COMPUTING IN SPACE AND TIME

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[1]

Abstract. Turing machine adequately accounts for the temporal aspect of real computing but fails to do so for certain spatial aspects of computing, in particular, in the case of distributed computing systems. This motivates a search for alternative models of computing, which could take such spatial aspects into account. I argue that a revision of the received views on the relationships between logic, computation and geometry may be helpful for coping with spatial issues arising in the modern computing.

1. Introduction

Computing takes time. For practical reasons it is crucial how much time it takes. Processing speed (usually measured in FLOPS or more generically in cycles per second) is a basic measure of computer hardware performance. During past several decades of continuing computer revolution the processing speed of hardware increased dramatically. Computation time (aka running time) is equally crucial for evaluating the software performance: invention of faster algorithms is going along with building faster processors.

Computing also takes space. It equally matters how much space it takes. Today's digital processors are smaller in size than their early prototypes by several orders of magnitude; the minimization of physical sizes of computing devices is a continuing technological trend. In order to see the role of space in computing more clearly it will be helpful to consider

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first a more traditional computing device like a set of pebbles (Latin *calculi*). Manipulating with pebbles requires a space. A basic trick of traditional computing is that this required computation space is more "handy" and usually significantly smaller than the space where live the counted objects. Suppose for example that one wants to count sheep in a herd. A reason for replacing sheep by pebbles is that one can manipulate with pebbles in space by far more easily and more effectively that one can do this with the real animals. Using pebbles one can even count stars and other objects on which one has no real control at all. Thus if we consider computing as a process applied to certain external objects rather than an abstract procedure on its own rights, then we can observe that scaling the system of objects under consideration down (or up in the case of a microscopic system) to the spatial and temporal scale where humans may possibly provide for an effective control, is an essential part of this process. A spatio-temporal scaling is equally at work when calculations are done symbolically.

If we now turn back to the modern electronic computing then we remark that the spatial issues enter into the picture in a number of new ways. Spatial issues are dealt by hardware engineers who seek to make handy human-scale computing devices as powerful as possible. The distributed computing made possible by the web technology involves the hardware scattered over the globe, so spatial issues (along with temporal ones) become in this case more pressing. The case of remote control of spacecrafts where the time of signals' traveling between different parts of the same information system becomes an essential factor, makes a link (practical rather than only theoretical!) between computing and relativistic space-time. These are but the most apparent ways (chosen by random) in which spatial and spatio-temporal issues can be relevant to modern computing; developers of information systems most certainly can specify more.

Theoretically spatial and spatiotemporal aspects of computing have been studied so far at least in these two directions. In 1969 Konrad Zusse proposed a powerful metaphor of "computing space" [25] which gave rise to an area of research known as *digital physics*, which explores the heuristic idea according to which the physical universe can be described as a computing device. Reciprocally, the fact that computing proceeds in the real physical space and time led a number of authors to reconsidering standard theoretical models of computation [12] and more recently gave rise to a new field of research named *spatial computing* [9].

2. Time and space estimations

A commonly used theoretical model of computation is the *Turing machine*, which can be described informally as follows. It consists of an infinite strip of tape divided into equal cells, which moves forward and backward with respect to a writing device. The device writes and rewrites symbols into the cells according to a table of rules. Thus the Turing machine presents computation as a discrete process divided into discrete atomic steps. According to this model every accomplished calculation is characterized by a certain finite number n of such steps. This apparently naive model (which can be given a more precise formal presentation) turns to be surprisingly effective for theorizing about computing in the context of modern information technology. The Turing machine model allows one to estimate the computing time straightforwardly. Given that a given algorithm A requires n Turing moves for accomplishing a given task T, and given that the CPU of one's computer makes m operations per second, one can estimate the required running time t as $\frac{n}{m}$. Since the Turing machine is an ideal theoretical model but not a real computing hardware, the above calculation is by far *too* straightforward. In the real life the exact number n as above is undetermined, so one can only estimate how n varies with the variation of parameters T such as the number of elements in sorting. But notwithstanding these details the very fact that the Turing machine, theoretically described algorithms, algorithms realized in a software and finally real CPUs all work step-by-step provides a firm ground for time estimations. The "internal time" of Turing machine measured in elementary moves of its tape turns to be a good theoretical model for the running time of real computers.

