Categorical Model Theory and the Semantic View of Theories

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Tarski-style Model theory

Functorial Semantics

Models of HoTT

Syntactic and Semantic Views of Theories

Constructive Axiomatic Method

Conclusions and Further Research

Satisfaction

$Model \models Theory$
in words: The Model *satisfies* (= makes true) the corresponding Theory.

$Interpretation : Signature \rightarrow Structure$
where a *signature* is a list of non-logical constants (provided with an additional information identifying each symbol as either an individual constant, or a function constant or a relation constant).

Assumptions 1-2

- A theory is a system of formal sentences (= sentential forms), which are satisfied in a model;
- Semantics of *logical* terms is rigidly fixed: interpretation concerns only *non-logical* terms.

Two distinct points of a straight line completely determine that line

If different points A,B belong to straight line a and to straight line b then a is identical to b.

Assumption 3

Structures are *set-theoretic* structures.

Tarski 1941

"For precision it may be added, that the considerations which we sketched here are applicable to any deductive theory in whose construction logic is presupposed, but their application to logic itself brings about certain complications which we would rather not discuss here."

Compare Tarski's topological semantics for Classical and Intuitionistic propositional calculi (1935). A mere meta-mathematical device?

Lawvere 1963: Functorial Semantics of Algebraic Theories

Idea: use categories instead of signatures (thus blurring the distinction btw. logical and non-logical terms)

Algebraic (Lawvere) theory: category LT with finite products and distinguished object X s.th. every object A in C is isomorphic to X^n for some finite number n.

Model: $LT \rightarrow SET$ that preserves finite limits.
Generalized Models: $LT \rightarrow C$ where C has finite limits.

Sketches (Ehresmann, Bar et al.

Observation: Even if (small) category C does not have (co)limits the presheaf category $\hat{C} = [C, SET]$ does. This allows for using *sketches* "instead of" theories.

Theories in the Categorical perspective (after Awodey & Bauer)

$$\mathit{Theory} \rightarrow \mathit{Category}$$

- cartesian theories (only $\wedge$ and $\top$)
- regular theories (only $\wedge$ and $\top$ and $\exists$)
  (*regular* category: finite completeness plus image factorization stable under pullbacks)
- coherent theories (plus $\vee$ and $\bot$)
  (*coherent* category: regularity plus unions stable under base change)
- geometric theories (plus infinitary $\bigvee$)
  (*geometric* category: infinitary coherent)

Syntactic aka Classifying aka "Walking" Categories

Idea: a category "freely generated from the given syntax"

contexts are objects, substitutions of variables are morphisms

Generic Models

Universal property: $Synt(T)$ is *initial* in $Mod(T)$

(Elephant D1.4, Th. 1.4.6)

Internal Language

$$Categories \underset{Synt}{\overset{Lang}{\rightleftarrows}} Theories$$

$$Model : T \to Lang(C)$$

(in Theories)

Problem:

It is not clear whether Tarski's notion of model based on the satisfaction relation and his T-schema covers the functorial notion(s) of model in all cases. Categorical model theory may need an independent philosophical underpinning.

Claim:

Existing models of Homotopy Type theory are not Tarskian models and cannot be described in terms of the satisfaction relation and the T-schema.

MLTT: Syntax

- 4 basic forms of judgement:
  (i) $A : TYPE$;
  (ii) $A \equiv_{TYPE} B$;
  (iii) $a : A$;
  (iv) $a \equiv_A a'$
- Context : $\Gamma \vdash$ judgement (of one of the above forms)
- no axioms (!)
- rules for contextual judgements; Ex.: dependent product :
  If $\Gamma, x : X \vdash A(x) : TYPE$, then $\Gamma \vdash (\Pi x : X)A(x) : TYPE$

MLTT: Semantics of $t : T$ (Martin-Löf 1983)

- t is an element of set T
- t is a proof (construction) of proposition T ("propositions-as-types")
- t is a method of fulfilling (realizing) the intention (expectation) T
- t is a method of solving the problem (doing the task) T (BHK-style semantics)

MLTT: Definitional aka judgmental equality/identity

$x, y : A$ (in words: x, y are of type A)

