DIAGRAMMATIC SYNTAX AND ITS CONSTRAINTS

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Content:

Diagrammatic logic 1

"The industrial demand greatly energized building formal semantics for diagrammatic languages in use, and an overwhelming amount of them was proposed. The vast majority of them employ the familiar first-order (FO) or similar logical systems based on string-based formulas, and fail to do the job because of unfortunate mismatch between the logical machineries they use for formalization and the internal logics of the domains they intend to formalize."

(Z. Diskin and U. Wolter (2009): *A diagrammatic logic for object-oriented visual modeling*)

Diagrammatic logic 2

"Parallelism of specification and graphical visualization is provided by the graph-based nature of the sketch logic, and sharply distinguishes sketches from those visual models which are externally graphical, yet internally are based on predicate-calculus-oriented string logics. The repertoire of graphical constructs used in these models has to be bulky since all kinds of logical formulas require its special visualization. Configurations/shapes of these visualization constructs can be rather arbitrary because there are no evident natural correlations between graphics and logical string-based formulas."

(Z. Diskin, Kadish B., F. Piessens, and M. Johnson (2000): *Universal arrow foundations for visual modeling*)

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- Philosophical prejudices can be eliminated only with philosophical means
- New applications require new foundations!

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1)-4): bookkeeping (syntax); 5): identity; 6): associativity

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Abstract categories are NOT categories of categories!

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Compare the case of internal models of ZF. A difference: BC is stronger than ETAC

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- B6) Object 4 of the form

$$\begin{tikzcd} & 3 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \arrow[ul] \end{tikzcd}$$

(notice the path

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- I4) $4 \to A$: associativity ($=$ an associative rhombus) in A

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CLAIM: "Purely categorical" foundations of mathematics must be built "purely categorically"; they require a new notion of axiomatic method. The standard Hilbert-style Axiomatic method is essentially set-theoretic and hence inappropriate in categorical foundations. Remind Mayberry's objection. Categories cannot be simply replaced by sets as far as one keeps the standard notion of Axiomatic method untouched.

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QUESTION: How to dispense with ETAC and make BT self-sustained?

Proposal:

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Read axioms B4-B6 as axioms of Category theory

Notice the "foundational circularity" like in the case of axiomatic theories of sets. Mayberry's objection doesn't go through!

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The existing version of Sketch theory relies upon set-theoretic foundations. The above proposal amounts to a synthesis of the Sketch theory with Lawvere's axiomatic approach. Axiomatic Category theory can play a role in the development of the categorical diagram-based logic similar to that played by the axiomatic Set theory in the development of the (family of) string-based logics.

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QUESTION: What is the exact relation between the string-based logic and Set theory?

Constraint 1:

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The standard notion of *space* implies the reversibility of spatial motions: any trajectory can be traced back.

Hence the spatial motion cannot represent the notion of categorical functor faithfully. A category should be thought of as a *space-time* rather than a mere space. This "dynamical" feature is not immediately present in diagrams.

Constraint 2:

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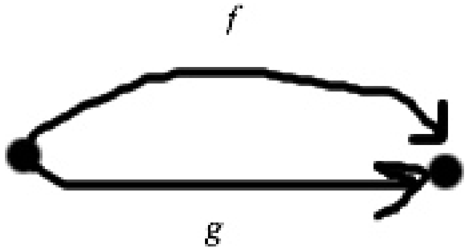
Our usual geometrical intuition allows for unlimited introduction of new points. In a categorical context this intuition becomes misleading.

Two functors may coincide on all their points but still be different. A transformation doesn't reduce to a set of momentary states.



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2) Syntax is a genuine part of mathematics. Cf. Postulates and Axioms in Euclid's *Elements*.