# Foundations of Axiomatic Mathematics

## Lecture 3: Axiomatic Method in Topos theory and in Homotopy Type theory

Autumn school Proof and Computation, 23rd to 26th September 2017

## Elementary Theory of Category of Sets (ETCS)

Lawvere 1964
The Idea (back to von Neumann in late 1920-ies): use functions and their composition instead of sets and the primitive membership relation $\in$ used in the ZF and its likes
Remark: The project as it stands is fully compatible with RAM; the proposed deviation from the standard approach amounts to a new choice of primitives.

# ETCS 1: ETAC

Elementary Theory of Abstract Categories (Eilenberg - MacLane)

- E1) $\Delta_i(\Delta_j(x)) = \Delta_j(x); i, j = 0, 1$
- E2) $(\Gamma(x, y; u) \wedge \Gamma(x, y; u')) \Rightarrow u = u'$
- E3) $\exists u \Gamma(x, y; u) \Leftrightarrow \Delta_1(x) = \Delta_0(y)$
- E4) $\Gamma(x, y; u) \Rightarrow (\Delta_0(u) = \Delta_0(x)) \wedge (\Delta_1(u) = \Delta_1(y))$
- E5) $\Gamma(\Delta_0(x), x; x) \wedge \Gamma(x, \Delta_1(x); x)$
- E6) $(\Gamma(x, y; u) \wedge \Gamma(y, z; w) \wedge \Gamma(x, w; f) \wedge \Gamma(u, z; g)) \Rightarrow f = g$

E1)-E4): bookkeeping (syntax); 5): identity; 6): associativity

# ETCS 2: Elementary Topos (anachronistically):

- finite limits;
- Cartesian closed (CCC): terminal object (1), binary products, exponentials;
- subobject classifier

$$\begin{tikzcd} U \arrow[r, "!"] \arrow[d, hook, "p"'] & 1 \arrow[d, "true"] \\ X \arrow[r, "\chi U"'] & 2 \end{tikzcd}$$

for all $p$ there exists a unique $\chi U$ that makes the square into a pullback

# ETCS 3: well-pointedness

for all $f, g : A \rightarrow B$, if for all $x : 1 \rightarrow A$ $xf = xg = y$ then $f = g$

$$
\begin{tikzcd}
 & 1 \arrow[dl, "x"'] \arrow[dr, "y"] & \\
A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B
\end{tikzcd}
$$

# ETCS 4: NNO

Natural Numbers Object: for all $t', f$ there exists unique $u$

$$
\begin{tikzcd}
1 \arrow[r, "t"] \arrow[dr, "t'"'] & \mathbb{N} \arrow[r, "s"] \arrow[d, "u"'] & \mathbb{N} \arrow[d, "u"] \\
 & A \arrow[r, "f"'] & A
\end{tikzcd}
$$

# ETCS 4: Axiom of Choice

Every epimorphism splits:
If $f : A \to B$ is epi then there exists mono $g : B \to A$ (called *section*) such that $gf = 1_B$

# The idea of internal logic: CCC

- Lawvere 1969: CCC is a common structure shared by (1) the simply typed $\lambda$-calculus (Schönfinkel, Curry, Church) and (2) Hilbert-style (and Natural Deduction style) Deductive Systems (aka Proof Systems).
- In other words CCC is "the" structure captured by the *Curry-Howard correspondence* or *Curry-Howard isomorphism*
- The CCC structure is *internal* for *Set* BUT is more general: *Cat* (of all *small* categories) is another example; any *topos* is CCC.
- Lawvere's Hegelian understanding of this issue: CCC is *objective* while usual syntactic presentations or logical calculi are only *subjective*. While syntactic presentations lay out only *formal* foundations, CCC lays out a *conceptual* foundation.

