Objecthood and Genetic Axiomatic Method in Categorical Mathematics

Andrei Rodin

24 апреля 2014 г.

Folk History

The modern era of mathematics begins at the turn of the 19th and the 20th century when mathematics cuts its traditional links with naive spatial and temporal intuitions and Non-Euclidean geometries and the Modern Abstract Algebra gain their rights in the family of mathematical disciplines.

Ancients

Definitions:

1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points.
4. A straight-line is whatever lies evenly with points

Moderns: 1899

Let us consider three distinct systems of things. The things composing the first system, we will call points and designate them by the letters A, B, C,. . . ; those of the second, we will call straight lines and designate them by the letters a, b, c,..; and those of the third system, we will call planes and designate them by the Greek letters α, β, γ . [..]

Folk Philosophy

"The formalist viewpoint just stated is a radical departure from the older notion that mathematics asserts "absolute truths", a notion that was destroyed once and for all by the discovery of Non-Euclidean geometry. This discovery has had a liberating effect on mathematics, who now feel free to invent any set of axioms they wish and deduce conclusions from them. In fact this freedom may account for the great increase in the scope and generality of modern mathematics." (Greenberg, Euclidean and Non-Euclidean Geometries, 1974)

Claim 1

Recent proposals in foundations of mathematics including

- Categorical and (more specifically) Topos-theoretic foundations (Lawvere) and
- Univalent Foundations (Voevodsky)

bring mathematics back to Euclid (in some important respect, which I specify in what follows).

Claim 2

This development helps to bridge the existing gap between pure and applied mathematics. It also helps to provide an answer to Wigner's question and make mathematics more effective in natural sciences. My aim is not only to describe but also push this development.

Modern (Formal Logical) Account

"I will fix the way I wish to use the term "object" and simultaneously say what I think useful in such abstract discussions [about objects in general] by saying that the usable general characterization of the notion of object comes from logic. We speak of particular objects by referring to them by singular terms [..]." (Ch. Parsons, Mathematical Thought and its Objects, 2008)

A Critique of Modern Account

"Here rises a problem that lies wholly outside the scope of "logistics" [= Formal Symbolic Logic]. All empirical judgements [..] must respect the limits of experience. What logistics develops is a system of hypothetical assumptions about which we cannot know, whether they are actually established in experience or whether they allow for some immediate or non-immediate concrete application. According to Russell even the general notion of magnitude does not belong to the domain of pure mathematics and logic but has an empirical element, which can be grasped only through a sensual perception. From the standpoint of logistics the task of thought ends when it manages to establish a strict deductive link between all its constructions and productions.

A Critique of Modern Account

Thus the worry about laws governing the world of objects is left wholly to the direct observation, which alone, within its proper very narrow limits, is supposed to tell us whether we find here certain rules or a pure chaos. [According to Russell] logic and mathematics deal only with the order of concepts and should not care about the order or disorder of objects. As long as one follows this line of conceptual analysis the empirical entity always escapes one's rational understanding. The more mathematical deduction demonstrates us its virtue and its power, the less we can understand the crucial role of deduction in the theoretical natural sciences. " (Cassirer 1907)

Remark

The Modern account of objecthood in mathematics makes the effectiveness of mathematics in natural sciences "unreasonable" (Wigner) - and for this very reason possibly less effective. Wigner's problem rises as a byproduct of the 20th century formalization of mathematics.

Ancient Account

"Give a philosopher the concept of triangle and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on his concept as long as he wants, yet he will never produce anything new. He can analyze and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts.

Ancient Account

But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle and obtains two adjacent angles that together are equal to the two right ones. [..] In such a way through a chain of inferences that is always guided by intuition, he arrives at a fully illuminated and at the same time general solution of the question." (Kant, Critique of Pure Reason, A 716 / B 744)

Euclid's Postulates 1-3

P1. Let it have been postulated to draw a straight-line from any point to any point.
P2. And to produce a finite straight-line continuously in a straight-line.
P3. And to draw a circle with any center and radius.

Remark

P1-3 are NOT propositional!

