DIAGRAMMATIC SYNTAX AND ITS CONSTRAINTS

ANDREI RODIN

1. Introduction: Applications and Foundations

In their recent papers Zynovy Duskin and his co-authors stress the following problem concerning diagrammatic tools currently used in Computer Science:

> The industrial demand greatly energized building formal semantics for diagrammatic languages in use, and an overwhelming amount of them was proposed. The vast majority of them employ the familiar first-order (FO) or similar logical systems based on string-based formulas, and fail to do the job because of unfortunate mismatch between the logical machineries they use for formalization and the internal logics of the domains they intend to formalize.[3]

> Parallelism of specification and graphical visualization is provided by the graph-based nature of the sketch logic, and sharply distinguishes sketches from those visual models which are externally graphical, yet internally are based on predicate-calculus-oriented string logics. The repertoire of graphical constructs used in these models has to be bulky since all kinds of logical formulas require its special visualization. Configurations/shapes of these visualization constructs can be rather arbitrary because there are no evident natural correlations between graphics and logical string-based formulas. [2]

The problem stressed in these passages has an interesting relevance to philosophy. The predicate-calculus-oriented string logic plays a prominent role in todays philosophy and particularly in the todays mainstream philosophical tradition called *Analytic*. When Analytic philosophers talk about Classical, Intuitionistic, Modal or another logic they always talk about some kind of predicate-calculus-oriented string logic. In particular, they view the predicate-calculus-oriented string logic as a basic element of foundations of mathematics (consider the case of axiomatic set-theories like ZF.) Classics of Analytic philosophy including Russell and Quine (to give only a couple of most influential names) argued that a particular kind of predicate-calculus-oriented string logic should be the basic logic underlaying our best scientific theories. The following proliferation of logical calculi gave rise to a more liberal attitude called logical pluralism (see [1]).

As any open-minded attitude the logical pluralism is a good thing. However by itself it can hardly support a system of logic that is not a predicate-calculus-oriented string logic; at best

it can only tolerate it. Since Founding Fathers of Analytic philosophy (many of which were also active logicians) explained us how a a predicate-calculus-oriented string logic connects basic metaphysical features of what there is (Quine) with basic features of how we reason about what there is (as these later features are expressed linguistically in our everyday talks) it is not so easy to imagine how to do logic differently! Earlier Aristotle and his followers made a similar job with a different system of logic called syllogistic. This historical example shows that the situation is not as hopeless as it might seem: there is a good chance that logico-metaphysical dogmas of Analytic philosophy will be soon destroyed by new scientific developments after the Scholastic dogmas of Aristotelian metaphysics.

The above remarks may suggest that philosophy is not particularly helpful for doing science in general and for doing logic in particular. I think however that this is a wrong conclusion. Its true that philosophical prejudices often become an obstacle for scientific progress. But its also true that philosophical prejudices can be effectively eliminated only with philosophical means. To eliminate old foundations when they turn into prejudices and replace them with new foundations is a job of philosophy. Such activity is an essential element of science.

In the rest of this paper I shall describe a philosophically-laden project aiming at making a diagram-based logical syntax into foundations of mathematics. I hope that the elimination of some old and recent prejudices about logic, which this project implies, can be helpful for solving technical problems stressed in the beginning of this Introduction.

2. Sets of sets and categories of categories

In his [5] Lawvere suggests to use The Category of Categories as a Foundation for Mathematics (this is the title). In order to understand Laweveres project of Category-theoretic foundations it is instructive to compare it with more familiar Set-theoretic foundations. As far as the mathematical notion of set is not used for foundational purposes people may talk about sets of various sorts of mathematical objects like numbers or points. But axiomatic theories of sets like ZF assume primitive objects of a single sort called sets and a single non-logical primitive relation of membership, so that every set within such a theory is a set of sets (with the only noticeable exception of the empty set for which the expression set of... is irrelevant). The idea of making ZF or a similar theory into foundations of mathematics amounts to an attempts to think of every mathematical object (like a number or a point) as a set of sets. Lawveres idea is similar in this respect. Mathematicians usually talk about categories of mathematical objects (or mathematical structures) introduced independently: this is the case of category of groups, category of topological spaces, category of sets, etc. Lawveres axiomatic theory of categories makes notions of category and functor into primitives; the usual categories just mentioned are supposed to be reconstructed from these primitive notions by axiomatic means. (Lawvere shows how it works for the category of sets.) Since Lawvere identifies a given category with its identity functor he doesnt actually

use two different primitive notions here, so the only primitive notion of his theory is that of functor.