# Internalization of quantifiers: Hypodoctrines

Suppose that we have a one-place predicate (a property) $P$, which is meaningful on set $Y$, so that there is a subset $P_Y$ of $Y$ (in symbols $P_Y \subseteq Y$) such that for all $y \in Y$ $P(y)$ is true just in case $y \in P_Y$.
Define a new predicate $R$ on $X$ as follows: we say that for all $x \in X$ $R(x)$ is true when $f(x) \in P_Y$ and false otherwise. So we get subset $R_X \subseteq X$ such that for all $x \in X$ $R(x)$ is true just in case $x \in R_X$.
Let us assume in addition that every subset $P_Y$ of $Y$ is determined by some predicate $P$ meaningful on $Y$. Then given morphism $f$ from "universe" $X$ to "universe" $Y$ we get a way to associate with every subset $P_Y$ (every part of universe $Y$) a subset $R_X$ and, correspondingly, a way to associate with every predicate $P$ meaningful on $Y$ a certain predicate $R$ meaningful on $X$.

# Internalization of quantifiers: Substitution functor

Since subsets of given set $Y$ form Boolean algebra $B(Y)$ we get a map between Boolean algebras:

$$f^* : B(Y) \longrightarrow B(X)$$

Since Boolean algebras themselves are categories $f^*$ is a functor. For every proposition of form $P(y)$ where $y \in Y$ functor $f^*$ takes some $x \in X$ such that $y = f(x)$ and produces a new proposition $P(f(x)) = R(x)$. Since it replaces $y$ in $P(y)$ by $f(x) = y$ it is appropriate to call $f^*$ *substitution* functor.

# Existential Quantifier as adjoint

The *left* adjoint to the substitution functor $f^*$ is functor

$$\exists_f : B(X) \longrightarrow B(Y)$$

which sends every $R \in B(X)$ (i.e. every subset of $X$) into $P \in B(Y)$ (subset of $Y$) consisting of elements $y \in Y$, such that *there exists* some $x \in R$ such that $y = f(x)$; in (some more) symbols

$$\exists_f(R) = \{y | \exists x(y = f(x) \wedge x \in R)\}$$

In other words $\exists_f$ sends $R$ into its *image* $P$ under $f$. One can describe $\exists_f$ by saying that it transformes $R(x)$ into $P(y) = \exists_f x P'(x, y)$ and interpret $\exists_f$ as the usual existential quantifier.

# Universal Quantifier as adjoint

The *right* adjoint to the substitution functor $f^*$ is functor

$$\forall_f : \ B(X) \longrightarrow B(Y)$$

which sends every subset $R$ of $X$ into subset $P$ of $Y$ defined as follows:

$$\forall_f(R) = \{y|\forall x(y = f(x) \Rightarrow x \in R)\}$$

and thus transforms $R(X)$ into $P(y) = \forall_f x P'(x, y)$.

# Prehistory of Internal Logic

- Boole 1847, Venn 1882: propositional logic as algebra and mereology of (sub) classes (of a given universe of discourse); logical diagrams
- Tarski 1938 topological interpretation of Classical and Intuitionistic propositional logic

While in Boole, Venn and Tarski an internal treatment is given only the propositional logic Lawvere develops a similar approach to the 1st-order logic. There is a significant technical and conceptual advance between the two cases.

# Quantifiers and Sheaves 1970

"The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. At the same time, in the present joint work with Myles Tierney there are important influences in the other direction: a Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for "the" category $\underline{S}$ of abstract sets to an arbitrary topos.

# Quantifiers and Sheaves 1970

We first sum up the principle contradictions of the Grothendieck-Giraud-Verdier theory of topos in terms of four or five adjoint functors [..] enabling one to claim that in a sense logic is a special case of geometry.

# Internal Logic (more formally)

- Syntax (Mitchell-Bénabou): A sorted language $L$ with lists of variables of every sort; sorts correspond to objects of the given topos; logical operations are compatible with usual operations with topos objects (produce, exponentiation, etc.)
- (External) Semantics (Kripke-Joyal): a formal satisfaction relation.

