This whole can be described as the current state of affairs in a broader sense of the word, which includes both the historical reflection upon the past and the projection towards the future of the given discipline. When I talk in this book about mathematics and its philosophy I think about these subjects in this way. When such a view is called historical this should mean the attention to development of the given discipline but not the exclusive attention to its past.

Although I write about logic and mathematics I don't use myself any formal logical or other mathematical means for expressing and justifying my arguments. A century ago this point would be hardly worth mentioning but since using formal methods in philosophy in general and in philosophy of mathematics in particular is nowadays popular (particularly in the philosophical school that calls itself *Analytic philosophy*) this point requires some explanations. Without going into a long discussion on this sensitive issue let me boldly express my believe that the natural language and the philosophical prose remain so far the best instruments for historical and philosophical work, or at least for the kind of such work that I want to do. The clarity and the exactness that formal methods bring to philosophy come with a price, which for my purposes is unacceptable. This price amounts to certain philosophical assumptions, without which these formal methods cannot work. I am not prepared to pay this price until I can see clearly these assumptions and thus know the price exactly. A philosophical and historical analysis of the notion of logical formalization is a part of my present project (see particularly Chaps. **3** and **10**). Even if a formal theory of formalization is possible I cannot see that it can be useful for this purpose. I shall not return to the question of using formal methods in philosophy in what follows but the reader will see that my analysis of the idea of logical formalization hardly supports the idea of using it as an universal instrument for philosophizing.

Although I am not going to use formal methods for philosophical purposes the reader will find below a lot of rudimentary mathematics. Since this book is about mathematics, and a part of this book is about very recent mathematics, which still remains a work in progress (see Sects. **7.9–7.10**), this is not surprising. So let me explain my strategy of presenting the relevant mathematical content and mention some mathematical prerequisites for reading this book. My intention is to make this book readable both for a working mathematician interested in philosophy and history of this discipline and for a philosopher like myself, who studies (or wants to study) mathematics and its history, and finds a broad philosophical inspiration in this discipline. To present a fragment of modern mathematics to a wider audience is a very challenging task, which normally should not be combined with any philosophical agenda. I certainly do have a philosophical agenda, which I have already outlined earlier in this Introduction. This is why writing this book I have tried to reduce the burden of explaining mathematics to minimum. At the same time I tried to avoid any *metaphoric* talk about mathematical concepts – even if some people would argue that any talk about mathematics outside the pure mathematics is doomed to be metaphoric. So I could not avoid the burden of explaining some mathematics completely but tried to use the most elementary examples and also tried to use some existing introductory expositions when such were available. In each particular case I refer to the existing mathematical literature and chose this literature accordingly to my specific purpose.

For the first superficial reading the given book is self-sustained and, as I hope, it gives a right idea of what I am after. A more attentive critical reading is by far more demanding. The ideal judge of this book is a working mathematician who is also a working philosopher and working historian of mathematics having some broader philosophical and scientific interests, which include some interest in physics, its history and its philosophy. I know several people who at some degree of approximation fit this description but I rather imagine an average reader of this book as a person like myself who during these recent years has learnt some philosophy, some mathematics and some history of both subjects, and who tries to make these ends meet. I shall say more about the mathematical prerequisites and give some suggestions for reading (in addition to references found in the main text) in the following summary of the Chapters.

Part **I** of this book treats the history of Axiomatic Method. As I have already explained this history is not only about the past. Only Chap. **2** on Euclid concerns what is indeed in the past (albeit in Sect. **2.5** I show that even in this case the past continues to live in the present); Chap. **3** on Hilbert treats (in the original historical context) what remains today the standard notion of Axiomatic Method; Chap. **5** on Lawvere treats what I suggest as a conceptual basis of the New Axiomatic Method. So these three Chapters of this book present, roughy, the past, the present and the anticipated future of the Axiomatic Method. Chapter **4** is reserved for studying the fate of Hilbert's Axiomatic Method in the twentieth century mathematics.