Lawveres theory begins like ZF or any other similar axiomatic theory: he assumes the Classical predicate calculus (that is, the usual variant of predicate-calculus-oriented string logic), choose few primitive non-logical predicates (namely, one assigning to a given functor its domain, another assigning to a given functor its co-domain and finally one assigning to a pair of appropriate functors a third functor called their composition) and writes down the usual axioms of category theory with these means. This first block of his first-order theory Lawvere calls the *elementary theory* of categories; I shall call it ET in what follows. A *category* this author defines as an arbitrary model of ET..

Noticeably ET by itself doesnt provide a notion of category which would allow for construing mathematical objects as categories of categories in a way similar to which they are construed as sets of sets. What we get with ET is a notion of category abstracted from the usual examples mentioned above, i.e. the notion of category of something. It can be a category of categories but it should not be. This feature is related to the following foundational difficulty. Arguably ET involves a primitive notion of class , which comes along with the very axiomatic setting underlying this theory: in order to interpret ET (i.e. to build its model) one should first of all consider a class of abstract individuals to be called *morphisms* or *functors* and then define suitable relations between these individuals. Even if classes involved here are not sets in the sense of ZF the notion of class qualify as settheoretic in a broader sense of the work. Hence, so the argument goes, ET cannot provide a self-standing foundation of mathematics; even if ET doesnt depend on ZF it depends on a primitive non-axiomatic notion of class or collection. [1]

Mayberry in his [6] puts the above argument forward and concludes that Lawveres idea of pure category-theoretic foundations is futile because a primitive non-axiomatic notion of set (class, collection) is indispensable in mathematics. I buy the argument but dont buy Mayberrys conclusion. The notion of class comes to ET with the standard axiomatic setting. Mayberrys conclusion follows only if one assumes that this axiomatic setting is the only possible one. As I shall argue in the next section this assumption is erroneous.

On the basis of ET Lawvere builds another theory, which he calls *basic theory* of categories (BC). Formally speaking BC is an extension of EC with some additional axioms. But it also involves a deep change of the whole foundational viewpoint as we shall now see. The idea behind BC is this: using the notion of category developed in ET conceive of category C of *all* categories; then pick up from C an arbitrary object A (i.e. an arbitrary category) and finally specify A as a category by internal means of C stipulating additional properties of C when needed. More precisely it goes as follows (I omit details and streamline the argument). Stipulate the existence of terminal object *1* in C, i.e. the object with exactly one incoming functor from each object of C. Then identify objects (= identity functors)

---

[1]Since objects in the obtained category are not necessary categories the term morphism is more appropriate in this context.

of A as functors of the form $1 \rightarrow A$. Stipulate also the existence of initial object 0, i.e. the object with exactly one outgoing functor into each object of $\boldsymbol{C}$. Then consider in $\boldsymbol{C}$ object 2 of the form $0 \rightarrow 1$ and stipulate for it some additional properties among which is following: 2 is universal generator which means that:

G (generator): for all f, g of the form:

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

and such that $f \neq g$ there exist x such that:

$$2 \overset{x}{\longrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

and $xf \neq xg$.
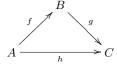
U (universal): if any other category N has the same property than there are y, z such that:

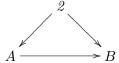$$A \underset{z}{\overset{y}{\rightleftarrows}} B$$

and $yz = 2$.

