Objecthood in Modern Logic

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Modern (Formal Logical) Account

"I will fix the way I wish to use the term "object" and simultaneously say what I think useful in such abstract discussions [about objects in general] by saying that the usable general characterization of the notion of object comes from logic. We speak of particular objects by referring to them by singular terms [..]." (Ch. Parsons, Mathematical Thought and its Objects, 2008)

A Critique of Modern Account

"Here rises a problem that lies wholly outside the scope of "logistics" [= Formal Symbolic Logic]. All empirical judgements [..] must respect the limits of experience. What logistics develops is a system of hypothetical assumptions about which we cannot know, whether they are actually established in experience or whether they allow for some immediate or non-immediate concrete application. According to Russell even the general notion of magnitude does not belong to the domain of pure mathematics and logic but has an empirical element, which can be grasped only through a sensual perception. From the standpoint of logistics the task of thought ends when it manages to establish a strict deductive link between all its constructions and productions.

A Critique of Modern Account

Thus the worry about laws governing the world of objects is left wholly to the direct observation, which alone, within its proper very narrow limits, is supposed to tell us whether we find here certain rules or a pure chaos. [According to Russell] logic and mathematics deal only with the order of concepts and should not care about the order or disorder of objects. As long as one follows this line of conceptual analysis the empirical entity always escapes one's rational understanding. The more mathematical deduction demonstrates us its virtue and its power, the less we can understand the crucial role of deduction in the theoretical natural sciences. " (Cassirer 1907)

Remark

The Modern treatment of objecthood as a trivial issue makes the effectiveness of applications of logic and mathematics in the real world (as experiences by humans) "unreasonable" (Wigner) - and for this very reason possibly less effective. Wigner's problem rises as a byproduct of the 20th century formalization of mathematics.

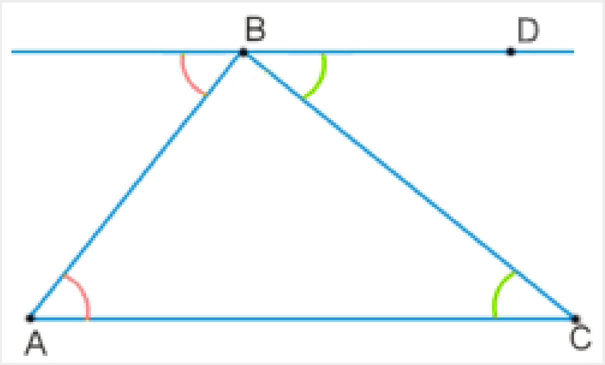
"Proof by Construction"

"Give a philosopher the concept of triangle and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on his concept as long as he wants, yet he will never produce anything new. He can analyze and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts.

"Proof by Construction"

But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle and obtains two adjacent angles that together are equal to the two right ones. [..] In such a way through a chain of inferences that is always guided by intuition, he arrives at a fully illuminated and at the same time general solution of the question." (Kant, Critique of Pure Reason, A 716 / B 744)

"Proof by Construction"



Euclid's Postulates 1-3

P1. Let it have been postulated to draw a straight-line from any point to any point.
P2. And to produce a finite straight-line continuously in a straight-line.
P3. And to draw a circle with any center and radius.

Remark

P1-3 are NOT propositions!

The Physical Value of Postulates 1-3 in Astronomy

P1-P2: light rays (= visual palps)
P3: (partly visible) motions of celestial bodies

Generation of Magnitudes in the 17th c. geometry

I am now entering into a new Field, whether more pleasant or fruitful, I cannot truly say, but yielding a most copious Variety which consequently is agreeable; and as it comprehends, for the most Part, the Original of Mathematical Hypotheses, from whence Definitions are formed and Properties flow, it must Necessarily be very useful too. What I mean is the Generation of Magnitudes, or the several Ways whereby the various Species of Magnitudes may be conceived to be generated or produced. Nor indeed is there any Magnitude given, but what may be conceived to be produced, and really is produced innumerable Ways; yet there may be brought under some general Heads [..] Among these Ways, or any other whatever, of generating Magnitudes, the Primary and Chief is that performed by local Motion. (Is. Barrow, Geometrical Lectures 1670, first lines of Lecture 1)

Ancient Objectivity (Kant)

Objectivity hangs on Objecthood: rules of object-building in logic and mathematics shared by all thinkers and correlated with physical principles at the fundamental level.