# External vs. Internal View

Def. $f : A \rightarrow B$ is *epic* iff for all $g, h$ $\qquad gf = hf$ implies $g = h$.
Def. Object $T$ is *terminal* if for all object $X$ there is unique arrow $X \rightarrow T$
Def. Arrow of the form $e : T \rightarrow A$ is called an *element of* $A$: $e =\in_T A$

Fact: $f : A \rightarrow B$ is epic iff it is internally onto: $y.B \vdash (\exists x.A) y = fx$

Warning: $y.B \vdash (\exists x.A) y = fx$ does not say that for each $y \in_T B$ there exists some $x \in_T A$ such that $y = fx$. Externally, $f$ is epic but not necessarily *split* epic.

Notice that $L$ has a double semantics informally described by Lawvere in the above quote: it is both logical and geometrical. The same quote makes it evident that the internal logic of topos served Lawvere as a key for his axiomatization of Grothendieck's topos concept.

# MLTT: Syntax

- 4 basic forms of judgement:
  (i) $A : TYPE$;
  (ii) $A \equiv_{TYPE} B$;
  (iii) $a : A$;
  (iv) $a \equiv_A a'$
- Context : $\Gamma \vdash$ judgement (of one of the above forms)
- no axioms (!)
- rules for contextual judgements; Ex.: dependent product :
  If $\Gamma, x : X \vdash A(x) : TYPE$, then $\Gamma \vdash (\Pi x : X)A(x) : TYPE$

# MLTT: Semantics of $t : T$ (Martin-Löf 1983)

- $t$ is an element of set $T$
- $t$ is a proof (construction) of proposition $T$ ("propositions-as-types")
- $t$ is a method of fulfilling (realizing) the intention (expectation) $T$
- $t$ is a method of solving the problem (doing the task) $T$ (BHK-style semantics)

# Sets and Propositions Are the Same

If we take seriously the idea that a proposition is defined by lying down how its canonical proofs are formed [. . . ] and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to an unnecessary duplication to keep the notions of proposition and set [. . . ] apart. Instead we simply identify them, that is, treat them as one and the same notion. (Martin-Löf 1983)

# MLTT: Definitional aka judgmental equality/identity

$x, y : A$ (in words: $x, y$ are of type $A$)

$x \equiv_A y$ (in words: $x$ is $y$ by definition)

# MLTT: Propositional equality/identity

$p : x =_A y$ (in words: $x, y$ are (propositionally) equal as this is evidenced by proof $p$)

# Definitional eq. entails Propositional eq.

$$\frac{x \equiv_A y}{p : x =_A y}$$

where $p \equiv_{x=_A y} refl_x$ is built canonically

# Equality Reflection Rule (ER)

$$\frac{p : x =_A y}{x \equiv_A y}$$

ER is not a theorem in the (intensional) MLTT (Streicher 1993).

# Extension and Intension in MLTT

- MLTT + ER is called *extensional* MLTT
- MLTT w/out ER is called *intensional*
  (notice that according to this definition intensionality is a negative property!)

## Higher Identity Types

- $x', y' : x =_A y$
- $x'', y'' : x' =_{x =_A y} y'$
- $\ldots$

# HoTT: the Idea

Types in MLTT are (informally!) modeled by spaces (up to homotopy equivalence) in Homotopy theory, or equivalently, by higher-dimensional groupoids in Category theory (in which case one thinks of $n$-groupoids as higher homotopy groupoids of an appropriate topological space).

# Homotopical interpretation of Intensional MLTT

- $x, y : A$
  $x, y$ are points in space $A$
- $x', y' : x =_A y$
  $x', y'$ are paths between points $x, y$; $x =_A y$ is the space of all such paths
- $x'', y'' : x' =_{x =_A y} y'$
  $x'', y''$ are homotopies between paths $x', y'$; $x' =_{x =_A y} y'$ is the space of all such homotopies
- ...

