Hilbertian Axiomatic Method
Lawvere on categorical foundations
Rethinking the Hilbertian Setting through the History of Geomet
A Sketch of categorical foundations of geometry

Towards categorical foundations of geometry:
What is geometrical object?

Andrei Rodin

31 mai 2010

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Content :

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Hilbertian Axiomatic Method

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Hilbertian Axiomatic Method

Veblen and Young, 1938, *Projective geometry*, vol. 1

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Hilbertian Axiomatic Method

Veblen and Young, 1938, *Projective geometry*, vol. 1

> *The starting point of any strictly logical treatment of*
> *geometry (and indeed of any branch of mathematics)*
> *must then be a set of undefined elements and relations,*
> *and a set of unproved propositions (=axioms) involving*
> *them, and from these all other propositions (theorems)*
> *are to be derived by the methods of formal logic.*
> *Moreover, since we assumed the point of view of fromal*
> *(i.e. symbolic) logic, the undefined elements are to be*
> *regarded as mere symbols devoid of content...*

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Hilbertian Axiomatic Method

> *We understand the term a <u>mathematical science</u> to mean*
> *<u>any set of propositions</u> arranged according to a sequence*
> *of <u>logical deduction</u>. ... Such a science is purely <u>abstract</u>.*
> *If any concrete system of things may be regarded as*
> *satisfying the fundamental assumptions (=axioms), this*
> *system is a concrete <u>application</u> or <u>representation</u> (=or*
> *model) of the abstract science.*

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Hilbertian Axiomatic Method

The notion of a <u>class</u> of objects is fundamental in logic and hence in any mathematical science. The object which make up the class are called the <u>elements</u> of the class. The notion of a class, moreover, and the relations of <u>belonging to a class</u> (being included in a class, being element of a class, etc.) are primitive notions of logic.

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Hilbertian Axiomatic Method

> *A set of assumptions (=axioms) is said to be categorical*
> *if there is essentially only one system for which the*
> *assumptions are valid, i.e. if any two such systems may*
> *be made simply isomorphic.*

(end of quote)

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Hilbertian Axiomatic Method

Some Basic Features :

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Hilbertian Axiomatic Method

Some Basic Features :

▶ The distinction between an "abstract" formal theory and its "concrete" models. The fundamental role of Logic.

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Hilbertian Axiomatic Method

Some Basic Features :

▶ The distinction between an "abstract" formal theory and its "concrete" models. The fundamental role of Logic.

▶ The fundamental role of the notion of a <u>class</u>. <u>Objects</u> are elements of classes.

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Hilbertian Axiomatic Method

Some Basic Features :

▶ The distinction between an "abstract" formal theory and its "concrete" models. The fundamental role of Logic.

▶ The fundamental role of the notion of a <u>class</u>. <u>Objects</u> are elements of classes.

▶ Structuralism : isomorphic models are essentially the same. The relation of being isomorphic plays the role of equality.

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Hilbertian Axiomatic Method

Claims :

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Hilbertian Axiomatic Method

Claims :

▶ Hilbertian Axiomatic Method implies set-theoretic foundations
(broadly conceived).

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▶ Hilbertian Axiomatic Method implies Structuralism (again broadly conceived)

▶ Genuine categorical foundations cannot rely onto Hilbertian Axiomatic Method but require a different method of theory-building.

▶ In particular, categorical foundations require a different notion of object.

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▶ In particular, categorical foundations require a different notion of object.

▶ Categorical foundations cannot be Structural foundations (leaving the issue aside).

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Lawvere on categorical foundations

"The Category of Categories as a Foundation for Mathematics",
1966 :

> *In the mathematical development of recent decades one
> sees clearly the rise of the conviction that the relevant
> properties of mathematical objects are those which can
> be stated in terms of their abstract structure rather than
> in terms of the elements which the objects were thought
> to be made of. The question thus naturally arises whether
> one can give a foundation for mathematics which
> expresses wholeheartedly this conviction concerning what
> mathematics is about, and in particular in which classes
> and membership in classes do not play any role.*

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Lawvere on categorical foundations

(1) Elementary Theory of Abstract Categories

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(1) Elementary Theory of Abstract Categories

(2) Basic Theory of Categories

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(1) Elementary Theory of Abstract Categories