Modern Objectivity (Frege and Aristotle)

Objectivity hangs on truth (factual and logical).

Hilbert&Bernays 1934

The term axiomatic will be used partly in a broader and partly in a narrower sense.We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics.

Hilbert&Bernays 1934

[F]or axiomatics in the narrowest sense, the *existential form* comes in as an additional factor. This marks the difference between the *axiomatic method* and the *constructive* or *genetic* method of grounding a theory. While the constructive method introduces the objects of a theory [..], an axiomatic theory [in the narrow sense of "axiomatic"] refers to a fixed system of things (or several such systems) [i.e. to one or several models].[..] This is an idealizing assumption that properly augments [?] the assumptions formulated in the axioms.

Hilbert&Bernays 1934

When we now approach the task of such an impossibility proof [= proof of consistency], we have to be aware of the fact that we cannot again execute this proof with the method of axiomatic-existential inference. Rather, we may only apply modes of inference that are free from idealizing existence assumptions.

Hilbert&Bernays 1934

Yet, as a result of this deliberation, the following idea suggests itself right away: If we can conduct the impossibility proof without making any axiomatic-existential assumptions, should it then not be possible to provide a grounding for the whole of arithmetic directly in this way, whereby that impossibility proof would become entirely superfluous?

Hilbert's answer is in negative because of his worries about infinities in Set theory and elsewhere in mathematics.

Comment 1

Genetic object-building is <u>not</u> wholly suppressed in the Hilbert-style Formal Mathematics but is

- limited to syntactic constructions
- isolated in a special area of Mathematics called *Meta*mathematics.

Comment 2

This "official" view poorly describes what mathematicians do in practice (cf. Group Theory). However just saying that in practice mathematicians work *informally* does not solve the problem!

Comment 3

The 20th c. showed no significant progress in the axiomatization of physics (Hilbert's 6th Problem). During this century FAM played no role at all in the mainstream research in physics and other natural sciences.

Motivation

The expression "Euclidean plane" is ambiguous.
In one sense it means a geometrical space studied in Planimetry where live circles, triangles, etc (EPLANE);
In a different sense it means an object living in the Euclidean 3-space (ESPACE)(eplane):

$$EPLANE \xrightarrow{eplane} ESPACE$$

Remarks:

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- There are many different eplanes living in ESPACE;
- Circles, etc. in ESPACE factor through EPLANE:

$$
\begin{tikzcd}
CIRCLE \arrow[r, "circle_1"] \arrow[d, "circle_2"'] & EPLANE \arrow[dl, "eplane"] \\
ESPACE &
\end{tikzcd}
$$

Generalization

General situation:

$$TYPE \xrightarrow{object} SPACE$$

Remarks:

Being a type and being a space are relational properties. Being an object is non-relational property.

Each object is of particular type and lives in a particular space.

Non-Euclidean examples:

$$HPLANE \xrightarrow{pseudosphere} ESPACE$$

(Beltramy)

$$EPLANE \xrightarrow{horisphere} HSPACE$$

(Lobachevsky)
Remark: Pseudosphere and horisphere are not types/spaces but objects (without ambiguity).

Observation

Objects of the same type look differently in different spaces:

$$\begin{tikzcd}
 & HSPACE \\
EPLANE \arrow[ur, "horisphere"] \arrow[dr, "eplane"'] & \\
 & ESPACE
\end{tikzcd}$$

Objects of different types in the same space look always differently.

Curry-Howard: Simply typed lambda calculus

Variable: $\dfrac{}{\Gamma, x : T \vdash x : T}$

Product: $\dfrac{\Gamma \vdash t : T \quad \Gamma \vdash u : U}{\Gamma \vdash \langle t, u \rangle : T \times U}$

$\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_1 v : T}$ $\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_2 v : U}$

Function: $\dfrac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x.t : U \to T}$

$\dfrac{\Gamma \vdash t : U \to T \quad \Gamma \vdash u : U}{\Gamma \vdash tu : T}$

Curry-Howard: Natural deduction

Identity: $\overline{\Gamma, A \vdash A}$ (Id)

Conjunction: $\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}$ (& - intro)

$\dfrac{\Gamma \vdash A \& B}{\Gamma \vdash A}$ (& - elim1); $\dfrac{\Gamma \vdash A \& B}{\Gamma \vdash B}$ (& - elim2)

Implication: $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}$ ($\supset$-intro)

$\dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$ ($\supset$-elim aka *modus ponens*)

Curry-Howard Isomorphism

$$\& \equiv \times$$

$$\supset \equiv \to$$

Brouwer-Heyting-Kolmogorov (BHK interpretation)

- proof of $A \supset B$ is a procedure that transforms each proof of A into a proof of B;
- proof of $A\&B$ is a pair consisting of a proof of A and a proof of B

Historical remark

Curry-Howard relates mathematical (λ-calculus) and meta-mathematical (natural deduction) concepts.