The Turing machine also helps for estimating space required for computing. This is done by the estimation of number m of the required cells. If one knows how m depends on parameters of the given task, one can estimate the volume of required memory, which in its turn provides a reasonable estimation of size of the real computing device. In that respect the time estimation and the space estimation are similar. However this very structural similarity between the temporal and the spatial sides of the Turing machine makes a big difference in how the Turing model of computation relates to computations made in the real world. Let me for the sake of the following argument assume that the physical time and space are classical (Newtonian). The elementary moves of the Turing machine can be identified with ticks of physical clocks and its tape divided into equal cells can be used as a ruler. The number of ticks is all one needs for measuring a time span between two events and the number of cells is all one needs for measuring the distance between two points in space. What makes the two cases very different is this: while the arithmetic of natural numbers in a sense comprises the formal structure of classical time (as already Kant rightly acknowledged) this is not the case regarding the formal structure of classical Euclidean 3D space. A fundamental property of this space which remains unaccounted in this way is the number if its dimensions. The one-dimensional Turing tape may serve as a good instrument for *testing* various spatial structures - Euclidean and beyond - but it cannot, generally, *represent* such structures (including their global topological properties) in the same direct way in which it represents the running time in real computers. Thus one can remark a sharp difference between the temporal and the spatial relevance of Turing model of computation: While this model adequately accounts for the temporal aspect of real computing (modulo usual reservations explaining the difference between a physical process and its theoretical model), it fails to do so with respect to certain spatial characteristics of modern computing devices.

Why this dissymmetry? Or perhaps it is more appropriate to ask why not? The success of Turing machine and other related models of computing (such as the lambda calculus) suggests seeing the running time as an essential feature of computing and seeing all spatial aspects of computing as non-essential. Even if Turing machine says nothing about spatial issues related to computing it is not obvious that these issues should be taken into account by a theory of computing at the fundamental level. Perhaps these spatial issues can be better accounted for separately after the basic model of computation is already fixed. The idea to unify spatial and temporal aspects of computing within the same fundamental theory may appear tempting (and natural from the point of view of today's physics) but it certainly needs further arguments in its favor.

In what follows I shall try to provide such arguments. I shall start with some historical observations concerning the relationships between space, geometry and computing. Then I consider a recent theory, which reveals a deep link between computing and geometry in a modern mathematical context. Finally I discuss some related philosophical issues concerning the relationships between the pure and the applied mathematics.

3. Historical Forms of Computing

Making history of a subject unavoidably involves projecting of the present state of this subject onto its past. These days by a computer one understands a digital electronic device, which inputs and outputs sequences of 0s and 1s; a peripheral hardware translates between between the 0-1 sequences and the data of different types including the data that can be received and/or outputed by human users immediately (such as strings of symbols and imagery). Looking for a close historical analogue of the modern computing one naturally points to arithmetical calculations in its various historical forms, some of which involve devices of abacus type [16]. However a closer examination shows that the historical forms of computing are more diverse. Suppose one needs to compute the hight of an equilateral triangle knowing its side. Such geometrical problems are common in building construction and many other practical affairs. If one has an electronic calculator at hand then to compute a decimal fraction approximating $\frac{\sqrt{3}}{2}$ is a reasonable solution. Otherwise one may use a more traditional tool such as the ruler and the compass for solving the problem geometrically. If the size of the figure in question does not allow one to apply these instruments directly one first solves the problem on a sheet of paper or another appropriate support, and then use a scaling technique (which typically but not necessarily involves arithmetical calculations) for applying this geometrical result in the given practical context [2] There is a tendency dating back to Plato to overlook or underestimate the computational aspect of the traditional elementary geometry as presented in Euclid's *Elements*. Whatever may be philosophical reasons behind it such an attitude is hardly appropriate as long when one studies the history of computing.