Instead of trying to reconstruct a general history of Axiomatic Method, I decided to choose these three key figures and look at the relevant parts of their work more attentively. Although a historical discussion on Euclid found in Chap. **2** may appear out of place in a book about today's mathematics it is important for me for several reasons. According to a common view (supported by Hilbert himself at some occasions), Hilbert's Axiomatic Method improves upon Euclid's method in terms of logical rigor and logical clarity. Of course, in such a general formulation this view can hardly be challenged. However in order to see how exactly this improvement on rigor and clarity has been achieved in the twentieth century we need first to study Euclid's method on its own rights. This requires some special hermeneutical techniques, which are well-known to historian of mathematics but are less familiar to logicians, mathematicians and philosophers who also write about this subject. We shall see that in some respects Euclid's and Hilbert's method are different in principle, so that the difference between these methods does not reduce to differences in degrees of continuous magnitudes like rigor and clarity. In addition to my attempt to reconstruct Euclid's mathematical reasoning in its proper terms (and in some terms borrowed from Greek philosophy) I explain in this Chapter the relevance of Euclid's geometry to Kant's philosophy of mathematics. In the end of this Chapter I point to some Euclidean patterns of reasoning in the recent mathematics. The main textual reference in this Chapter is obviously Euclid's *Elements*, which is now available in a new English translation (Euclid 2011). An interested reader who would like to study the history of Greek mathematics more broadly and would like to better understand Euclid's special place in this history (this is an important subject that I wholly skip in this book) is advised to begin with (Heath 1981, 2003) and then study more recent secondary literature.

Chapter **3** on Hilbert is also written in a historical style and contains extended quotes from Hilbert's writings. Although I leave outside the scope of my discussion most of the contemporary context of Hilbert's work I follow the development of Hilbert's own ideas rather closely and distinguish in it several stages. In its narrow historical aspect my treatment of Hilbert's work contains nothing original. However I also make an attempt to reconstruct the history of some relevant notions (or at least to keep track of their changing meaning) including the notion of being formal. This historical discussion is combined with an explanation of Hilbert's Formal Axiomatic Method, which can be used by a non-mathematical reader for the first acquaintance with this basic method of modern mathematical reasoning. Someone well acquainted with this method will find here an analysis of certain assumptions required by this method, which remain tacit when this method becomes an intellectual habit and is used automatically. I shall pay a lot of attention to philosophical remarks made by Hilbert in his presentations of Axiomatic Method trying to reconstruct Hilbert's thinking and its philosophical motivation. I also discuss in this Chapter some related subjects including the notion of logicality, diagrammatic and symbolic thinking and some others. This Chapter presents (in its historical original form) the core notion of modern Formal Axiomatic Method, which I contrast in what follows to more traditional Euclid's method, on the one hand, and to some later versions of Axiomatic Method including the anticipated New Axiomatic Method, on the other hand.

The main suggested reading for Chap. **3** is Hilbert's *Foundations of Geometry*, which exist in multiple editions including the English edition (Hilbert 1950) and some later English editions. I highly recommend this reading also to a non-mathematical reader of this book because the real subject-matter of this short masterpiece is the Axiomatic Method itself rather than geometry, and so this short book can be used as a shortcut to the modern style of mathematical thinking. For a later more developed systematic presentation of Formal Axiomatic Method and its underlying philosophy I refer the reader to Tarski's textbook (1941). This textbook presents in a very clear form a philosophical view on logic and mathematics that I discuss in my present book.