The Physical Value of Postulates 1-3 in Astronomy

P1-P2: light rays (= visual palps)
P3: (partly visible) motions of celestial bodies

Euclid's Common Notions (Axioms)

A1. Things equal to the same thing are also equal to one another.
A2. And if equal things are added to equal things then the wholes are equal.
A3. And if equal things are subtracted from equal things then the remainders are equal.
A4. And things coinciding with one another are equal to one another.
A5. And the whole [is] greater than the part.

Shared Structure of Problems and Theorems: Proof by Construction

"Every Problem and every Theorem that is furnished with all its parts should contain the following elements:

- an enunciation
- an exposition
- a specification
- a construction [regulated by Postulates]
- a proof [based on Definitions, Hypotheses and Axioms]
- and a conclusion.

(Proclus, Commentary on Euclid, circa 450 A.D.)

Friedman on Kant on Euclid

Euclidean geometry [..] is not to be compared with Hilbert's axiomatization [of Euclidean geometry], say, but rather with Frege's *Begriffsschrift*. It is not a substantive doctrine, but a form of rational representation: a form of rational argument and inference.

Generation of Magnitudes in the 17th c. geometry

I am now entering into a new Field, whether more pleasant or fruitful, I cannot truly say, but yielding a most copious Variety which consequently is agreeable; and as it comprehends, for the most Part, the Original of Mathematical Hypotheses, from whence Definitions are formed and Properties flow, it must Necessarily be very useful too. What I mean is the Generation of Magnitudes, or the several Ways whereby the various Species of Magnitudes may be conceived to be generated or produced. Nor indeed is there any Magnitude given, but what may be conceived to be produced, and really is produced innumerable Ways; yet there may be brought under some general Heads [..] Among these Ways, or any other whatever, of generating Magnitudes, the Primary and Chief is that performed by local Motion. (Is. Barrow, Geometrical Lectures 1670, first lines of Lecture 1)

Ancient Objectivity (Kant)

Objectivity hangs on Objecthood: rules of object-building are based on (or at least correlated with) fundamental physical principles and shared by all thinkers.

Modern Objectivity (Frege and Aristotle)

Objectivity hangs on truth (including factual and logical truth).

Hilbert&Bernays 1934

The term axiomatic will be used partly in a broader and partly in a narrower sense.We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics.

Hilbert&Bernays 1934

[F]or axiomatics in the narrowest sense, the *existential form* comes in as an additional factor. This marks the difference between the *axiomatic method* and the *constructive* or *genetic* method of grounding a theory. While the constructive method introduces the objects of a theory [..], an axiomatic theory [in the narrow sense of "axiomatic"] refers to a fixed system of things (or several such systems) [i.e. to one or several models].[..] This is an idealizing assumption that properly augments [?] the assumptions formulated in the axioms.

Hilbert&Bernays 1934

When we now approach the task of such an impossibility proof [= proof of consistency], we have to be aware of the fact that we cannot again execute this proof with the method of axiomatic-existential inference. Rather, we may only apply modes of inference that are free from idealizing existence assumptions.

Hilbert&Bernays 1934

Yet, as a result of this deliberation, the following idea suggests itself right away: If we can conduct the impossibility proof without making any axiomatic-existential assumptions, should it then not be possible to provide a grounding for the whole of arithmetic directly in this way, whereby that impossibility proof would become entirely superfluous?

Hilbert's answer is in negative because of his worries about infinities in Set theory and elsewhere in mathematics.

Comment 1

Genetic object-building is not wholly suppressed in Formal Mathematics but

- limited to syntactic constructions
- isolated in a special area of Mathematics called *Meta*mathematics.

Comment 2

This “official” view poorly describes what mathematicians do in practice (cf. Group Theory). However just saying that in practice mathematicians work *informally* does not solve the problem!

Comment 3

The 20th c. showed no significant progress in the axiomatization of physics (Hilbert's 6th Problem). During this century FAM played no role at all in the mainstream research in physics and other natural sciences.