The combination of ruler and compass works as a simple *analogue* computer. While modern *digital* computers use the idea of symbolic *encoding*, the analogue computers exploit the idea of *analogy* between different physical processes. This latter idea can be made more precise through the concept of *mathematical form* (which, of course, in its turn needs further specifications which I omit here). Different physical processes, including those having very different physical nature, happen to share the same mathematical form; in many case, they may be adequately described by the same mathematical tools such as differential equations. Let P be a class of processes sharing the same mathematical form F. Now the idea of analogue computing can be formulated as follows: choose in P an appropriate process C (for "computation"), which is artificially reproducible, well-controllable and conceptually transparent; then use C as a standard representation for F. What one learns about F through C applies to all other processes in P disregarding their specific physical nature. In the above example F is the geometrical form of equilateral triangle and C is the standard construction of such a triangle by the ruler and the compass.

Analogue computers have been largely superseded by their digital rivals at some time in early 1960-ies (or earlier on some accounts [16]). A thorough discussion on digital and analogue computing is out of place here but I shall point to one advantage of the digital computing which obviously contributed to its success. It consists in its *universality*. What

[2] The Euclidean space is the only one among Riemannian spaces of constant curvature, which allows for a simple linear scaling. The importance of scaling in practical matters provides, in my view, a plausible explanation why the Euclidean geometry for many centuries was considered as the only "true" theory of space. The fact that the linear scaling property implies Euclid's Fifth Postulate (aka the Parallel Postulate) was first realized by Wallis in 1693 [2].

we want to call a computer is not just a device that allows one to simulate physical processes and technological procedures of some particular type P as described above but rather a universal toolkit, which allows for simulating processes and procedures of many different sorts. The ruler and the compass meet this requirement only to a certain degree. These instruments can be used for solving a large class of geometrical problems but this class turns to be limited in a way, which from the practical viewpoint may appear very strange and even arbitrary. Why the trisection of a given segment is doable but the trisection of a given angle is not? Why a regular hexagon can be so easily computed but a regular heptagon cannot be computed by these means at all? Today we know good theoretical answers to these questions but they don't make the ruler and the compass more useful than they are. Now consider the claim according to which all relevant mathematical procedures and mathematical structures serving as mathematical expressions of various physical "analogies" in the analogue computing as explained above, can *in principle* be encoded into (i.e., represented with) 0-1 sequences and operations with these things. This claim is problematic from a theoretical point of view (not all mathematical theories currently used in physics are constructive and moreover computable); it also involves very strong theoretical idealizations, which make many theoretical possibilities, which this idea implies, physically unfeasible. The more computing power we get the more such limitations become visible. However the idea of a single universal model of calculation appears so attractive and so promising that our technological development largely follows it anyway.

4. Geometrical Characteristic

Let me now turn from the history of computing to the history of ideas about computing. Leibniz is commonly and rightly seen as a forerunner of modern computing; his ideas about this subject he put under the title of *Universal Characteristic*, which he described as a hypothetical symbolic calculus for solving problems in all areas of human knowledge. Although this idea sounds appealing in the modern context to reconstruct it precisely is a laborious historical task; moreover so since this idea never achieved in Leibniz a stable and accomplished form. I shall discuss here only one specific aspect of this general idea, which is relevant to my argument, namely the notion of *Geometrical* Characteristic [10], for partial English translation see [11].