In Chap. **4** I talk about applications of Hilbert's Axiomatic Method in the twentieth century mathematics and stress the fact that it has hardly ever been used in its original form and for its originally intended purpose. I discuss from this point view some formal studies of axiomatic set theories, Bourbaki's *Elements of Mathematics Bourbaki:1939–1988* and more specifically an unpublished Bourbaki's draft (Bourbaki 1935–1939). My main observation amounts to saying that both the modern set theory and Bourbaki's structural mathematics can be described in Hilbert's terms as a *metatheory* or in Tarski's terms as a *model theory* of certain Hilbert-style axiomatic theory or, more typically, of a number of such theories. Since this metatheory or model theory itself is developed by some other means (i.e., *not* axiomatically in Hilbert's sense) one can say that the mainstream mathematics widely applies Hilbert's Formal Axiomatic Method only with a pinch of salt. In the mainstream structural mathematics of the twentieth century this method serves as a method of definition and constructing new concepts rather than method of building deductive theories. On the basis of this observations I criticize Hilbert's Axiomatic Method arguing that it is not apt to support mathematical theories useful in the modern physics. Finally I consider in this Chapter Tarski's topological model of intuitionistic propositional logic (Tarski 1956) and stress its unusual character: although, technically speaking, there is no big difference between modeling a given formal theory and modeling a given logical calculus, philosophically it makes a huge difference and requires a rethinking of the whole idea of Axiomatic Method. Although Tarski himself does not draw from this work such far-reaching conclusions I use this example in the following Chapter as a historical prototype of the New Axiomatic Method.

In addition to the literature referred to in Chap. **4** I suggest reading the classical introduction (Bar-Hillel et al. 1973) to the modern axiomatic set theory including its last philosophical chapter, and Galileo's *Two New Sciences* (Galilei 1974) where the author stresses the constructive experimental character of the New Science against the background of the earlier Scholastic patterns of doing science.

Chapter **5** plays a central role in this book because here I first introduce the notion of category and discuss a new notion of Axiomatic Method, which emerges in category theory and, more specifically, in categorical logic. Although categorical logic is already a well established subject (see Marquis and Reyes 2012 for a historical introduction) I decided to follow here the pattern of the first two Chapters and focus my attention on the work of one particular person, namely Lawvere, who founded this discipline in 1960s; as before I combine here a historical and a systematic orders of presentation and pay a minute attention to Lawvere's philosophical comments found throughout his writings. After presenting Lawvere's categorical axiomatization of (the category of) sets (Lawvere 1964) and of the category of categories (Lawvere 1966a), which gives the first idea of using the category theory for axiomatization, I turn to Lawvere's critique of the standard Formal Axiomatic Method as "subjective" and explain his idea of *objective* conceptual logic realized by category-theoretic means. I begin this latter discussion by considering two Lawvere's papers (Lawvere 1966b, 1967) that mark the birth of the categorical logic, and in the same context explain Lawvere's notion of quantifiers as adjoint functors to the substitution functor. Then I make a digression on Curry's *combinatorial logic*, type theory and the so-called *Curry-Howard correspondence*, and show how these conceptual developments combine in Lawvere's notion of Cartesian closed category. Then after a brief discussion on Lawvere's notions of hyperdoctrine (that conceptually connects to the discussion on homotopy type theory found in Sect. **7.9**) and functorial semantics (further discussed in Sect. **10.2**) I turn to philosophical issues and discuss the role of Hegel's dialectical logic in Lawvere's thinking, which Lawvere stresses himself at many instances. Here I provide a philosophical reconstruction of Hegel's distinction between the *objective* and the *subjective* logic and then describe how this philosophical distinction is realized by Lawvere with the technical means of categorical logic. This discussion helps me then for interpreting the groundbreaking paper (Lawvere 1970b) where Lawvere suggests his axiomatization of topos theory and demonstrates the strength of his notion of internal logic of a given category. In the last Chap. **10** I use Lawvere's axiomatization of topos theory as a basic example of the new axiomatic approach, which I try to describe in general terms under the title of New Axiomatic Method.