$x \equiv_A y$ (in words: x is y by definition)

MLTT: Propositional equality/identity

$p : x =_A y$ (in words: x, y are (propositionally) equal as this is evidenced by proof p)

Definitional eq. entails Propositional eq.

$$\frac{x \equiv_A y}{p : x =_A y}$$

where $p \equiv_{x =_A y} refl_x$ is built canonically

Equality Reflection Rule (ER)

$$\frac{p : x =_A y}{x \equiv_A y}$$

ER is <u>not</u> a theorem in the (intensional) MLTT (Streicher 1993).

Extension and Intension in MLTT

- MLTT + ER is called *extensional* MLTT
- MLTT w/out ER is called *intensional*
  (notice that according to this definition intensionality is a negative property!)

Higher Identity Types

- $x', y' : x =_A y$
- $x'', y'' : x' =_{x =_A y} y'$
- ...

HoTT

"The central new idea in homotopy type theory is that types can be regarded as spaces in homotopy theory, or higher-dimensional groupoids in category theory." (HoTT Book 2013).

One more item to the above list of interpretations? NOT just that.

The homotopical semantics of MLTT, which is used in HoTT, is not compatible with the informal semantics of MLTT proposed by Martin-Löf in 1983!

h-stratification in MLTT

- (i) Given space A is called *contractible* (aka space of h-level -2) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- (ii) We say that A is a space of h-level $n+1$ if for all its points x, y path spaces $paths_A(x, y)$ are of h-level n.

h-hierarchy

(-2) single point pt;

(-1) the empty space $\emptyset$ and the point pt : truth values aka *classical* or "mere" propositions

(0) sets (discrete point spaces)

(1) (flat) path groupoids but no non-contractibe surfaces

(2) 2-groupoids (paths and surfaces but no non-contractible volumes)

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(n) n-groupoids

- ...

(ω) ω-groupoids

The above stratification of types is a robust mathematical structure in MLTT discovered via the homotopic interpretation of MLTT syntax. MLTT intended semantic does not take this structure into account. HoTT semantics does.

Competing approaches to modeling HoTT (an open controversy

- Awodey: classifying categories, natural models
- Voevodsky: (1) an interpretation of rules MLTT in the category of simplicial sets (2009);
  (2) contextual categories (Cartmell), C - systems, the Initiality Conjecture stands open.

Since HoTT is *not* a system of sentences (propositions) but also comprises non-propositional types Tarski's notion of model based on the satisfaction relation may account at most for the propositional fragment (h-level - 1) of HoTT/MLTT.

Theories, which are *not* systems of sentences are less exotic than one could think. The geometrical theory of Euclid's *Elements*, Books 1-4, is another example.

Euclid's *Common Notions* aka *axioms* and *Postulates* are rules but not axioms in the modern sense of the term. Euclid's axioms do not reduce to propositions at the pain of Carrol Parodox!

Aristotle uses the term "axiom" for rules of inference such as the "perfect syllogism"

Arguably a typical scientific theory is not a system of proposition either: the "Non-Statement View of Theories" of P. Suppes, B. van Fraassen et al.

(Once) Received view

A theory is a set of propositions (expressed in a formal language L). It can be possibly (but not generally) represented through a list of *axioms* and described as the deductive closure of these axioms (Carnap).

Non-statement view

A theory is a class of models but not an axiom system, nor its deductive closure. (Suppes, Sneed, Stegmüller, Balzer, Moulines, van Fraassen)

Suppes 2002: Term "model" is used in logic and science similarly. One and the same theory may allow for many different axiomatizations.

Hans Halvorson

"Scientific Theories" (forthcoming). The controversy between syntactic and semantics approaches is artificial. There is a duality between syntax and semantics (Lawvere 1963, Awodey&Forssell 2013). Suppes' Bourbaki-style semantic presentation of theories is language-dependent and gives little or no advantage. The "semantic view" arguments is are no longer relevant in the presence of formal theories of semantics.

