Categorical Model Theory and the Semantic View of Theories

Andrei Rodin

EUTypes and SSTT meeting in Ljubljana 2017

30 January 2017

Satisfaction

$Model \models Theory$
in words: The Model *satisfies* (= makes true) the corresponding Theory.

$Interpretation : Signature \rightarrow Structure$
where a *signature* is a list of non-logical constants (provided with an additional information identifying each symbol as either an individual constant, or a function constant or a relation constant).

Assumptions 1-2

- A theory is a system of formal sentences (= sentential forms), which are satisfied in a model;
- Semantics of *logical* terms is rigidly fixed: interpretation concerns only *non-logical* terms.

Two distinct points of a straight line completely determine that line

If different points A,B belong to straight line a and to straight line b then a is identical to b.

Assumption 3

Structures are *set-theoretic* structures.

Tarski 1941

"For precision it may be added, that the considerations which we sketched here are applicable to any deductive theory in whose construction logic is presupposed, but their application to logic itself brings about certain complications which we would rather not discuss here."

Compare Tarski's topological semantics for Classical and Intuitionistic propositional calculi (1935). A mere meta-mathematical device?

Lawvere 1963: Functorial Semantics of Algebraic Theories

Idea: use categories instead of signatures (thus blurring the distinction btw. logical and non-logical terms)

Algebraic (Lawvere) theory: category LT with finite products and distinguished object X s.th. every object A in C is isomorphic to X^n for some finite number n.

Model: $LT \rightarrow SET$ that preserves finite limits.
Generalized Models: $LT \rightarrow C$ where C has finite limits.

Theories in the Categorical perspective (after Awodey & Bauer)

$$Theory \rightarrow Category$$

- cartesian theories (only $\wedge$ and $\top$)
- regular theories (only $\wedge$ and $\top$ and $\exists$)
  (*regular* category: finite completeness plus image factorization stable under pullbacks)
- coherent theories (plus $\vee$ and $\bot$)
  (*coherent* category: regularity plus unions stable under base change)
- geometric theories (plus infinitary $\bigvee$)
  (*geometric* category: infinitary coherent)

Syntactic aka Classifying aka "Walking" Categories

Idea: a category "freely generated from the given syntax"

contexts are objects, substitutions of variables are morphisms

Generic Models

Universal property: $Synt(T)$ is *initial* in $Mod(T)$

(Elephant D1.4, Th. 1.4.6)

Internal Language

$$Categories \underset{Synt}{\overset{Lang}{\rightleftarrows}} Theories$$

$$Model : T \rightarrow Lang(C)$$

(in Theories)

Problem:

It is not clear whether Tarski's notion of model based on the satisfaction relation and his T-schema covers the functorial notion(s) of model in all cases. Categorical model theory may need an independent philosophical underpinning.

Claim:

Existing models of Homotopy Type theory are not Tarskian models and cannot be described in terms of the satisfaction relation and the T-schema.

Semantics of $t : T$ according to Martin-Löf (1984, p.5)

"Each form of judgement admits of several different readings"

- t is an element of set T
- t is a proof (construction) of proposition T ("propositions-as-types")
- t is a method of fulfilling (realizing) the intention (expectation) T
- t is a method of solving the problem (doing the task) T (BHK-style semantics)

Martin-Löf on sets and propositions (1984, p.13)

If we take seriously the idea that a proposition is defined by laying down how its canonical proofs are formed ... and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to unnecessary duplication to keep the notions of proposition and set (and the associated notions of proof of a proposition and element of a set) apart. Instead, we simply identify them, that i s, treat them as one and the same notion. This is the formulae-as-types (propositions-as-sets) interpretation on which intuitionistic type theory is based.

HoTT

"The central new idea in homotopy type theory is that types can be regarded as spaces in homotopy theory, or higher-dimensional groupoids in category theory." (HoTT Book 2013).

One more item to the above list of interpretations? NOT just that.