Motivation

The expression "Euclidean plane" is ambiguous.
In one sense it means a geometrical space studied in Planimetry where live circles, triangles, etc (EPLANE);
In a different sense it means an object living in the Euclidean 3-space (ESPACE)(eplane):

$$EPLANE \xrightarrow{eplane} ESPACE$$

Remarks:

Remarks:

- There are many different eplanes living in ESPACE;

Remarks:

- There are many different eplanes living in ESPACE;
- Circles, etc. in ESPACE factor through EPLANE:

$$\begin{tikzcd} CIRCLE \arrow[r, "circle_1"] \arrow[d, "circle_2"'] & EPLANE \arrow[dl, "eplane"] \\ ESPACE & \end{tikzcd}$$

Generalization

General situation:

$$TYPE \xrightarrow{object} SPACE$$

Remarks:

Being a type and being a space are relational properties. Being an object is non-relational property.

Each object is of particular type and lives in a particular space.

Non-Euclidean examples:

$$HPLANE \xrightarrow{pseudosphere} ESPACE$$

(Beltramy)

$$EPLANE \xrightarrow{horisphere} HSPACE$$

(Lobachevsky)
<u>Remark</u>: Pseudosphere and horisphere are not types/spaces but objects (without ambiguity).

Observation

Objects of the same type look differently in different spaces:

$$\begin{tikzcd}
 & HSPACE \\
EPLANE \arrow[ur, "horisphere"] \arrow[dr, "eplane"'] & \\
 & ESPACE
\end{tikzcd}$$

Objects of different types in the same space look always differently.

Curry-Howard: Simply typed lambda calculus

Variable: $\dfrac{}{\Gamma, x : T \vdash x : T}$

Product: $\dfrac{\Gamma \vdash t : T \quad \Gamma \vdash u : U}{\Gamma \vdash \langle t, u \rangle : T \times U}$

$\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_1 v : T}$ $\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_2 v : U}$

Function: $\dfrac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x.t : U \to T}$

$\dfrac{\Gamma \vdash t : U \to T \quad \Gamma \vdash u : U}{\Gamma \vdash tu : T}$

Curry-Howard: Natural deduction

Identity: $\overline{\Gamma, A \vdash A}$ (Id)

Conjunction: $\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A\&B}$ (& - intro)

$\dfrac{\Gamma \vdash A\&B}{\Gamma \vdash A}$ (& - elim1); $\dfrac{\Gamma \vdash A\&B}{\Gamma \vdash B}$ (& - elim2)

Implication: $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}$ ($\supset$-intro)

$\dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$ ($\supset$-elim aka *modus ponens*)

Curry-Howard Isomorphism

$$\& \equiv \times$$

$$\supset \equiv \rightarrow$$

Brouwer-Heyting-Kolmogorov (BHK interpretation)

- proof of $A \supset B$ is a procedure that transforms each proof of A into a proof of B;
- proof of $A\&B$ is a pair consisting of a proof of A and a proof of B

Historical remark

Curry-Howard relates mathematical (λ-calculus) and meta-mathematical (natural deduction) concepts.

Historical remark

Foundational consideration played a crucial role in this story from the outset (Schönfinkel, Curry, Church, Kolmogorov, Lawvere, Lambek). The expression “Curry-Howard isomorphism”, which suggests that we have here an unexplained/surprising formal coincidence, is due to Howard 1969. The *true* history (and the true meaning) still waits to be explored.

Lawvere and Lambek 1969

The structure behind the Curry-Howard isomorphism is precisely captured by the notion of *Cartesian closed category* (CCC), which is an (abstract) category with the terminal object, products and exponentials.

Examples: Sets, Boolean algebras

Simply typed lambda-calculus / natural deduction is the *internal language* of CCC.

- Objects: types / propositions
- Morphisms: terms / proofs

Lawvere's philosophical motivation

- objective invariant structures vs. its subjective syntactical presentations
- objective logic vs. subjective logic (Hegel)

From Moders to Ancients

The concept of CCC was discovered by Lawvere when he tried to axiomatize Set theory as a (first-order) theory of the category of sets (replacing $\in$ in its role of non-logical primitive by functions: ETCS.) This discovery marks Lawvere's shift from Hilbert to Euclid: instead of "using" the external (classical) FOL he now aims at building FOL internally as a part of his target axiomatic theory!