Leibniz builds his idea of Geometrical Characteristic upon Descartes' *Analytic Geometry*. In its original form (unlike its usual modern presentations) this latter concept has little to do with the arithmetization of geometry through a coordinate system. It has been rather conceived by Descartes as a geometrical application of a general algebraic theory of magnitude. This general algebra of magnitudes was supposed to cover both arithmetic (the case of discrete magnitude) and geometry (the case of continuous magnitude). As Leibniz stresses in his Geometrical Characteristic paper the general algebra of magnitudes cannot be a sufficient foundation of geometry because this general algebra treats only metrical properties of geometrical objects while these objects also have relational *positional* properties (which we call today topological). Leibniz tries to push Descartes' project further forward by mastering a more advanced algebraic theory capable to account for positional properties of geometrical configurations along with their metrical properties. He conceives here of a possibility of replacing traditional geometrical diagrams with appropriate symbolic expressions and appropriate syntactic procedures on such expressions, which would express the positional properties directly, without using the Cartesian algebra of magnitudes. For this end Leibniz observes that the traditional geometrical letter notation (as in Euclid) is not wholly arbitrary but has a certain syntactic structure, which reflects certain positional properties. For example when one denotes a given triangle ABC the syntactic rules require A, B, C to be the names of this triangle's vertices, and AB, BC, AC be the names of its three sides. Leibniz' idea is to develop this sort of syntax into a full-fledged symbolic calculus similar (on its syntactic side) to Descartes' algebraic calculus [3].

[3]An attempt to develop geometry systematically on an algebraic basis (in the form of a general theory of magnitude) has been made by Descartes's follower Antoine Arnauld [1]. This Arnauld's work was carefully studied by Leibniz and contributed to his thinking about geometrical matters. Another name for the same circle of Leibniz' ideas, which connects them to Descartes' work more directly, is *Analysis Situs* (Situational

Leibniz's idea of Geometrical Characteristic is interesting because it directly links the symbolic computation to the geometrical reasoning on a fundamental theoretical level - while in the mainstream 20th-century theoretical works on computation by Church and others such a link appears to be wholly absent. However in the 19th century the idea of Geometrical Characteristic has a rich history, which involves works of Grassmann [5], Peano [17][18] and other important contributors. Even if in the 20th century this circle of ideas did not make part of the mainstream research in the theory of symbolic computing (which in this century was largely monopolized by logicians) it continued to develop during this century within other mathematical disciplines including algebraic geometry. Tracing this history of ideas continuously up to the present is a challenging historical task, which I leave for another occasion. In this paper I shall only briefly describe what I see as the latest episode of this history, which establishes a new surprising theoretical link between geometry and computing in today's mathematical and logical setting.

5. Univalent Foundations

The *Univalent Foundations* of mathematics (UF) is an ongoing research project headed by Vladimir Voevodsky and his collaborators in Princeton Institute for Advanced Study; this project is closely related to the recently emerged mathematical discipline of *Homotopy Type theory* (HoTT). The backbone of UF/HoTT is a detailed formal correspondence between a type calculus due to Martin-Löf (MLTT) [15] and a geometrical theory (in a broad sense of "geometrical) known as *Homotopy theory* (HT); see [19] for a systematic introduction and further references. A role in the discovery of this correspondence was played by the concept of infinite-dimensional *groupoid* first introduced by Grothendieck in 1983 [6]; more historical details are found in [21], Ch. 7.

For my present argument it is essential to take into account the specific character of correspondence between the type calculus MLTT and the geometrical theory HT, which gives

Analysis); under this latter name this circle of Leibniz' ideas plays a prominent role in the early history of modern *topology* [3].