The homotopical semantics of MLTT, which is used in HoTT, is not compatible with the informal semantics of MLTT proposed by Martin-Löf in 1984!

h-stratification in MLTT

- (i) Given space A is called *contractible* (aka space of *h*-level -2) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- (ii) We say that A is a space of *h*-level $n+1$ if for all its points x, y path spaces $paths_A(x, y)$ are of *h*-level n.

h-hierarchy

(-2) single point *pt*;

(-1) the empty space ∅ and the point *pt* : truth values aka *classical* or "mere" propositions

(0) sets (discrete point spaces)

(1) (flat) path groupoids but no non-contractibe surfaces

(2) 2-groupoids (paths and surfaces but no non-contractible volumes)

-
-

(n) n-groupoids

- ...

(ω) ω-groupoids

The above stratification of types is a robust mathematical structure in MLTT discovered via the homotopic interpretation of MLTT syntax. MLTT intended semantic does not take this structure into account. HoTT semantics does.

Competing approaches to modeling HoTT (an open controversy

- Awodey: classifying categories, natural models
- Voevodsky: (1) an interpretation of rules MLTT in the category of simplicial sets (2009);
  (2) contextual categories (Cartmell), C - systems, the Initiality Conjecture stands open.

Since HoTT is *not* a system of sentences (propositions) but also comprises non-propositional types Tarski's notion of model based on the satisfaction relation may account at most for the propositional fragment (h-level - 1) of HoTT/MLTT.

Theories, which are *not* systems of sentences are less exotic than one could think. The geometrical theory of Euclid's *Elements*, Books 1-4, is another example.

Euclid's *Common Notions* aka *axioms* and *Postulates* are rules but not axioms in the modern sense of the term. Euclid's axioms do not reduce to propositions at the pain of Carrol Parodox!

Euclid's geometry does not use axioms!

Postulates 1-3:

(P1) to draw a straight-line from any point to any point;
(P2) to produce a finite straight-line continuously in a straight-line;
(P3) to draw a circle with any center and radius.

Postulates 1-3:

	input	output
P1	two points	segment
P2	segment	extended segment
P3	segment and its endpoint	circle

Euclid's geometry does not use axioms!

Common Notions (aka Axioms) 1-3:

(A1) Things equal to the same thing are also equal to one another;
(A2) If equal things are added to equal things then the wholes are equal;
(A3) If equal things are subtracted from equal things then the remainders are equal.

Aristotle uses the term "axiom" for *rules* of inference such as his "perfect syllogism". Euclid similarly uses his"axioms" as rules rather than hypothetical propositions. The possible interpretation of A1-3 as hypothetical propositions leads to an infinite regress ("Carrol paradox").

Objection:

(thanks to Dimitris Tsementzis)

The above argument relies on the informal interpretation of (-1) types as propositions, which is controversial and needs not to be respected in the model theory of HoTT.

Interpreting syntactic *rules*

according to Tarski:
any model that satisfies p and satisfies $p \rightarrow q$ also satisfies q;

in other words q is a *logical* (semantic) consequence of premises $p, p \rightarrow q$.

$$\begin{array}{rl} & p \\ & \underline{p \rightarrow q} \\ \therefore & q \end{array}$$

Tarskian conception of modeling only allows one to check whether a given syntactic rule interprets as a valid rule of logical inference (assuming Tarski's semantic view of logical inference). This framework does not allow for modeling a syntactic rule (or a system of such rules) with a (system of) rule(s) applied to operations with certain semantic entities such as sets or any other mathematical objects. However the notion of "modeling rules" plays a key role in the current discussion on the model theory of HoTT.

Arguably a typical *scientific* theory is not a system of propositions either: the "Non-Statement View of Theories" of P. Suppes, B. van Fraassen et al.

(Once) Received view

A theory is a set of propositions (expressed in a formal language L). It can be possibly (but not generally) represented through a list of *axioms* and described as the deductive closure of these axioms (Carnap).

Non-statement view

A theory is a class of models but not an axiom system, nor its deductive closure. (Suppes, Sneed, Stegmüller, Balzer, Moulines, van Fraassen)

Suppes 2002: Term "model" is used in logic and science similarly. One and the same theory may allow for many different axiomatizations.