Higher-order generalization: Hyperdoctrines (Lawvere)

- Quantifiers as adjoints to substitution; hyperdoctrines (1969)
- Toposes (1970)
- *Locally* Cartesian closed categories (LCCC) (1972)

Lawvere on logic and geometry

The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. At the same time, in the present joint work with Myles Tierney there are important influences in the other direction: a Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for "the" category $\underline{S}$ of abstract sets to an arbitrary topos .

Lawvere on logic and geometry (continued)

We first sum up the principle contradictions of the Grothendieck-Giraud-Verdier theory of topos in terms of four or five adjoint functors [..] enabling one to claim that in a sense logic is a special case of geometry. (Lawvere 1970)

Lawvere's axioms for topos

(Elementary) topos is a category which

Lawvere's axioms for topos

(Elementary) topos is a category which

- has finite limits

Lawvere's axioms for topos

(Elementary) topos is a category which

- has finite limits
- is CCC

Lawvere's axioms for topos

(Elementary) topos is a category which

- has finite limits
- is CCC
- has a subobject classifier

Constructive Proof theory

"[P]roof and knowledge are the same. Thus, if proof theory is construed not in Hilbert's sense, as metamathematics, but simply as a study of proofs in the original sense of the word, then proof theory as the same as theory of knowledge, which, in turn, is the same as logic in the original sense of the word, as the study of reasoning, or proof, not as metamathematics." (Martin-Löf 1983)

MLTT (Martin-Löf 1980): key features

MLTT (Martin-Löf 1980): key features

- double interpretation of types: "sets" and propositions

MLTT (Martin-Löf 1980): key features

- double interpretation of types: "sets" and propositions
- double interpretation of terms: elements of sets and proofs of propositions

MLTT (Martin-Löf 1980): key features

- double interpretation of types: “sets” and propositions
- double interpretation of terms: elements of sets and proofs of propositions
- higher orders: dependent types (sums and products of families of sets)

MLTT (Martin-Löf 1980): key features

- double interpretation of types: “sets” and propositions
- double interpretation of terms: elements of sets and proofs of propositions
- higher orders: dependent types (sums and products of families of sets)
- MLTT is the internal language of LCCC (Seely 1983)

MLTT: two identities

MLTT: two identities

- Definitional identity of terms (of the same type) and of types: $x = y : A$; $A = B : type$ (substitutivity)

MLTT: two identities

- Definitional identity of terms (of the same type) and of types: $x = y : A$; $A = B : type$ (substitutivity)
- Propositional identity of terms x, y of (definitionally) the same type A:
  $Id_A(x, y) : type$;
  Remark: propositional identity is a (dependent) type on its own.

MLTT: Higher Identity Types

- $x', y' : Id_A(x, y)$
- $Id_{Id_A}(x', y') : type$
- and so on

Idea:

The central new idea in homotopy type theory is that types can be regarded as spaces in homotopy theory, or higher-dimensional groupoids in category theory. (HoHT Book 2013)

Fundamental group

Fundamental group G^0_T of a topological space T:

- a base point P;
- loops through P (loops are circular paths $l : I \to T$);
- composition of the loops (up to homotopy only! - see below);
- identification of homotopic loops;
- independence of the choice of the base point.

Fundamental (1-) groupoid

G^1_T:

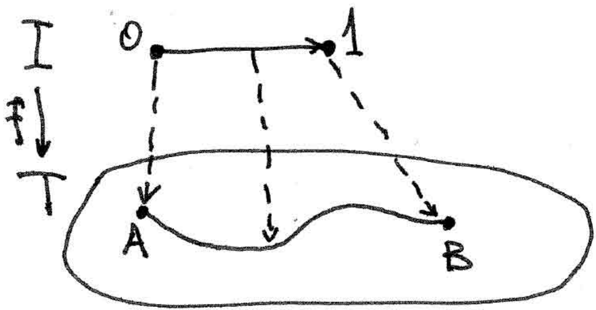
- all points of T (no arbitrary choice);
- paths between the points (embeddings $s : I \rightarrow T$);
- composition of the *consecutive* paths (up to homotopy only! - see below);
- identification of homotopic paths;

Since not all paths are consecutive G^1_T contains more information about T than G^0_T!