rise to UF/HoTT. It is terminologically correct to say that HT provides a *model* of MLTT using the term "model" as it is usually understood in the Model theory. However if one compares how the concept of model is used here with standard examples of models like Beltrami-Klein or Poincaré models of Hyperbolic geometry (HG), one immediately notices a striking difference. A standard axiomatic presentation of HG contains *non-logical* terms like "point", "lies between" , etc, and certain *logical* terms like "and", "if then", "therefore" etc. As far as one thinks about formal axiomatic theories and their models along the pattern provided by Hilbert in his classical [7], one assumes that the meaning of logical terms is fixed (and commonly understood), while the non-logical terms are place-holders, which get definite semantic values only under this or that possible *interpretation*; when such an interpretation turns the axioms of the given theory into true statements this interpretation qualifies as a model of that theory. The distinction between logical and non-logical terms is of a major epistemic significance here because it usually (and certainly in Hilbert) goes along with the assumption according to which logical concepts are more epistemically reliable than mathematical (and in particular geometrical) ones. The assumption about the epistemic primacy of logic provides a ground for the claim that a Hilbert-style formal axiomatic presentation of a given mathematical theory is a genuine epistemic gain rather than just one's favorite style of writing mathematical textbooks.

In UF/HoTT the above familiar pattern of axiomatic thinking does not apply. For HoTT provides geometrical interpretations for those terms of MLTT, which by all usual accounts qualify as *logical*. The most interesting (both mathematically and philosophically) case in point is the concept of *identity* (as in MLTT without the additional axioms of extensionality which makes the identity concept in this theory trivial) and its homotopical interpretation. By all usual accounts (including Frege's classics [4]) the concept of identity is logical. In HoTT it receives a highly non-trivial geometrical (homotopical) interpretation in the form of *fundamental groupoid* of a topological space, which is the groupoid of *paths* between points of this space. This construction of usual "flat" fundamental groupoid of paths is further extended onto that of infinite-dimensional *higher homotopy groupoid*, which accounts to *higher identity types* appearing in MLTT. This geometrical interpretation makes intelligible the complexity of the identity concept as in MLTT, which otherwise may appear as unnecessary technically complicated and conceptually opaque. So in this case a logical concept is analyzed and clarified by geometrical means rather than the other way round. The reciprocal epistemic impact of logic onto geometry in UF/HoTT is also significant (see below) but it would be clearly wrong to see this impact as one-sided. What brings an epistemic gain in this case is a cross-fertilization of logic and geometry rather than an one-sided influence.

UF/HoTT has a special relevance to computing, which I am now going to describe. MLTT has been designed from the outset as a formal calculus apt for computer implementations. It is a constructive theory in the strong sense of being Turing computable. The homotopical interpretation of MLTT makes possible to see MLTT (possibly with some additional axioms such as the Univalence Axiom) as a formal version of HT and use program languages based on MLTT for computing in HT. The fact that MLTT has been designed as a general formal constructive framework rather than a formal version of any particular mathematical theory suggests that UF/HoTT may serve as a foundation of all mathematics and that its computational capacities can be used also outside HT, ideally everywhere in mathematics and mathematically-laden sciences. Realization of this project remains a work in progress.

Like Leibniz's Geometrical Characteristics UF/HoTT can be seen as a theoretical means for reducing geometrical constructions to symbolic expressions, which can be managed by the Turing Machine. However the link between geometry and computing established in UF/HoTT can be also explored in the opposite direction and provide a theoretical ground for attributing to computations a geometrical (topological) structure. I submit that such a notion of internal geometrical structure of computing may be used for designing distributed computing systems and coping with other spatial aspects of modern computing mentioned above. This is, of course, nothing but a bold speculation *à la* Leibniz, which I cannot support by any specific technical argument. Instead I shall discuss certain related philosophical issues. One's stance towards these issues can make the above guess to appear more reasonable or, on the contrary, less reasonable and direct one's technical efforts accordingly. Leibniz's example demonstrates that in the past philosophical speculations played a role in later technological developments. I cannot see a reason why this should not work today and in the future.