(2) Basic Theory of Categories

(3) Stronger Theory of Categories

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Elementary Theory of Abstract Categories (ETAC)

Axioms of ETAC :

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Elementary Theory of Abstract Categories (ETAC)

Axioms of ETAC :

- E1) $\Delta_i(\Delta_j(x)) = \Delta_j(x); i, j = 0, 1$

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Elementary Theory of Abstract Categories (ETAC)

Axioms of ETAC :

- E1) $\Delta_i(\Delta_j(x)) = \Delta_j(x); i, j = 0, 1$
- E2) $(\Gamma(x, y; u) \wedge \Gamma(x, y; u')) \Rightarrow u = u'$

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- E4) $\Gamma(x,y;u) \Rightarrow (\Delta_0(u) = \Delta_0(x)) \wedge (\Delta_1(u) = \Delta_1(y))$
- E5) $\Gamma(\Delta_0(x),x;x) \wedge \Gamma(x,\Delta_1(x);x)$

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- E5) $\Gamma(\Delta_0(x), x; x) \wedge \Gamma(x, \Delta_1(x); x)$
- E6) $(\Gamma(x, y; u) \wedge \Gamma(y, z; w) \wedge \Gamma(x, w; f) \wedge \Gamma(u, z; g)) \Rightarrow f = g$

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1)-4) : bookkeeping (syntax) ; 5) : identity ; 6) : associativity

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Lawvere on categorical foundations

"The Category of Categories as a Foundation for Mathematics",
1966 :

> By a category we of course understand (intuitively) any
> structure which is an interpretation of the elementary
> theory of abstract categories, and by a functor we
> understand (intuitively) any triple consisting of two
> categories and a rule T which assigns, to each morphism
> x of the first category, a unique morphism xT of the
> second category in such a way that ...

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Mayberry's objection :

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Mayberry's objection :

In order to build a model of ETAC (or any other first-order theory) one needs a primitive notion of *collection* (class, set). Similarly in the case of axiomatic set theories. Hence categorical foundations cannot be *ultimate* self-standing foundations.

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A Brief History of Non-Euclidean Geometry (1)

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A Brief History of Non-Euclidean Geometry (1)

▶ Since Antiquity people doubted the validity of P5 because unlike other principles of Euclidean geometry this Postulate was not self-evident. (The popular view according to which the "usual" geometrical intuition is Euclidean doesn't stand against this historical evidence.)

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- ▶ Boliay and Lobachevsky (1820-1830ies) like their predecessors aimed at a general notion of geometry *independent* of P5, not at an alternative geometry. They called it *absolute* geometry (Boliay) or *pangeometry* (Lobachevsky).

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- ▶ This general theory split itself into parts in a rather unusual way (Unification Problem)

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A Brief History of Non-Euclidean Geometry (2)

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A Brief History of Non-Euclidean Geometry (2)

▶ Independently Gauss developed a theory of curve surfaces (1827); later Riemann generalized it to a theory of curve spaces of higher dimensions (1854).

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A Brief History of Non-Euclidean Geometry (2)

- ▶ Independently Gauss developed a theory of curve surfaces (1827); later Riemann generalized it to a theory of curve spaces of higher dimensions (1854).

- ▶ Beltrami in 1868 identified Lobachevskian spaces as Riemannian manifolds of constant negative curvature. In particular, he discovered that the Lobachevskian 2-space *is* a surface (*pseudosphere*) living in the Euclidean 3-space.

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- ▶ Klein in 1971 developed a general theory of spaces of constant curvature (that also included the elliptic case).

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- ▶ Klein in 1971 developed a general theory of spaces of constant curvature (that also included the elliptic case).

- ▶ Hilbert in 1899 proposed a novel Axiomatic Method that met certain epistemological concerns about the new geometry.

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Unification Problem

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Unification Problem

Where live different geometrical spaces ? How they co-exist ?

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Where live different geometrical spaces ? How they co-exist ?

Hilbert's answer : They coexist in the space of logical possibilities

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Unification Problem

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Hilbert's answer : They coexist in the space of logical possibilities

<u>Note :</u> The "space of logical possibilities" is a mere metaphor

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Unification Problem

The Hilbertian Axiomatic Method solves the Problem but...

