CATEGORICAL LOGIC AND HEGELIAN DIALECTICS

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Content:

Motivations

Admittedly a bit of provocation :) however:

- NOT an external comparison of two different theories of logic (one of which is mathematical while the other is not)
- BUT taking seriously Lawvere's philosophical motivation behind his *invention* of categorical logic in mid-1960ies
- AND equally - behind Lawvere's further work in the field including his axiomatization of topos theory.

Quantifiers and Sheaves 1970

"The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. [..] We first sum up the principle contradictions of the Grothendieck-Giraud-Verdier theory of topos in terms of four or five adjoint functors, significantly generalizing the theory to free it from reliance on an *external* notion of infinite limit [..]."

Quantifiers and Sheaves 1970

"When the main contradictions of a thing have been found, the scientific procedure is to summarize them in slogans which one then constantly uses as an ideological weapon for the further development and transformation of the thing. Doing this for "set theory" requires taking account of the experience that the main pairs of opposing tendencies in mathematics take the form of adjoint functors and frees us of the mathematically irrelevant traces ($\in$) left behind by the process of accumulating ($\cup$) the power set (P) at each stage of the metaphysical "construction". Further, experience with sheaves, [..], etc., shows that a "set theory" for geometry should apply not only to *abstract* sets divorced from time, space, ring of definition, etc., but also to more general sets which do in fact develop along such parameters."

Reservations

- Categorical logical as any other system of mathematical logic can be presented in purely mathematical terms without any philosophical motivation or interpretation.
- Categorical logic as any other system of mathematical logic is open to different philosophical interpretations.
- However Lawvere's original philosophical motivation behind the building of Categorical logic needs a special attention, at least for a historical reason.

Reservations

- It may appear that Lawvere's Hegelian remarks found throughout his mathematical papers are superficial and reflect nothing but an intellectual fashion or a political engagement. I shall show that in fact Hegel's philosophical logic substantially drives the whole of Lawvere's research in Categorical logic.
- Lawvere suggests to consider particular category-theoretic constructions as formal mathematical expressions of certain Hegelian notions, in particular, to consider adjoint functors as expressing the Hegelian notion of dialectical contradiction. I shall *not* talk about this aspect of Lawvere's work but focus on more general questions concerning Lawvere's philosophical understanding of Categorical logic including the question of origin and purpose of logic.

Project

- Ontological grounding of logic (after the model of Aristotle)
- Not merely postulative/dogmatic but deductive (after the model of Kant's transcendental deduction)
- Not merely descriptive, not linguistic.

In short: to ground ontologically Kant's (transcendental) deduction of categories without restoring the Aristotelean or Wolfean dogmatism.

Being and Nothing

"Being, pure being, without any further determination. In its indeterminate immediacy it is equal only to itself.[..] Being, the indeterminate immediate, is in fact nothing, and neither more nor less than nothing." STUCK??

"Pure Being and pure nothing are, therefore, the same. [..] But it is equally true that they are not undistinguished from each other, that, on the contrary, they are not the same, that they are absolutely distinct, and yet that they are unseparated and inseparable and that each immediately vanishes in its opposite. Their truth is therefore, this movement of the immediate vanishing of the one in the other: becoming, a movement in which both are distinguished, but by a difference which has equally immediately resolved itself."

Further Categories

becoming
something (thisness)
quality
finitude
infinity
one
many
quantity
measure
essence
. . . .
subjectivity

Objective and Subjective logic

"What is to be considered is the whole Notion, firstly as the Notion in the form of being, secondly, as the Notion; in the first case, the Notion is only *in* itself, the Notion of reality or being; in the second case, it is the Notion as such, the Notion existing *for* itself (as it is, to name concrete forms, in thinking man, and even in the sentient animal and in organic individuality generally [..]). Accordingly, logic should be divided primarily into the logic of the Notion as being and of the Notion as Notion - or, by employing the usual terms (although these as least definite are most ambiguous) into 'objective' and 'subjective' logic. "

Elementary Theory of Category of Sets (ETCS)