6. Geometrical thinking

Is it reasonable to expect that geometrical methods may help one to cope with spatial issues arising in engineering (including IT engineering)? Two centuries ago the answer in positive would be a matter of course. However today we live with a very different received view on the nature and the subject-matter of mathematics. This modern vision has been strongly influenced by Hilbert's notion of axiomatic theory and stabilized at some point in the mid-20th century. A concise presentation of this received view is found in Professor Mainzer recent monograph [13], where he describes a "mathematical universe" of "proper worlds and structures existence of which is thought of solely in terms of accepted axioms and logical proofs" , (*op. cit.*, p. 280, my translation from German). I shall call this view the *standard picture* (SP) for further references. It should be understood that SP is not a description of what mathematicians are doing in their everyday work but rather a judgement on what the pure mathematics really *is* in the proper philosophical analysis. Elementary arithmetical calculations like $7 + 5 = 12$ at the first glance do not look like logical inferences. In order to fit $7 + 5 = 12$ into SP one needs to make a judgement like this: this calculation is ultimately justified by a logical inference, which is made explicit by through a logical reconstruction of arithmetic, i.e., through presenting this traditional mathematical discipline in the modern axiomatic form of Peano Arithmetic or similar. Such a gap between the current mathematical practice and SP exists in all areas of today's mathematics including mathematical logic itself. It is a controversial matter among philosophers whether or not such a gap is tolerable.

SP implies that there is no direct connection between the "proper worlds" of mathematical structures and the material world in which we live, act and develop our technologies. How it happens that some of these structures play a significant role in natural sciences and technologies constitutes a philosophical puzzle famously called by Wigner [24] the "unreasonable effectiveness " of mathematics. This puzzle has a number of plausible solutions compatible with SP (including one explained in Professor Mainzer's book, ch. 14), which I shall not discuss here. Instead I shall try to revise SP and briefly present a different understanding of modern mathematics, which establishes (or rather re-establishes) a stronger conceptual connection between mathematics, natural sciences and technology. Such a link was taken for granted by many philosophers, mathematicians and scientists in the past but was later lost of view in popular 20-th century accounts of the so-called "non-Euclidean revolution" [23] of the mid 19-th century. Without going into a thorough historical discussion of this matter I shall try to show here that the results of this alleged revolution have been largely misconceived and somewhat exaggerated.

SP comes with the following assumption, which at the first glance may look merely technical but in fact is epistemically important: an axiomatic presentation of mathematical (and in fact also all other) theories involves a definite *symbolic syntax*. So in addition to the ideal existence "in terms of accepted axioms and logical proofs" all mathematical objects and structures enjoy within in SP a more palpable form of existence, namely, the existence in the form of symbolic presentations. Hilbert, who was a pioneer of formal axiomatic method, described this double form of mathematical existence explicitly and ontologically qualified mathematical symbols as the only "real" mathematical objects, while the rest of mathematical objects on his account were merely "ideal" [8]. Accordingly, he exempted a part of mathematics from SP and called this special part *metamathematics*. Hilbert conceived of metamathematics as a foundational discipline, which allows one to develop the rest of mathematics safely using symbolic logical methods. Hilbert hoped that the metamathematics would reduce to a theoretically transparent and wholly unproblematic fragment of *finitary* mathematics.

Thanks to Gödel and others we know today that Hilbert was seriously mistaken here; for this reason mathematicians and logicians today usually feel free to apply in mathematical logic and in metamathematics any sort of mathematics that may prove useful, i.e., that may prove some non-trivial results. Yuri Manin expresses this changed attitude by saying that "good metamathematics is a good mathematics rather than shackles on good mathematics " ([14], p. 2). As we have seen HoTT applies the Homotopy theory (HT) for a similar purpose: it provides a geometrical model for a symbolic calculus (MLTT) the intended semantic of which is logical (in a broad sense of the word). A logical inference in HoTT is a different name for a geometrical construction. The "existence of mathematical structures" in HoTT is as much logical as it is geometrical. This feature of HoTT does not square with SP. One may point, however, to a theoretical possibility to disregard all such specific features and treat HoTT by standard logical methods along with the rest of mathematics. Such a possibility exists. But as I have already explained, resources of HoTT can be also used more specifically for developing a competing foundational project (UF) based on different epistemic principles. I shall not discuss these principles here systematically but only briefly touch upon the issue of relationships between mathematics and the physical world.