# Point

**Definition**

*Space $S$ is called contractible or space of h-level (-2) when there is point $p : S$ connected by a path with each point $x : A$ in such a way that all these paths are homotopic (i.e., there exists a homotopy between any two such paths).*

# Homotopy Levels

## Definition

*We say that $S$ is a space of h-level $n+1$ if for all its points $x, y$ path spaces $x =_S y$ are of h-level $n$.*

# Cummulative Hierarchy of Homotopy Types

- -2-type: single point *pt*;
- -1-type: the empty space $\emptyset$ and the point *pt*: truth-values aka (mere) propositions
- 0-type: sets: points in space with no (non-trivial) paths
- 1-type: flat groupoids: points and paths in space with no (non-trivial) homotopies
- 2-type: 2-groupoids: points and paths and homotopies of paths in space with no (non-trivial) 2-homotopies
- ...

Propositions-as-**Some**-Types !

## Which types are propositions?

Def.: Type $P$ is a *mere proposition* if $x, y : P$ implies $x = y$ (definitionally).

# Truncation

Each type is transformed into a (mere) proposition when one ceases to distinguish between its terms, i.e., *truncates* its higher-order homotopical structure.

Interpretation: Truncation reduces the higher-order structure to a single element, which is **truth-value**: for any non-empty type this value is **true** and for an empty type it is **false**.
The reduced structure is the structure of **proofs** of the corresponding proposition.
To treat a type as a proposition is to ask whether or not this type is instantiated without asking for more.

- Thus in HoTT "merely logical" rules (i.e. rules for handling propositions) are instances of more general formal rules, which equally apply to non-propositional types.
- These general rules work as rules of building models of the given theory from certain basic elements which interpret primitive terms (= basic types) of this given theory.
- Thus HoTT qualify as *constructive* theory in the sense that besides of propositions it comprises non-propositional objects (on equal footing with propositions rather than "packed into" propositions as usual!) and formal rules for managing such objects (in particular, for constructing new objects from given ones). In fact, HoTT comprises rules with apply *both* to propositional and non-propositional types.

# Syntactic and Semantic (aka Non-Statement) Views on Theories

Syntactic View: A direct Hilbert-style axiomatization of Physical and other scientific theories (since 1900: Hilbert, Rudolf Carnap, Carl Gustav "Peter" Hempel and Ernest Nagel)

Semantic View: A typical scientific theory should be identified with a class with (set-theoretic) models rather than with a particular axiomatic presentation in a formal language (since late 1950-ies: Evert Beth, Patrick Suppes, Bas van Fraassen)

# Problem:

None of the above two approaches support an adequate representation of scientific *methods* including methods of justification of scientific claims. This concerns both logical and (particularly) extra-logical methods such as methods of conducting observations and staging experiments.

# Cassirer 1907 contra Russell 1903)

Die Sorge um die Gezetzlichkeit der Welt der Objekte dagegen bleibt gänzlich der direkten Beobachtung überlassen, die allein uns innerhalb ihrer eigenen, sehr eug gestecken Grenzen zu lehren vermag, ob auch hier bestimmte Regelmássigkeiten sich finden, oder aber ein reines Chaos herrscht. Logik and Mathematik haben es nur mit Ordung der Begriffe zu thun; die Ordnung oder Verwirrung unter den Gegenstánden ficht sie nicht an und braucht sie nicht zu beirren. So beliebt, wieweit man auf diesem Standpunkt die Analyse der Begriffe auch treiben mag, das empirische Sein ein ewig unbegriffenes Problem. Je deutlicher der Wert und die Kraft der Deduktion im Gebiete der Mathematik sich vor uns offenbart, um so weniger verstehen wir die gewaltige und entscheidende Bedeutung, die der Deduktion im Gebiet der theoretischen Naturwissenschaft zufält.