This allows Lawvere to identify internal functors (morphisms) of A as functors of the form $2 \rightarrow A$ in C. For the fact that 2 is the universal generator (that is unique up to isomorphism as follows from the above definition) assures that categories are determined "arrow-wise": two categories coincide if and only if they coincide on all their arrows. This new definition of functor also allows one to make a sense of the notion of a component of a given functor of the form $h \colon A \to B$, which in ET is understood as a map m sending a particular morphism f of A into a particular morphism g of B. In BC m turns into this commutative triangle:
[2]

----

[2]A categorical diagram is said to commute or be commutative when the compositions of all morphisms shown at the given diagram produce other morphisms shown at the same diagram in appropriate places, so that any ambiguity about results of the compositions is avoided. For example, saying that the triangle
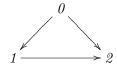
$$\begin{array}{ccc} & B & \\ {}^{f}\nearrow & & \searrow^{g} \\ A & \underset{h}{\longrightarrow} & C \end{array}$$

is commutative is simply tantamount to saying that $fg = h$. Morphisms resulting from composition of shown morphisms can be omitted at a commutative diagram when this doesnt lead to an ambiguity. For example, saying that this square

$$
\begin{array}{ccc}
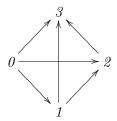 & 2 & \\
A & \longrightarrow & B
\end{array}
$$

This, once again, significantly changes the whole perspective: categories and functors are no longer built "from their elements" but rather "split them into" their elements when appropriate.

Further consider this triangle which Lawvere denotes 3:

$$
\begin{array}{ccc}
 & 0 & \\
1 & \longrightarrow & 2
\end{array}
$$

(it should satisfy a universal property which I omit). 3 serves for defining composition of morphisms in our "test-category" A as a functor of the form $3 \to A$ in C. Finally, in order to assure the associativity of the composition Lawvere introduces category 4 that is pictured as follows:

$$
\begin{array}{ccc}
 & 3 & \\
0 & \longrightarrow & 2 \\
 & 1 &
\end{array}
$$

(The associativity concerns here the path $0 \to 1 \to 2 \to 3$.)

This construction provided with appropriate axioms makes A into an "internal model" of ET in the following precise sense: If F is any theorem of ET then "for all A, A satisfies F" is a theorem of BT.. [3]

_____

$$
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\uparrow{\scriptstyle f} & & \uparrow{\scriptstyle i} \\
C & \xrightarrow{h} & D
\end{array}
$$

is commutative is tantamount to saying that $fg = hi$.

[3]Isbell in his review [4] of Lawvere's [5] points to a technical flaw in Lawveres proof of this theorem . This flaw is fixed, in particular, in [7].

3. Purifying categorical foundations

As we have seen Lawveres construction involves two different steps, which correspond to ET and BT. At the first step Lawvere formalizes usual axioms and definitions of the informal abstract Category theory with usual means, namely with Classical predicate calculus. At the second step Lawvere uses categorical resources obtained with ET for introduction (or rather re-introduction) of basic categorical concepts including the basic concept of functor (remind that in BC an internal functor of a test-category is defined as a commutative triangle of a special form). As I have already explained this apparently strange move allows Lawvere to treat any category as a category of categories in a way similar to which in axiomatic theories of sets people treat any set as a set of sets. Thus the two layers of Lawveres construction correspond to two different foundational projects. The first one (ET) is standard: it is designed after the example of ZF or any similar first-order theory. The second (BT) involves an original method of building a theory with Category-theoretic means. The combination of the two methods suggested in [5] qualifies as a sound a pragmatic solution. However it is natural to ask whether or not the latter categorical method may work independently and so provide purely categorical foundations, which wouldnt involve any primitive notion of class.

In order to get an answer let us see more precisely what the first standard step of Lawvere's construction serves for. First of all it introduces the preliminary notion of category, which is used at the second step. I suggest that this preliminary notion of category can be assumed informally and then made more precise by means of BC. An analogy with Set theory is once again helpful. Axiomatic set theories use a preliminary informal notion of collection and then make it more specific and more precise; they dont produce sets *ex nihilo*. Categorical foundations, in my view, should be built similarly: one begins with an informal intuitive notion of category and then use it for making this very notion more precise and better behaved. Such circularity is not vicious in the case of sets and it is not vicious in the case of categories either.