As we have seen Hilbert in 1927 believed that finite strings of symbols are privileged mathematical objects, which serve as a unique join between the abstract mathematics and the concrete material world. Even if modern presentations of SP don't make the same point explicitly, they need to use this assumption tacitly because it is enforced by the current standard of formal logical rigor, which requires using symbols. But since Hilbert's project of building mathematical foundations on the basis of finitary mathematics is given up, I can see no further reason to justify the aforementioned assumption either. Mathematically speaking, the combinatorics of symbols is important but it does not play a distinguished role in mathematical matters - whether one provides it with one's favorite logical semantics or not. Epistemologically speaking, there is no reason to consider symbols as the sole tool, which connects the human cognition and the human history with the outer world. The geometrical intuition is another obvious candidate. One should keep in mind that the implementation of mathematical ideas in physics and technology is never a straightforward matter. It is not straightforward even in the Euclidean case, and it is by far less straightforward in the case of modern geometry. Nevertheless I cannot see that the modern geometry in this respect so drastically differ from the traditional Euclid's geometry, as proponents of the non-Euclidean Revolution often tend to say. Mathematics in general and geometry in particular is a cognitive activity rooted in human material practices and experiences, which on this basis explores further theoretic possibilities by modeling them conceptually. Even if testing of such newly discovered theoretic possibilities against new experiences and new practices belongs not to the pure mathematics but rather to science and technology, there is no reason, in my view, to think of mathematics as a genuinely independent discipline exploring its own "proper world". Human experiences and practices do not, generally, simply guide one's "choice of axioms" for developing on this basis some useful mathematical theories, as SP suggests, but rather help one to build conceptual frameworks, in which certain axioms and certain inferences from these axioms can be later established [4].

On this - admittedly merely speculative - ground I suggest that HoTT indeed qualifies as a reasonable candidate for a theory of spatial computing or at least for a fragment of such a theory. In fact, it appears as the only such candidate since no other mathematical theory treating the concept of computing *geometrically*, to the best of my knowledge, is presently known.

7. Conclusion

Computing is an old and very important channel, which connects the research in pure mathematics (when such an activity is practiced in a society) with the society's economy, administration, political institutions, technology and natural science. Professor Mainzer

[4] For the notion of geometrical intuition in modern mathematical contexts see [20]. For more details concerning the role of geometrical modeling in axiomatic theories see [22]

[13] provides a detailed account of how this channel functions in today's information societies. In particular, he shows how today's standard picture of mathematics fits the contemporary ideas about computing and its implementation in the existing computing technology. On my part, I tried to suggest a revision of this standard picture and offer a different view on mathematics and computing, which, as I believe, may help one to cope with some technological challenges related to the spatial aspect of computing technologies.

In this context I argued that Hilbert's view on what is real and what is ideal in mathematics is biased. However important is the historical impact of alphabetic writing techniques on mathematics, it is certainly not the only thing, which connects mathematics to the material world and to human material practices. However impressive is the implementation of these techniques in the modern digital computing it would be wrong to isolate these specific techniques from other mathematically-laden material practices and technologies and think of symbolic proceeding (possibly providing it with one's favorite logical semantics) as a unique and exceptional channel that links mathematics to the material works. Among other things such an ideological focusing on the symbolic processing and on the Turing model of computation artificially isolates the temporal aspect of computing from the spatial one and thus makes it more difficult to theorize mathematically about spatial aspects of computing.

As a possible remedy I pointed to the ongoing research in Univalent Foundations and Homotopy Type theory, which provide a surprising conceptual link between geometry and computing. Whether or not this theory may indeed help one to cope with distributed information systems and long-distance control at the present stage of research is wholly unclear, and in any event there is a very long way to it. However I tried to demonstrate using this example that the contemporary mathematics - by which I here mean the very edge of the ongoing mathematical research - can be more friendly to technological implementations in general and to computer implementations in particular than suggests the popular picturing of this mathematics as exceedingly abstract and wholly detached from all other human affairs.

