Proofs and Fundamentals: Sample Exercises for the final exam

21 мая 2018 г.

1. Which of the following formulae are tautologies, which are contradictions and which are neither?:
   (a) $P \wedge (\neg P \vee \neg Q)$
   (b) $(\neg X \vee Y) \leftrightarrow (X \to Y)$
   (c) $(L \to (M \to N)) \to (L \to (M \to N))$
   (d) $((P \leftrightarrow \neg Q) \vee P) \vee Q$

2. Give a formal derivation using only *modus ponens* and *modus tollens*:

   (a) $\dfrac{[P \to Q]; [Q \to R]; [\neg R]}{\neg P}$

   (b) $\dfrac{[\neg P][\neg P \to \neg R]; [Q \to R];}{\neg Q}$

3. Are the given sets equal?

   a) $\{x, y\}$ and $\{y, x\}$
   b) $\{x, \{y\}\}$ and $\{y, \{x\}\}$

4. Prove using truth-tables:

   (a) $\bar{A} \cap \bar{B} = \overline{A \cup B}$ (de Morgan Law)
   (b) If $A \cap B = A \cup B$ then $A = B$

5. Simplify the given expressions

   a) $C \cup \overline{B \cap C}$
   b) $(A \cap B) \cup (A \cap \neg B)$

6. Do the two given sets have the same cardinality? (The answer needs a proof.)

   (a) The set of integers and the set of positive integers.

   (b) The set of positive rational numbers smaller than 1 and the set of positive real numbers smaller than 1.

7. Is the given relation an equivalence, a partial order or neither? (The answer needs to be justified with a proof.)
   (a) $m, n \in \mathbb{N}$, m is divisible by n
   (b) $m, n \in \mathbb{Z}, \ m = m \cdot n$

8. Is the given function invertible? If it is then construct its inverse. If not then provide a proof that it has no inverse.
   (a) $f : \mathbb{N} \to \mathbb{N}$; $f(n) = 2n + 3$, where $\mathbb{N}$ is the set of natural numbers
   (b) $f : \mathbb{Q} \to \mathbb{Q}$; $f(n) = 2n + 3$, where $\mathbb{R}$ is the set of rational numbers

9. Prove by induction:
   (a) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

   (b) For all natural n number $7^n - 1$ is divisible by 6.

10. Does the given set with operation form a group? Explain why.
    (a) $\mathbb{Z}$ with the standard product $\times$
    (b) the set of all rotations of given circle around its centre with composition of those rotations
    (c) set of all subsets (aka the powerset) of given set U with the binary operation of intersection of these subsets