Indeed, the semantic view can be understood as an attempt to pave the wide gap between logicians' and physicists' theory concepts. Unlike Halvorson I do not believe that the standard theories of formal semantics actually do the job. However think that a HoTT-based theory of semantics may help to solve the problem (more shortly).

Hans Halvorson

"Claim [in support of the semantics view] : Scientists often deal with collections of models that are not elementary classes, i.e. aren't the collection of models of some set of first-order sentences."

Hans Halvorson: comment

"This claim is strange, for it seems to indicate that scientists work with classes of L-structures (for some language L) that are not elementary classes (i.e. not the classes of models of a set of first-order sentences). I happen to know of no such example. Certainly, scientists work with classes of models that are not in any obvious sense elementary classes, but largely because they haven't been given a precise mathematical definition."

In my view the above claim is justified. The notion of model as a truth-maker does not fully characterize a typical scientific model.

E.g. the model of Solar System of Newton's *Principia* (evidently makes true certain sentences but also) comprises a non-propositional structure, which allows one to synthesize planets' curvilinear orbits from their supposed elementary (infinitesimal) rectilinear motions (and, dually, analyze the curvilinear orbits into infinitesimal rectilinear elements). There is no straightforward way to represent such a structure via a logical structure (albeit roundabout ways to this exist). Moreover, for epistemological reasons such a translation can be <u>un</u>desirable.

Desiderata for a formal framework for Physics

- support deduction from first principles (first elements), including non-propositional ones (primitive objects, types, etc.);
- combine logical rules with constructive rules for non-propositional objects;
- support thought-experimentation and experimental design.

Constructive axiomatic method (arXiv:1408.3591 forthcoming in Logique et Analyse)

Theories satisfying the above desiderata I shall call *constructive* axiomatic theories.

This use of the term "constructive" has a historical grounding (ex. Hilbert&Bernays 1934) but is not standard. This notion of being constructive does not fix any specific set of rules.

Slogan:

A theory is primarily a system of <u>rules</u> and interpretations of those rules rather than a system of axioms and their interpretations. (Cf. Aristotle's use of the term "axiom".)

However a constructive theory, generally, allows for using axioms (e.g. univalence) as well as higher-order principles with corresponding derived rules (higher inductive types: interval, circle, 2-sphere, truncation, localization, spectrification)

"Low-level" physical models of constructive theories are concrete *methods* for conducting particular experiments (Fraassen's "experimental design") and making observations.

Even if the sole purpose of experiments and observation is to give yes-no answers to certain questions an experiment and an observation need to be *designed*. Experimental design cannot be effective if it is done only by trials and errors.

Hintikka 2011 on the Axiomatic Method

"The class of structures that the axioms are calculated to capture can be either given by intuition, freely chosen or else introduced by experience."

"[N]ew logical principles are not dragged [. . .] by contemplating one's mathematical soul (or is it a navel?) but by active thought-experiment by envisaging different kinds of structures and by seeing how they can be manipulated in imagination. [. . .] [M]athematical intuition does not correspond on the scientific side to sense-perception, but to experimentation."

Thought-experimentation without following certain adopted rules (which are <u>not</u> fixed once and for all) is trivial and does not meet scientific needs. Material experimentation is always rule-based, which allows experiments to be reproducible.

Conclusions:

- A theory is, generally, not a system of propositions; it is rather a system of rules applied to propositions along with some non-propositional objects. Examples : Euclid, HoTT , Newton
- Functorial models are not, generally, Tarskian models - as evidenced by HoTT and its known models. In functorial models the distinction between logical and non-logical terms does not apply in its usual form. The generalized satisfaction relation between a constructive theory and its models is expressed in terms of *rule-following* (technically expressed as functoriality) rather than truth-evaluation. It does not reduce to the standard Tarskian notion of satisfaction for sentences at the pain of Carrol paradox.

Conclusions:

- The Non-Statement aka Semantic view of theories is not appropriately captured by the standard methods of formal semantics. The proposed concept of Constructive Axiomatic theory and its model (motivated by HoTT and its model theory) is a viable alternative.

Further Research

Formal reconstruction of physical and other scientific theories in a constructive axiomatic setting.

THANK YOU!