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The Dogma of Extensionality (extended abstract)

We call *extensionalism* a methodology of science which prescribes to account for any investigated system (to use the latter term very widely) in terms of its elements, as for example a gas is accounted for in terms of its molecules. From the point of view of strong extensionalism the final goal of science is to discover *Urelemente* of Nature. A weaker version of extensionalism gives up the hope to find the *Urelemente* and relies on elements which allow better accounts for possibly wider classes of phenomena. A non-reductionistic version of weak extensionalism also allows a hierarchy where the same entities can be treated both as basic elements within some accounts and as complex constructions within other accounts without a possibility to reduce the former accounts to the latter.

Although the extensionalism, at least in its weaker versions, is widely accepted in contemporary science in some cases it is apparently hardly applicable. Particularly in biology it is normally easier to account for this or that biochemical mechanism in terms of its function within given cell or whole organism than to explain away global properties of the cell or the organism in terms of interactions between molecules. Within physical field theories various local phenomena are explained away in terms of global features of an underlying field. In such cases elements of a system are accounted for in terms of the system (particularly in terms of its functions) but not the other way round. The latter methodology we call *intensionalism*.

Extensionalism is supported by atomistic or particularistic ontologies while intensionalism is supported by holistic ones. The old ontological discussion started in the early era of Ancient philosophy gave no decisive argument to prefer one of the two options and hence to choose between extensionalism and intensionalism *a priori*. A rational position is seemingly to consider both approaches as equally sound and mutually dual. Nevertheless nearly all the mathematical tools used in science (including field theories) are inherently atomistic and extensionalistic. Particularly *point* is considered as a basic (or even *the* basic) geometrical object since Euclid's time. In the Set Theory which is widely considered as a basis for almost all the contemporary mathematics the extensionality is taken as an axiom. Thus the mentioned domination of extensionalism in the contemporary science is seemingly due to the only fact that it has much better formal tools in its disposal. But this fact is apparently historically contingent and depended on an unjustified choice for ontological atomism at some very early stage of history of mathematics.

To fulfil the gap we suggest an intensionalistic version of the Set Theory. Instead of the Axiom of Extensionality which allows to identify sets by their elements, or in symbols

(1) $\forall x\forall y(\forall z(z\in x\leftrightarrow z\in y)\rightarrow x=y)$,

the Axiom of Intensionality is introduced which allows to identify elements by sets (to which the element(s) belong(s)), or in symbols

(1') $\forall x\forall y(\forall z(x\in z\leftrightarrow y\in z)\rightarrow x=y)$.

Note that within the new axiomatic system the primitive symbol $\in$ gains a new meaning which however can be also loosely expressed in English as *belongs to*. Given translation of *belongs to set B* as *has a property B* the Axiom of Intensionality is translated into Leibniz Law with the following reservation: the identity in question is also that of a *property*, not of a substance. The following axioms of the proposed system are obtained by similar duality from "constructive" axioms of ZF (ZF2-4). Unlike ZF which allows an atomic set (i.e. a set without elements, namely *the* empty set) and does not allow sets which are not elements of other sets (i.e. using the standard terminology does not allow a *class* to be a set) the new proposed axioms the other way round do not allow atoms but do allow sets which are not elements (for different reasons we call them *worlds*). Namely the Axiom of Pairing of ZF which allows to build a set consisting of two given different sets (as its elements), or in symbols

(2) $\forall a\forall b(a\neq b\rightarrow\exists p\forall x(x\in p\leftrightarrow(x=a\vee x=b)))$

is replaced by the duality by the Axiom of Linkage which says that every couple of different sets share an element (which is uniquely defined by the Intensionality provided it belongs to the given sets and *only* to them). In symbols

(2') $\forall a\forall b(a\neq b\rightarrow\exists p\forall x(p\in x\leftrightarrow(x=a\vee x=b)))$.

According the latter axiom all the considered (intensional) sets are linked or make a "bundle".

To word the next axiom we introduce one more term (which is actually redundant like the term *element* in ZF). Namely if $A\in B$ we shell call B an *area* of A.

Then we replace the Axiom of Union of ZF which guarantees the existence of a set consisting of elements of elements of given set, or in symbols

(3) $\forall a(\exists b(b\in a)\rightarrow\exists y\forall x(x\in y \leftrightarrow\exists z\ (x\in z\ \&\ z\in a)))$,

by the Axiom of Intersection which guaranties the existence of an element belonging to all the areas of areas of given element (and only to them), or in symbols

(3') $\forall a(\exists b(a\in b)\rightarrow\exists y\forall x(y\in x \leftrightarrow\exists z\ (z\in x\ \&\ a\in z)))$.

(3) allows to “put together” elements of elements of a set. Dually (3’) allows to “link together” areas of areas of an element.