# Cassirer 1907 contra Russell 1903)

From the standpoint of logistics [= formal mathematics] the task of thought ends when it manages to establish a strict deductive link between all its constructions and productions. Thus the worry about laws governing the world of objects is left wholly to the direct observation, which alone, within its proper very narrow limits, is supposed to tell us whether we find here certain rules or a pure chaos. [According to Russell] logic and mathematics deal only with the order of concepts and should not care about the order or disorder of objects. As long as one follows this line of conceptual analysis the empirical entity always escapes one's rational understanding. The more mathematical deduction demonstrates us its virtue and its power, the less we can understand the crucial role of deduction in the theoretical natural sciences.

# Non-Propositional Content in Science

HoTT supports a strong version of Non-Sentence View of Theories by providing a precise sense in which a theory, generally, does not reduce to the set of its propositions.

HoTT also supports a *Constructive View* of theories according to which the non-propositional *Knowldge How* is a part of scientific and technical knowledge, which is at least as much important as the propositional *Knowledge That*). HoTT provides a model of how the two sorts of knowledge relate to each other.

# Mathematical Modeling in Science

[E]xperience with sheaves, [..], etc., shows that a "set theory" for geometry should apply not only to *abstract* sets divorced from time, space, ring of definition, etc., but also to more general sets which do in fact develop along such parameters. (Lawvere 1970 inspired by Hegel)

Logical and mathematical concepts must no longer produce instruments for building a metaphysical "world of thought": their proper function and their proper application is only within the empirical science. (Cassirer 1907)

# Suppes' Lesson

A formal representational framework for Science and Technology should include a formal semantic part rather than apply syntactic structures to material contents directly.

Suppes and his followers use Set theory for that purpose with a relatively little success — at least if this success is measured by the role of formal approaches in the mainstream scientific research. The homotopical semantics can be more appropriate of the task.

# Open Problem

It appears that we still miss a good replacement of Tarski's notion of model, which could work with HoTT and CAM more generally. Tarski's notion of satisfaction in its original does not make the whole job in such a context because it involves the concept of truth-evaluation and no alternative notion of model is universally accepted.

The Model theory of HoTT is presently a subject of active research. This research revises basic conceptual issues such as the concept of model itself.

# Conclusion 1

The constructive axiomatic architecture is rooted in history (Euclid) as well as in the recent successful practice of axiomatizing geometrical theories (ET, HoTT).

# Conclusion 2

As the examples of ET and HoTT clearly demonstrate CAM involves a pattern of relationships between Logic and Geometry, which is quite unlike the corresponding pattern used in RAM. RAM-based axiomatic architecture leaves no room for a conceptual linking of geometrical principles to logical ones. Geometrical axioms appear here as very specific formal principles put on the top of logical principles and motivated solely by unspecified references to spatial experiences and intuitions. The CAM-based axiomatic architecture, in its turn, presents geometrical principles as a generalization of logical principles: in a CAM-based geometrical theory such as HoTT "logic is a special case of geometry".

# Conclusion 3

RAM proved effective as a very specific representational tool for meta-mathematical studies. But it appeared to be nearly useless for more general epistemic purposes, for which this method was originally designed or tentatively applied later. This includes the formal proof-checking, developing formal standards in scientific communication and education, developing a software for computer-based Knowledge Representation. Today's science and mathematics applies little of RAM-based methods and of logical methods more generally. Even in CS and software engineering the role of logical approaches appears rather modest.

# Conclusion 3 (continued)

CAM already has a better performance and a better record in this respect. Its traditional informal version proved effective in mathematics (Euclid) and physics (Newton, Clausius). Today's proof-assistances such as COQ are CAM-based rather than RAM-based. There are reasons to expect that CAM-based logical methods (and perhaps HoTT more specifically) will apply in today's science and technology (including IT) more effectively that the standard RAM-based methods. In any event it is worth trying.

THANK YOU