References

[1] A. Arnauld. *Nouveaux Eléments de Géométrie*. Guillaume Desprez, Paris, 1683.

[2] R. Bonola. *Non-Euclidean Geometry*. Dover Publications, 1955.

[3] V. de Risi. *Geometry and Monadology: Leibnizs Analysis Situs and Philosophy of Space*. Birkhauser, 2007.

[4] G. Frege. über sinn und bedeutung. *Zeitschrift für Philosophie und philosophische Kritik*, 100:25–50, 1892.

[5] H. Grassmann. *Geometrische Analyse, geknupft an die von Leibniz erfundene Geometrische Charakteristik*. Leipzig: Weidmann, 1847.

[6] A. Grothendieck. *Pursuing Stacks (Letter to Quillen*. (unpublished manuscript, 1983.

[7] D. Hilbert. *Grundlagen der Geometrie*. Leipzig, 1899.

[8] D. Hilbert. Die grundlagen der mathematik. *Abhandlungen aus dem mathematische Seminar der Hamburgischen Universität*, 6(1/2):65–85, 1927.

[9] O. Michel J.-L. Giavitto and A. Spicher. Unconventional and nested computations in spatial computing. *International Journal of Unconventional Computing*, 9(1-2):71–95, 2013.

[10] G. Leibniz. Characteristica geometrica. *in G. I. Gerhardt (ed.). Leibnizens Mathematische Schriften. Halle. 1849-1863*, 5:141–168, 1679.

[11] G. Leibniz. Characteristica geometrica. *in M. Dascal, Leibniz: Language, Signs and Thought, John Benjamins B.V. 1987, p. 167-174*, 5:141–168, 1987.

[12] G. Longo and Th. Paul. The mathematics of computing between logic and physics. *in B. Cooper and A. Sorbi (eds.) Computability in Context, Imperial College Press*, pages 243–274, 2009.

[13] K. Mainzer. *Die Berechnung der Welt: Von der Weltformel zu Big Data*. C.H. Beck, 1899.

[14] Yu. Manin. *Foundations as Superstructure*. arXiv:1205.6044, 2012.

[15] P. Martin-Löf. *Intuitionistic Type Theory (Notes by Giovanni Sambin of a series of lectures given in Padua, June 1980)*. Napoli: BIBLIOPOLIS, 1984.

[16] G. O'Regan. *A Brief History of Computing*. Springer, 2012.

[17] G. Peano. *Calcolo Geometrico secondo l'Ausdehnungslehre di H. Grassmann, preceduto dalle operazioni della logica deduttiva*. Torino: Fratelli Bocca Editori, 1888.

[18] G. Peano. *Geometric Calculus according to the Ausdehnungslehre of H. Grassmann, translated by Lloyd C. Kannenberg*. Birkhäuser, 2000.

[19] Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study (Princeton); available at http://homotopytypetheory.org/book/, 2013.

[20] A. Rodin. How mathematical concepts get their bodies. *Topoi*, 29(1):53–60, 2010.

[21] A. Rodin. *Axiomatic Method and Category Theory (Synthese Library vol. 364)*. Springer, 2014.

[22] A. Rodin. *On Constructive Axiomatic Method*. arXiv:1408.3591v3, 2015.

[23] R.J. Trudeau. *The non-Euclidean Revolution*. Birkhüaser, 1986.

[24] E. Wigner. The unreasonable effectiveness of mathematics in the natural sciences. *Communications on Pure and Applied Mathematics*, 13:1–14, 1960.

[25] K. Zuse. *Rechnender Raum*. Friedrich Vieweg & Sohn, Braunschweig, 1969.

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