To make a replacement for the Axiom of Power-set of ZF we first need to define the dual notion for that of subset. We shall call *y* a *subelement* of *z* iff every area of *y* is an area of *z*, or in symbols

$\forall x\ (y\in x\rightarrow z\in x)$

Then we replace the Axiom of Power-set which allows the set of all the subsets of given set, or in symbols

(4) $\forall a\exists y\forall x(x\in y\leftrightarrow x\subseteq a)$

by the Axiom of Root which guarantees that any subelement of every element of given set in its turn has an certain element (which we call *root*), i.e. that no subelement is atomic. In symbols

(4’) $\forall a\exists y\forall x(y\in x\leftrightarrow a\subseteq x)$

(Here $a\subseteq x$ stands for *a is subelement of x*.) Since by the above definition every set is its own subelement (4’) forbids atomic sets (elements) at all. (Dually (4) forbids sets which are not elements which leads to “Cantor’s Paradox”).

Axioms (1-4) are called by Fraenkel and Bar-Hillel “expansive” since they guarantee the existence of sets which are “more extensive” than those *supposed* by the axioms (*Foundation of Set Theory*, ch.2, #3). Dually we might call the proposed axioms (1’-4’) *compressing* since they guarantee the existence of sets which are “more intensive” than those supposed by the latter axioms, that is which belong to more sets. While (1-4) allow a constructive approach when the (atomic) empty set is taken as a basic element (like a brick) for a wide (and in a certain sense unlimited) range of complex extensive constructions (starting with finite ordinals obtained by von Neuman’s method), the proposed (1’-4’) allow a dual approach: we might start with a set which is not an element (a *world*) and develop its “internal structure” also in a certain sense unlimitedly.

To complete the proposed system in the line of ZF we need first of all to find a dual notion for that of *predicate* used in the fifth Axiom of Subsets of ZF. This opens a new, purely logical, field for a further study. As a preliminary remark we only note that perhaps it is impossible to gain a consistently intensional system keeping the standard way of building a formal system which is to start with a list of primitive symbols and use them as *Urelemente* for further constructions. Such an approach itself might be called extensionalistic (in the sense exposed in the beginning of the paper). Perhaps for the intended purpose we should rather fix (formally) some global features of a desired system allowing any “safe” changes of alphabets and vocabularies at all levels. From this point of view the above attempt to dualize the axioms 1-4 of ZF using the same formal tool should be considered as a preliminary one. The following three final remarks are in order.

1) the concept of intensionality applied here does not rely immediately on that of a property, predicate, meaning or Frege's *Sinn*;
2) while a suitable weakening of (1) allows *Urelemente*, i.e. the plurality of atoms, the dual weakening of (1') allows the plurality of worlds.
3) while elements of a (standard extensional) set are naturally thought of as *objects* (remind school examples of sets of apples and similar things) there are strong reasons (which however cannot be discussed in full here) to interpret considered intensional sets as *events*.

(The last remark is based on the following intuition: while objects are observed externally (from outside, ultimately from nowhere) events are observed internally (from inside which means locally). To illustrate this: an object is said to be here and now when the speaker can point it by a finger (which still needs a distance) while an event is said to be here and now rather when the speaker is involved in it (or at least when the event occurs "around" the speaker, think about a rain). What is observed when one, say, moves inside a building, is rather the history of the journey, i.e. an event which might involve many smaller objects but not the whole building as an object. To perceive the globe as an object one needs to go up into the space (and ultimately to reach Heavens to be able to see the whole thing at once), otherwise one relies on histories of her/his own or somebody else's journeys. The vagueness of those intuitions can be at least partly erased by applying the Guassian concepts of external and internal geometrical spaces and by analyzing the event ontology of General Relativity.

Supposing this we come to some sort of ontological dualism according to which every "normal" entity like a building or the globe can be observed both from inside and outside and correspondingly to be described both as an object and an event. There are (or rather we might think about) two exceptions though. *Atoms* can be observed *only* from outside while *worlds* only from inside (the latter might be supposed to be by definition). Thus the standard extensional set theoretical approach which is ultimately supposed to build everything from the only atom (the empty set) is suitable to account for things *externally*, i.e. as objects, while the proposed dual intensional approach which is ultimately supposed to develop everything from the only *world* is suitable to account for things *internally*, i.e. as events. It is needed to add that there is no (and obviously can be no) empirical evidence nor metaphysical necessity for both atoms and worlds *to exist*; we should rather use both notions very cautiously as (extremely strong) conceptual devices which might be often helpful but are dangerous if misused. By the way this seemingly fits Kant's position concerning his first two *antinomies* of *Critique of Pure Reason* (the first one about atoms and the second one about the world); he argues that such metaphysical concepts as those of atom and world (together with those of God and Freedom considered in the next two antinomies) are rather of practical than of purely theoretical use. The question whether the extensional approach might suppose no atoms and whether the intensional approach might suppose no worlds is a matter for a further study.)