Hans Halvorson

"Scientific Theories" (forthcoming). The controversy between syntactic and semantics approaches is artificial. There is a duality between syntax and semantics (Lawvere 1963, Awodey&Forssell 2013). Suppes' Bourbaki-style semantic presentation of theories is language-dependent and gives little or no advantage. The "semantic view" arguments is are no longer relevant in the presence of formal theories of semantics.

Indeed, the semantic view can be understood as an attempt to pave the wide gap between logicians' and physicists' theory concepts. Unlike Halvorson I do not believe that the standard theories of formal semantics actually do the job. However think that a HoTT-based theory of semantics may help to solve the problem (more shortly).

Hans Halvorson

“Claim [in support of the semantics view] : Scientists often deal with collections of models that are not elementary classes, i.e. aren’t the collection of models of some set of first-order sentences.”

Hans Halvorson: comment

"This claim is strange, for it seems to indicate that scientists work with classes of L-structures (for some language L) that are not elementary classes (i.e. not the classes of models of a set of first-order sentences). I happen to know of no such example. Certainly, scientists work with classes of models that are not in any obvious sense elementary classes, but largely because they haven't been given a precise mathematical definition."

In my view the above claim is justified. The notion of model as a truth-maker does not fully characterize a typical scientific model.

E.g. the model of Solar System of Newton's *Principia* (evidently makes true certain sentences but also) comprises a non-propositional structure, which allows one to synthesize planets' curvilinear orbits from their supposed elementary (infinitesimal) rectilinear motions (and, dually, analyze the curvilinear orbits into infinitesimal rectilinear elements). There is no straightforward way to represent such a structure via a logical structure (albeit roundabout ways to this exist). Moreover, for epistemological reasons such a translation can be <u>un</u>desirable.

Desiderata for a formal framework for Physics

- support deduction from first principles (first elements), including non-propositional ones (primitive objects, types, etc.);
- combine logical rules with constructive rules for non-propositional objects;
- support thought-experimentation and experimental design.

Constructive axiomatic method (arXiv:1408.3591 forthcoming in Logique et Analyse)

Theories satisfying the above desiderata I shall call *constructive* axiomatic theories.

This use of the term "constructive" has a historical grounding (ex. Hilbert&Bernays 1934) but is not standard. This notion of being constructive does not fix any specific set of rules.

Slogan:

A theory is primarily a system of rules and interpretations of those rules rather than a system of axioms and their interpretations. (Cf. Aristotle's use of the term "axiom".)

However a constructive theory, generally, allows for using axioms (e.g. univalence) as well as higher-order principles with corresponding derived rules (higher inductive types: interval, circle, 2-sphere, truncation, localization, spectrification)

"Low-level" physical models of constructive theories are concrete *methods* for conducting particular experiments (Fraassen's "experimental design") and making observations.

Even if the sole purpose of experiments and observation is to give yes-no answers to certain questions an experiment and an observation need to be *designed*. Experimental design cannot be effective if it is done only by trials and errors.

Scientific truths (= scientifically relevant true propositions) are, generally, proof-relevant. Standard formal logical approaches in science including the standard version of the *semantic view* simply ignore this fundamental feature of scientific theories. Scientific knowledge does not reduce to the *knowledge-what* but also involves a lot of *knowledge-how*.

The *logical* knowledge-how, i.e., the capacity to make logical inferences, is not the only type of knowledge-how that is essential in science. The capacity of proving propositions (not simply by inferring them logically from some other propositions but also) by using some higher-order non-propositional constructions is equally essential.

Scientific truths (= scientifically relevant true propositions) are, generally, proof-relevant. Standard formal logical approaches in science including the standard version of the *semantic view* simply ignore this fundamental feature of scientific theories. Scientific knowledge does not reduce to the *knowledge-what* but also involves a lot of *knowledge-how*.

The *logical* knowledge-how, i.e., the capacity to make logical inferences, is not the only type of knowledge-how that is essential in science. The capacity of proving propositions (not simply by inferring them logically from some other propositions but also) by using some higher-order non-propositional constructions is equally essential.

Further Research

Formal reconstruction of physical and other scientific theories in a constructive axiomatic setting.

THANK YOU!