For a better understanding of Chap. **5** it would be useful if the reader get some knowledge of basic category theory beforehand (albeit this is not an absolutely necessary requirement and the reader can also follow references during the reading). For a non-mathematical reader or a reader with a modest mathematical background I recommend (Lawvere and Schanuel 1997; Lawvere and Rosebrugh 2003) co-authored by Lawvere as a very accessible introduction into the subject. For a mathematical reader not familiar with categorical logic I recommend (MacLane and Moerdijk 1992) that covers most of the mathematical material that I discuss in this Chapter (but unfortunately skips hyperdoctrines). There is a huge gap in terms of required mathematical skills between these two suggested readings and by the present day this gap has not been yet filled in spite of many very valuable attempts such as Reyes et al. (2004). I believe that there is a principle and not only technical and pedagogical difficulty involved with the project of writing a fairly elementary introduction to category, topos theory and categorical logic. The problem is that the elementary introductions like Lawvere and Schanuel (1997), Lawvere and Rosebrugh (2003), and Reyes et al. (2004) begin with considering the category of finite sets, which are first introduced naively as bags of dots and then are treated in terms of their maps. Although such an introduction is geometrical in its character the basic geometry reduces here to the geometry of bags of dots, which is a geometry of a very special sort. A genuine continuous geometry appears then only at the much more advanced level and in a much more abstract form of Grothendieck topology and Grothendieck topos, which are systematically treated in MacLane and Moerdijk (1992) and other books of the same advanced level. So it still remains, in my view, a challenging task to follow Hilbert's example and rewrite Euclidean or other simple intuitive geometry in new categorical terms. Voevodsky Univalent Foundations discussed in Sect. **7.10** appear to be a step in this direction.

Talking about elementary introductions to category theory and topos theory I would like also to mention (1992) by McLarty. The expression "elementary' theory" in the title does not stand for being easy to grasp by a beginner but is used in the technical sense of being a first-order theory in the sense of modern logic and the standard Formal Axiomatic Method. This book is a systematic presentation of category and topos theory which fully complies with the requirement of Formal Axiomatic Method and at the same time treats the internal logic of a given topos and the idea of internal description of a given topos with its internal language. So for a logically-minded philosopher habituated to formal methods this book may also serve as an introduction into the subject. I would like to stress however that since in the present book I discuss specific features of Lawvere's axiomatic thinking, which fall apart from the standard Formal Axiomatic Method, studying McLarty's book does not replace studying Lawvere's original works even if, formally speaking, McLarty's book fully covers the same subject.

Part **II** is devoted to the notion of identity (in mathematics). This may appear as a side subject with respect to the general theme of this book but it is actually not. A mathematical logicist argues like this: in order to build a mathematical theory in an axiomatic form one needs first to fix some basic logical notions like that of being the *same* (or being equal). Unless this is done beforehand and quite independently from the content of any particular mathematical theory, so the argument goes, no axiomatic construction of mathematical theories is possible. A similar point can be made, of course, about other logical notions including logical connectives "and", "or", the notion of logical inference, of truth-value, etc. This standard logicist argument does not go through in the case of categorical logical, or at least it does not go through immediately, because the categorical logic *internalizes* the logical notions, i.e., reconstructs them in terms of a given mathematical theory (see Sects. **5.9** and **10.3**). This applies to logical connectives, the relation of inference, quantifiers, truth-values and to some other logical notions. It also applies to the logical identity relation but this case turns to be both more difficult and more mathematically and philosophically interesting than other cases. So I treat it systematically in the two consequent Chapters making the Part **II**.

In Chap. **6** I consider the question of identity/equality in mathematics in general beginning with some naive observations and historical examples. In particular, I briefly consider Plato's view according to which the mathematical equality is a weak form of strict identity: while the latter applies only the ideal world of Forms the former applies in the world of mathematics, which takes an intermediate position between the world of immutable Forms and the world of changing material beings. Plato's theory is an echo of the modern mathematical structuralism discussed later in Chap. **9**. In Chap. **6** I also show the significance of discussions about identity in mathematics in Frege's and Russell's works for establishing the logicist view on mathematics in the end of the nineteenth and the beginning of the twentieth century. Then I turn to more theoretical subjects including a discussion on classes and individuals, and a discussion of the distinction between logical extension and logical intension. This Chapter resumes with a discussion on Martin-Löf's intuitionistic type theory (Martin-Löf 1984) that provides a theory of identity types, which is very non-trivial in the intensional case. I compare Martin-Löf's approach to identity with Frege's approach and reconsider Frege's famous *Venus* example through the optics of Martin-Löf's type theory.