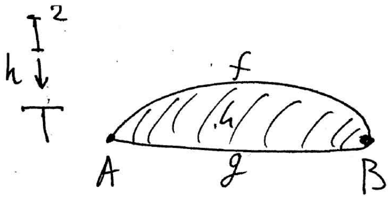
Path Homotopy and Higher Homotopies

$s : I \rightarrow T$, $p : I \rightarrow T$ where $I = [0, 1]$: paths in T
$h : I \times I \rightarrow T$: homotopy of paths s, t if $h(0 \times I) = s$, $h(1 \times I) = t$
$h^n : I \times I^{n-1} \rightarrow T$: n-homotopy of $n-1$-homotopies h_0^{n-1}, h_1^{n-1} if
$h^n(0 \times I^{n-1}) = h_0^{n-1}$, $h^n(1 \times I^{n-1}) = h_1^{n-1}$;
<u>Remark:</u> Paths are zero-homotopies

Path Homotopy and Higher Homotopies



Homotopy categorically and Categories homotopically



$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

Higher Groupoids and Omega-Groupoids (Grothendieck 1983)

- all points of T (no arbitrary choice);
- paths between the points ;
- homotopies of paths
- homotopies of homotopies (2-homotopies)
- higher homotopies up to n-homotopies
- higher homotopies ad infinitum

G^n_T contains more information about T than G^{n-1}_T!

Composition of Paths

Concatenation of paths produces a map of the form $2I \longrightarrow T$ but not of the form $I \longrightarrow T$, i.e., not a path. We have the whole space of paths $I \longrightarrow 2I$ to play with! But all those paths are homotopical. Similarly for higher homotopies (but beware that n-homotopies are composed in n different ways!)
On each level when we say that $a \oplus b = c$ the sign $=$ hides an infinite-dimensional topological structure!

Grothendieck Conjecture:

G_T^ω contains all relevant information about T; an omega-groupoid is a complete algebraic presentation of a topological space.

Homotopy Type theory

- Groupoid model of MLTT: basic types are groupoids, terms are their elements, dependent types are fibrations of groupoids (families of groupoids indexed by groupoids - rather than families of sets indexed by sets). Extensionality one dimension up. (Streicher 1993).
- Higher (homotopical) groupoids model higher identity types. Intensionality all way up (Voevodsky circa 2008).

Voevodsky on Univalent Foundations

Whilst it is possible to encode all of mathematics into Zermelo-Fraenkel set theory, the manner in which this is done is frequently ugly; worse, when one does so, there remain many statements of ZF which are mathematically meaningless. [..]

Voevodsky on Univalent Foundations (continued)

Univalent foundations seeks to improve on this situation by providing a system, based on Martin-Löf's dependent type theory whose *syntax is tightly wedded to the intended semantical interpretation* in the world of everyday mathematics. In particular, it allows the direct formalization of the world of homotopy types; indeed, these are the basic entities dealt with by the system. (Voevodsky 2011)

h-levels

- (i) Given space is called A *contractible* (aka space of h-level 0) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- (ii) We say that A is a space of h-level $n+1$ if for all its points x, y path spaces $paths_A(x, y)$ are of h-level n.

h-universe

h-universe

- Level 0: up to homotopy equivalence there is just one contractible space that we call "point" and denote pt;

h-universe

- Level 0: up to homotopy equivalence there is just one contractible space that we call "point" and denote pt;
- Level 1: up to homotopy equivalence there are two spaces here: the empty space $\emptyset$ and the point pt. (For $\emptyset$ condition (ii) is satisfied vacuously; for pt (ii) is satisfied because in pt there exists only one path, which consists of this very point.) We call $\emptyset, pt$ *truth values*; we also refer to types of this level as *properties* and *propositions*. Notice that h-level n corresponds to the logical level $n-1$: the propositional logic (i.e., the propositional segment of our type theory) lives at h-level 1.