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Unification Problem

The Hilbertian Axiomatic Method solves the Problem but... only in principle. In particular, it doesn't explain why Riemann-Klein's generalisation of Euclidean geometry is viable while Lobachevsky-Bolyai's generalisation (that includes the Euclidean case along with the Hyperbolic case) is not.

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Multiple Representations Phenomenon :

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Multiple Representations Phenomenon :

- Projective duality

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Multiple Representations Phenomenon :

- ▶ Projective duality
- ▶ Analytic geometry (real and complex geometry)

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- ▶ Projective duality
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Multiple Representations Phenomenon :

- ▶ Projective duality
- ▶ Analytic geometry (real and complex geometry)
- ▶ Beltrami's pseudosphere
- ▶ Lobachevsky's *horosphere* (a surface in Lobachevskian 3-space that is a non-standard model of Euclidean 2-space, i.e. Euclidean plane)

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Multiple Representations Phenomenon :

The Hilbertian approach accounts for the Phenomenon through
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about 100 years of development such objects (and sets thereof) are
no longer conceived as *abstract*.... The (? !) set-theoretic universe
is made into a new "absolute space" of geometry (and of
mathematics in general !).

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An alternative approach : some historical hints :

The notion of *Euclidean plane* has a triple presence in the
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► the *type* that such objects instantiate.

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- ▶ a particular object (an *eplane*) that lives in the Euclidean 3-space (the *ESPACE*) and studied in the Euclidean Stereometry.
- ▶ the *type* that such objects instantiate.

The notion of *intrinsic geometry* (of a geometrical object living in some space) generalises upon the above example.

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An alternative approach : objects as maps

An eplane is a map of the form :
$e : \text{EPLANE} \rightarrow \text{ESPACE}$

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An alternative approach : objects as maps

An eplane is a map of the form :
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A pseudosphere is a map of the form
$p : \text{HPLANE} \rightarrow \text{ESPACE}$

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$e : \text{EPLANE} \rightarrow \text{ESPACE}$
A pseudosphere is a map of the form
$p : \text{HPLANE} \rightarrow \text{ESPACE}$
A horosphere is a map of the form
$h : \text{EPLANE} \rightarrow \text{HSPACE}$

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An alternative approach : objects as maps

An eplane is a map of the form :

e : EPLANE $\rightarrow$ ESPACE

A pseudosphere is a map of the form

p : HPLANE $\rightarrow$ ESPACE

A horosphere is a map of the form

h : EPLANE $\rightarrow$ HSPACE

Generally an *object* is a map from a *type* to a *space* of representation.

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An alternative approach : objects as maps

Objects of the same type look similarly (or are of the same <u>form</u>)
only if they are also represented in the same space.

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representation space by default, i.e. assumes it as *absolute*. The
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of *type* <u>and</u> the role of *space*. Generally, given a map (i.e. an
object) o : A→B we shall call A a *type* and call B a *space*. Thus
the notions of type and space are relational and mutually dual. (I
shall use the word *unit* for both.) A duality between Logic and
Geometry ?

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An alternative solution of the Unification Problem :

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An alternative solution of the Unification Problem :

Geometrical spaces (or more precisely *units* live in a space of *maps* between such things, i.e. in a *category of geometrical objects*.

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An alternative solution of the Unification Problem :

Geometrical spaces (or more precisely *units* live in a space of *maps* between such things, i.e. in a *category of geometrical objects*. The term *objects* as it is usually used in the Category theory is a misname ! It reflects the old Hilbertian way of thinking about categories.

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An alternative account of the Multiple Representation Phenomenon :

Different types are differently "represented" in different spaces. A given type *qua* type is described in terms of such different "images" (outgoing maps) ; *qua* space is described in terms of it "contents", i.e. in terms of incoming maps. No need for an "abstract structure". There is a duality between the "content" and the "representation" : a given *unit* has multiple "contents" as well as multiple "representations". The former feature is well known since Euclid's times ; the latter feature was discovered only in 19th century when the notion of absolute space was given up.

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What replacement for Hilbert's Axiomatic Method ?

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What replacement for Hilbert's Axiomatic Method ?

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What replacement for Hilbert's Axiomatic Method ?

This remains an open question...

but here is a sketch...

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to be elaborated with Sketch theory (but without using any "base" category like that of sets !)

for all $p_1, p_2 : P \to L$ there is unique $d : p_1 \to p_2$