Lawvere 1964

Idea (back to von Neumann late 1920-ies): functions instead of $\in$'s

Remark: everything stems from Lawvere's Thesis of 1963

ETCS 1: ETAC

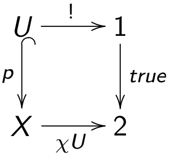
Elementary Theory of Abstract Categories

- E1) $\Delta_i(\Delta_j(x)) = \Delta_j(x); i, j = 0, 1$
- E2) $(\Gamma(x, y; u) \wedge \Gamma(x, y; u')) \Rightarrow u = u'$
- E3) $\exists u \Gamma(x, y; u) \Leftrightarrow \Delta_1(x) = \Delta_0(y)$
- E4) $\Gamma(x, y; u) \Rightarrow (\Delta_0(u) = \Delta_0(x)) \wedge (\Delta_1(u) = \Delta_1(y))$
- E5) $\Gamma(\Delta_0(x), x; x) \wedge \Gamma(x, \Delta_1(x); x)$
- E6) $(\Gamma(x, y; u) \wedge \Gamma(y, z; w) \wedge \Gamma(x, w; f) \wedge \Gamma(u, z; g)) \Rightarrow f = g$

E1)-E4): bookkeeping (syntax); 5): identity; 6): associativity

ETCS 2: Elementary Topos (anachronistically):

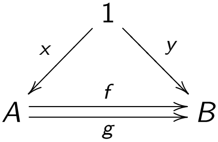
- finite limits;
- Cartesian closed (CCC): terminal object (1), binary products, exponentials;
- subobject classifier



for all p there exists a unique χU that makes the square into a pullback

ETCS 3: well-pointedness

for all $f, g : A \to B$, if for all $x : 1 \to A$ $xf = xg = y$ then $f = g$



ETCS 4: NNO

Natural Numbers Object: for all t', f there exists unique u

$$\begin{array}{ccccc} 1 & \xrightarrow{t} & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \\ & \searrow^{t'} & \downarrow u & & \downarrow u \\ & & A & \xrightarrow[f]{} & A \end{array}$$

ETCS 4: Axiom of Choice

Every epimorphism splits:
If $f : A \rightarrow B$ is epi then there exists mono $g : B \rightarrow A$ (called *section*) such that $gf = 1_B$

No Categorical logic and no Hegelian dialectic so far but..

The idea of internal logic: Prehistory

- Boole 1847, Venn 1882: propositional logic as algebra and mereology of (sub) classes (of a given universe of discurse); logical diagrams
- Tarski 1938 topological interpretation of Classical and Intuitionistic propositional logic

Remark: Consider the similarity with Hegel's approach: logic is not god-given but appears as a feature of the given subject-matter. However in Boole, Venn and Tarski such an internal treatment concerns only propositional logic

The idea of internal logic: CCC

- Lawvere 1969: CCC is a common structure shared by (1) the simply typed λ-calculus (Schönfinkel, Curry, Church) and (2) Hilbert-style (and Natural Deduction style) Deductive Systems (aka Proof Systems).
- The fact that such a common structure exists is often *misleadingly* called *Curry-Howard correspondence* or *Curry-Howard isomorphism* (my upcoming talk of 14th March)
- The CCC structure is *internal* for *Set* BUT is more general: *Cat* (of all *small* categories) is another example; any *topos* is CCC.
- Lawvere: CCC is *objective* (in the Hegelian sense) while usual syntactic presentations or logical calculi are only *subjective*. While syntactic presentations lay out only *formal* foundations, CCC lays out a *conceptual* foundation.

Internalization of quantifiers: adjoint functors

An *adjoint situation* (aka *adjunction*; Kan 1958) is a pair of categories A, B with two functors f, g going in opposite directions:

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

with natural transformations $\alpha : A \rightarrow fg$ and $\beta : gf \rightarrow B$ such that $(g\alpha)(\beta f) = g$ and $(\alpha f)(g\beta) = f$ such that the following two triangles commute:

$$\begin{array}{ccc} g & \xrightarrow{g\alpha} & gfg \\ & {\scriptstyle 1_g}\searrow & \downarrow{\scriptstyle \beta g} \\ & & g \end{array} \qquad \begin{array}{ccc} f & \xrightarrow{\alpha f} & fgf \\ & {\scriptstyle 1_f}\searrow & \downarrow{\scriptstyle f\beta} \\ & & f \end{array}$$

Simple fact: A given functor has at most one (up to unique isomorphism) left adjoint and one right adjoint.