Chapter **7** continues to treat the issue of identity but this time with new approaches coming from category theory and some related fields. In the beginning of this Chapter I stress the conceptual similarity and the conceptual difference between the logical notion of relation and geometrical notion of transformation aka mapping or simply map. On this basis I re-introduce the notion of category with a naive geometrical example, stress the geometrical origin of categorical thinking and the relationships between category theory and Klein's *Erlangen Program*. (I come back to this topic in Sect. **9.6**). Then I turn to more advanced geometrically motivated categories and show how they realize the idea of identity as a map (rather than a relation). In particular, I consider Bénabou's *fibered categories* (Bénabou 1985) and higher categories (aka *n*-categories) – first in an abstract form and then in the geometrical form of homotopy categories. So I approach the hot subject of *homotopy type theory*, which brings together identity types of Martin-Löf's type theory and the geometrical approaches to identity and the homotopical higher category theory. When I began to study these two subjects about 10 years ago the precise mathematical connection between them was not yet established and the mathematical discipline of homotopy type theory did not yet exist. So it was for me a great relief to learn that these ideas combine not only at the level of speculative philosophy but also in precise mathematical terms. I conclude this Chapter with a presentation of Voevodsky's new foundations of mathematics that he calls Univalent Foundations (Voevodsky 2010, 2011; Voevodsky et al. 2013). In Chap. **10** I refer to the Univalent Foundations as an example of a new form of axiomatic presentation along with the example of Lawvere's axiomatic topos theory.

As a general mathematical reading for Part **II** I recommend Leinster's book (2004) on higher category-theory, which has great pedagogical advantages, Granstrom's book (2011) on type theory, which also provides a philosophical perspective on this theory, Jacob's book (1999) that stresses the link between categorical logic and type theory. The homotopy type theory has been not yet exposed in textbooks but there are very clear expository papers and the collective monograph (Awodey and Warren 2009; Awodey 2010; Voevodsky et al. 2013).

Last Part **III** of the book treats two different subjects, which fall under the scope of Hegel-Lawvere's distinction between objective and subjective features of logic and mathematics. In Chap. **8** I discuss the issue of mathematical intuition from a historical perspective and argue using some historical examples that mathematical intuitions change through the historical time at least as rapidly as do mathematical concepts. The main purpose of this Chapter is to refute the popular opinion according to which mathematics always develops by increasing its degree of abstractness and according to which the highly abstract character of modern mathematical concepts does not allow for a faithful intuitive representation in principle. I suggest an alternative picture of the historical development of mathematics where concepts and intuitions develop side-by-side but sometimes the conceptual development takes over the intuitive development and sometimes, on the contrary, the intuitive development takes over the conceptual one.

I expect that a phenomenologically-minded philosophical reader may object that what I discuss is not the strict philosophical notion of intuition but rather a commonsensical meaning of the word "intuition" as a bunch of helpful analogies borrowed from the everyday life or elsewhere. I argue in this Chapter that the changing mathematical intuition that I describe qualifies at least as intuition in Kant's sense of the term. The lack of discussion of Husserl's views is indeed a significant lacuna of this Chapter that I cannot easily fix. So I leave it for a future work.

Although I wholly share Lawvere's Hegelian view concerning the objective character of scientific logic (which perfectly squares with Cassirer's view on the place and the role of mathematics and logic expressed in the above epigraph) I also stress the role of the subjective intuition because it provides the necessary link that connects the pure mathematics to the individual sensual experience to the scientific empirical methods to the whole body of empirical science. Without such a link Hegel's objective dialectical logic too easily turns into a new form of speculative dogmatic metaphysics wholly detached from reality. One may suggest that since the dogmatic dialectics is an obvious oxymoron it cannot refer to anything real. But the dialectical logic quite rightly protects one from such naive conclusions made on abstract logical grounds: as a matter of painful historical fact the examples of dogmatic misuse of philosophical dialectics are abound.[1]