h-universe

h-universe

- Level 2: Types of this level are characterized by the following property: their path spaces are either empty or contractible. So such types are disjoint unions of contractible components (points), or in other words *sets* of points. This will be our working notion of set available in this framework.

h-universe

- Level 2: Types of this level are characterized by the following property: their path spaces are either empty or contractible. So such types are disjoint unions of contractible components (points), or in other words *sets* of points. This will be our working notion of set available in this framework.
- Level 3: Types of this level are characterized by the following property: their path spaces are sets (up to homotopy equivalence). These are obviously (ordinary flat) *groupoids* (with path spaces hom-sets).

h-universe

- Level 2: Types of this level are characterized by the following property: their path spaces are either empty or contractible. So such types are disjoint unions of contractible components (points), or in other words *sets* of points. This will be our working notion of set available in this framework.
- Level 3: Types of this level are characterized by the following property: their path spaces are sets (up to homotopy equivalence). These are obviously (ordinary flat) *groupoids* (with path spaces hom-sets).
- Level 4: 2-groupoids

h-universe

- ..
- Level n+2: n-groupoids
- ..
- ω-groupoids
- ω-groupoids $(\omega + 1 = \omega)$

How it works

Let $isc\text{ontr}(A)$ and $isaprop(A)$ be formally constructed types " A is contractible" and "A is a proposition" (for formal definitions see Voevodsky:2011, p. 8). Then one formally deduces (= further constructs according to the same general rules) types $isaprop(iscontr(A))$ and $isaprop(isaprop(A))$, which are non-empty and thus "hold true" for each type A; informally these latter types tell us that for all A "A is contractible" is a proposition and "A is a proposition" is again a proposition.

How it works

With the same technique one defines in this setting type $weq(A, B)$ of *weak equivalences* (i.e., homotopy equivalences) of given types A, B (as a type of maps $e : A \rightarrow B$ of appropriate sort) and formally proves its expected properties. These formal proves involve a *different* type $isweq(A, B)$ of h-level 2, which is a proposition saying that A, B are homotopy equivalent, i.e., that type $weq(A, B)$ is inhabited.)

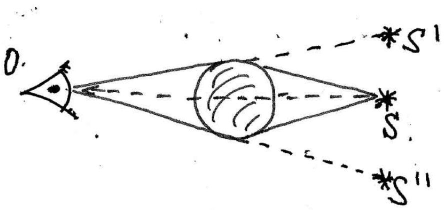
Axiom of Univalence

Homotopically equivalent types are (propositionally) identical. This means that the universe *TYPE* of homotopy types is construed like a homotopy type (and also modeled by ω-groupoid).
Axiom of Univalence is the <u>only</u> axiom of Univalent Foundations on the top of MLTT.

Naive stuff

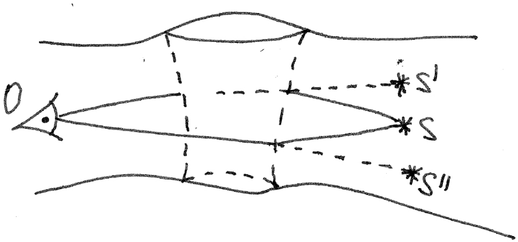
Identity through time

Naive stuff



Gravitational lensing

Naive stuff



Wormhole lensing

Serious stuff

Topos Physics:
A. Döring, Ch. Isham: 'What is a Thing?': Topos Theory in the Foundations of Physics (2008): http://arxiv.org/abs/0803.0417
Univalent Physics:
Urs Schreiber: Quantization via Linear homotopy types (Feb. 2014) http://arxiv.org/abs/1402.7041

Conclusion

The (really) Modern Axiomatic Method is the Good Old Genetic Axiomatic Method of Euclid, Newton and Clausius!

productFlyer_978-3-319-00403-7.pdf

http://arxiv.org/abs/1210.1478

THANK YOU!