However this doesnt solve the whole problem since BT like ET uses standard logical means; remind that the former is an axiomatic extension of the latter. Although axioms of BT are represented with categorical diagrams it is assumed that they can be always rewritten with the standard logical syntax. Commenting on the role of diagrams in his theory Lawvere says:

> Commutative diagrams in general are regarded as abbreviated formulas, signifying the usual indicated systems of equations. For example, our statement above of the associativity axiom becomes transparent on contemplating the following commutative diagram...

(The associativity axiom is first written as an equation; the corresponding diagram follows the quote.)

If the diagram makes the meaning (or more precisely the intended meaning) of the associativity axiom transparent what is the reason for writing this axioms also in the form of equation? The obvious answer is this: because the equation unlike the diagram can be treated as a formula of the Classical first-order calculus; the equational form of the axiom allows one to apply to the given this standard logical machinery. In order to treat diagrams similarly one need a system of diagrammatic logical syntax, which could replace the standard linear syntax. With such a diagrammatic syntax and an informal notion of category BT can be indeed built independently of ET. This would provide a purely categorical method of theory-building and purely categorical foundations of mathematics, which dont involve any primordial notion of collection or class.

4. Some constraints of diagrammatic representation

Diagrams seem to have a stronger intuitive appeal than strings of formulae written with usual symbols. Nevertheless diagrams like symbols need to be interpreted. The idea that the meaning of a given diagram can be captured through a pure contemplation is erroneous. Lets consider the case of categorical diagrams where categorical morphisms (functors) are represented by arrows. There are at least two problems with this representation. Any trajectory in a space can be followed in both directions; any spatial motion can be reversed. At least this is how we usually think about a space. But categorical morphisms are, generally, non-reversible. This is why we represent them by arrows rather than by lines. To be read correctly a categorical diagram should be thought of as a spatio-temporal object rather than a merely spatial object.

The other problem is this. Different geometrical figures have different points; when the figures coincide on all their points they are the same. But two morphisms coinciding an all their points are, generally, different. (By a point in this latter context I understand as usual any morphism from the terminal object of a given category.) This means that one cannot add points for free to a given categorical diagram as we usually do this with geometrical diagrams. Although categorical morphisms can be conceived of as transformation (in the most general sense of the work) the idea that a given transformation can be fully analyzed in terms of momentary states should be definitely abandoned in this case.

5. Conclusion

The fact that a diagrammatic syntax is both a matter of industrial demand and an open problem in foundations of mathematics seems me significant. A foundational turn is helpful because it allows one to learn how to think about old notions in a new way. I suspect that the claim that every mathematical object is a category may sound for a working mathematician or computer scientist as another example of a philosophical absurdity just like the earlier claim that every mathematical object is a set. Nevertheless it would be quite hard to imagine how computer science and computer technology develop without modern logical means, which in their turn emerged as a part and parcel of the foundational revolution in

mathematics in the first half of 20th century. The ongoing revision of foundations through Category theory and related field may have a similar long-term effect.

References

[1] J.P. Beall and G. Restall. *Logical Pluralism*. Oxford University Press, 2006.
[2] Z. Diskin, Kadish B., F. Piessens, and M. Johnson. Universal arrow foundations for visual modeling. In *Diagrams'2000: Proc. 1st Int. Conference on the theory and application of diagrams, Edinburgh (Scotland)*, pages 1–21, 2000.
[3] Z. Diskin and U. Wolter. A diagrammatic logic for object-oriented visual modeling, forthcoming. To appear in ENTCS.
[4] J.R. Isbell. Review of lawvere:1966. *Mathematical Reviews*, 34, 1967.
[5] F.W. Lawvere. The category of categories as a foundation for mathematics. In *Proceedings of the La Jolla Conference on Categorical Algebra*, pages 1–21, 1966.
[6] J.P. Mayberry. *The foundations of mathematics in the theory of sets*. Cambridge University Press, 2000.
[7] C. McLarty. Axiomatizing a category of categories. *The Journal of Symbolic Logic*, 56(4), 1991.