In Chap. **9** I discuss structuralism including its mathematical variety. Considering structuralism as a suggestive idea rather than a system of stable philosophical views I argue against the received view according to which category theory brings about a new variety of structuralism and provides a new framework for developing structural mathematics. I recognize the role of structural thinking in the development of category theory and describe this role in this Chapter. In particular, I elaborate on Eilenberg and Mac Lane's idea of category theory as a continuation of Klein's *Erlangen Program* (Eilenberg and MacLane 1945). This very analogy allows me to specify the crucial difference between Klein's structural thinking and new categorical thinking: when groups are generalized up to categories the notion of invariant structure is replaced by the notion of covariant or contravariant functor. I argue that the structuralist thinking about functoriality in terms of preservation of invariant structures is, generally, inappropriate; then I suggest a different philosophical view (or rather another suggestive idea) where the notion of functoriality (i.e., of co- and contravariance) becomes central. Although this conceptual development begins with a mere generalization of the structuralist *Erlangen Program* it brings about a new view, which is very unlike the structuralist view. In the end of this Chapter (Sect. **9.8**) I suggest a purely geometrical way of thinking about categories alternative to the more convenient way of thinking about categories as categories of structures. The basic idea here is thinking of geometrical objects as maps from types (of geometrical objects) to spaces. I demonstrate this approach with some elementary examples from the twentieth century geometry. Thus in my suggested post-structuralist picture the notion of object (this time understood as a map) becomes once again central.

The conceptual change described in Chap. **9** affects not only the choice of structures explored with the Formal Axiomatic Method but also this method itself. So in the concluding Chap. **10** I make the long-promised attempt to describe the New Axiomatic Method more systematically. I first describe the two basic functions of Axiomatic Method, which Lawvere calls the *unification* and the *concentration*. Here I contrast the unificatory strategy of the New Method to the more traditional unificatory strategy of Formal Axiomatic Method, which has a structuralist and a logicist underpinning. Then I describe the *concentration* part, which turns to be

[1]Unlike the older forms of dogmatism the more recent dialectical dogmatism does not use any fixed system of beliefs but enforces a permanent organized change of one's beliefs on changing pragmatic grounds (political, economical, etc.).

more traditional and in a new form reproduces some features of Euclid's Axiomatic Method. The most original part of the New Axiomatic Method is, of course, its logical part, which involves the notion of internal logic. Generalizing on works of Lawvere and Voevodsky I describe here in general terms a way of using the internal logic of some given category (which is construed in intuitive geometrical terms at the first step of the axiomatic construction) for improving upon the construction of this very category and providing it with some deductive structure. This way of using logic for building mathematical theories suggests a new way of thinking about the role of logic in mathematical theories, which is very unlike Hilbert's and Tarski's.

In my suggested approach logic is designed along with the rest of conceptual construction rather than used as a ready-made foundation for making further mathematical constructions. One may think that the freedom of making up logical calculi added to the freedom of making up new axiomatic mathematical theories (assured already by Hilbert) only reinforce the inflation of the "metaphysical world of thought". In fact the New Axiomatic Method prevents this inflation in two different ways. First, by taking into account the objective meaning of the category of interest (which can be, for example, a spatiotemporal category used in physics) and, second, by requiring the relevant logic to be the internal logic of this given category. While the former feature is at some degree also compatible with the standard Formal Axiomatic Method the latter feature is a genuinely original contribution of the New Method. The New Method no longer reduces the function of logical formalization to a logical censorship; instead logic is used here as a flexible tool for the internal conceptual reconstruction.

An important part of my argument consists of pointing to Lawvere's and Voevodsky's works as applications of this New Method, and stressing the fact that in both cases it allows for a remarkable conceptual simplification and clarification of otherwise difficult and conceptually problematic theories. Since in both cases the relevant logic is internal with respect to its base category this logic inherits the objective meaning of this base category. This allows me to suggest that the New Axiomatic Method may help to bridge the gap between mathematics and physics created and justified by the standard Formal Axiomatic Method and by the logicist view on mathematics that underpins this standard method. Notwithstanding my critique of Hilbert's version of Axiomatic Method developed throughout in this book, I believe (contra Arnold) that Hilbert was perfectly right when he described this method as "the basic instrument of all research" (Hilbert 1927, p. 467) and when he said that "[t]o proceed axiomatically means [..] nothing else than to think with consciousness" (Hilbert 1922, p. 1120).