Internalization of quantifiers: Translating properties

Suppose that we have a one-place predicate (a property) P, which is meaningful on set Y, so that there is a subset P_Y of Y (in symbols $P_Y \subseteq Y$) such that for all $y \in Y$ $P(y)$ is true just in case $y \in P_Y$.
Define a new predicate R on X as follows: we say that for all $x \in X$ $R(x)$ is true when $f(x) \in P_Y$ and false otherwise. So we get subset $R_X \subseteq X$ such that for all $x \in X$ $R(x)$ is true just in case $x \in R_X$. Let us assume in addition that every subset P_Y of Y is determined by some predicate P meaningful on Y. Then given morphism f from "universe" X to "universe" Y we get a way to associate with every subset P_Y (every part of universe Y) a subset R_X and, correspondingly, a way to associate with every predicate P meaningful on Y a certain predicate R meaningful on X.

Internalization of quantifiers: Substitution functor

Since subsets of given set Y form Boolean algebra $B(Y)$ we get a map between Boolean algebras:

$$f^* : B(Y) \longrightarrow B(X)$$

Since Boolean algebras themselves are categories f^* is a functor. For every proposition of form $P(y)$ where $y \in Y$ functor f^* takes some $x \in X$ such that $y = f(x)$ and produces a new proposition $P(f(x)) = R(x)$. Since it replaces y in $P(y)$ by $f(x) = y$ it is appropriate to call f^* *substitution* functor.

Existential Quantifier as adjoint

The *left* adjoint to the substitution functor f^* is functor

$$\exists_f : B(X) \longrightarrow B(Y)$$

which sends every $R \in B(X)$ (i.e. every subset of X) into $P \in B(Y)$ (subset of Y) consisting of elements $y \in Y$, such that *there exists* some $x \in R$ such that $y = f(x)$; in (some more) symbols

$$\exists_f(R) = \{y | \exists x(y = f(x) \wedge x \in R)\}$$

In other words $\exists_f$ sends R into its *image* P under f. One can describe $\exists_f$ by saying that it transformes $R(x)$ into $P(y) = \exists_f x P'(x, y)$ and interpret $\exists_f$ as the usual existential quantifier.

Universal Quantifier as adjoint

The *right* adjoint to the substitution functor f^* is functor

$$\forall_f : \ B(X) \longrightarrow B(Y)$$

which sends every subset R of X into subset P of Y defined as follows:

$$\forall_f(R) = \{y | \forall x(y = f(x) \Rightarrow x \in R)\}$$

and thus transforms $R(X)$ into $P(y) = \forall_f x P'(x, y)$.

Moral

A higher-order logic is <u>also</u> internal for *Set* - in fact, <u>not only</u>

Toposes

- Grothendieck toposes (=toposes of sheaves on sites)are generalized (topological) spaces
- Elementary toposes (Lawvere) are axiomatized Grothendieck toposes
- Lawvere's axiomatization recovers the *internal* logic of a given topos rather than puts the topos concept into a pre-established logical framework.
- In THAT sense Lawvere's logical reasoning follows Hegel's pattern.
- The same pattern of axiomatic thinking is present in the recent Voevodsky's axiomatization of higher homotopy theory with Martin-Löf's type theory (the resulting theory is called the homotopy type theory).

Categories of Space and Quantity 1992

It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. In turn the explicit attention by mathematicians to such philosophical questions is necessary to achieve the goal of making mathematics (and hence other sciences) more widely learnable and useable. Of course this will require that philosophers learn mathematics and that mathematicians learn philosophy.

Another way to develop Kant

Logical and mathematical concepts must no longer produce instruments for building a metaphysical "world of thought": their proper function and their proper application is only within the empirical science. (Cassirer 1907)
Cf. the fate of geometry in GR