Historical remark

Foundational consideration played a crucial role in this story from the outset (Schönfinkel, Curry, Church, Kolmogorov, Lawvere, Lambek). The expression "Curry-Howard isomorphism", which suggests that we have here an unexplained/surprising formal coincidence, is due to Howard 1969. The *true* history (and the true meaning) still waits to be explored.

Lawvere and Lambek 1969

The structure behind the Curry-Howard isomorphism is precisely captured by the notion of *Cartesian closed category* (CCC), which is an (abstract) category with the terminal object, products and exponentials.

Examples: Sets, Boolean algebras

Simply typed lambda-calculus / natural deduction is the *internal language* of CCC.

- Objects: types / propositions
- Morphisms: terms / proofs

Lawvere's philosophical motivation

- objective invariant structures vs. its subjective syntactical presentations
- objective logic vs. subjective logic (Hegel)

Lawvere on logic and geometry

The unity of opposites in the title [Quantifiers and Sheaves] is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. [..] [A] Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for "the" category $\underline{S}$ of abstract sets to an arbitrary topos . We first sum up the principle contradictions of the Grothendieck-Giraud-Verdier theory of topos [..] enabling one to claim that in a sense logic is a special case of geometry. (Lawvere 1970)

Lawvere's axioms for topos

(Elementary) topos is a category which

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(Elementary) topos is a category which

- has finite limits
- is CCC
- has a subobject classifier

Remark

Lawvere's axioms for topos (including the topos of sets) do not "describe defining properties" of a topos with some pre-established logical framework - but (re)construct a topos genetically with its specific internal logic. So logic and geometry turn to be two complimentary aspects of the same object, viz. a topos.

Constructive Proof theory

"[P]roof and knowledge are the same. Thus, if proof theory is construed not in Hilbert's sense, as metamathematics, but simply as a study of proofs in the original sense of the word, then proof theory as the same as theory of knowledge, which, in turn, is the same as logic in the original sense of the word, as the study of reasoning, or proof, not as metamathematics." (Martin-Löf 1983)

MLTT (Martin-Löf 1980): key features

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- double interpretation of types: "sets" and propositions
- double interpretation of terms: elements of sets and proofs of propositions
- higher orders: dependent types (sums and products of families of sets)
- MLTT is the internal language of LCCC (Seely 1983)

MLTT: two identities

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- Definitional identity of terms (of the same type) and of types: $x = y : A$; $A = B : type$ (substitutivity)

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- Definitional identity of terms (of the same type) and of types: $x = y : A$; $A = B : type$ (substitutivity)
- Propositional identity of terms x, y of (definitionally) the same type A:
  $Id_A(x, y) : type$;
  Remark: propositional identity is a (dependent) type on its own.

MLTT: Higher Identity Types

- $x', y' : Id_A(x, y)$
- $Id_{Id_A}(x', y') : type$
- and so on

Idea:

The central new idea in homotopy type theory is that types can be regarded as spaces in homotopy theory, or higher-dimensional groupoids in category theory. (HoTT Book 2013)

Voevodsky on Univalent Foundations

Whilst it is possible to encode all of mathematics into Zermelo-Fraenkel set theory, the manner in which this is done is frequently ugly; worse, when one does so, there remain many statements of ZF which are mathematically meaningless. [..]

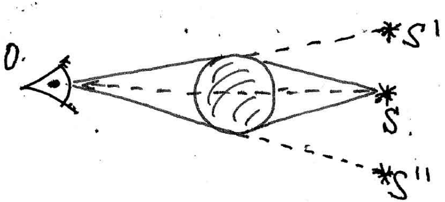
Voevodsky on Univalent Foundations (continued)

Univalent foundations seeks to improve on this situation by providing a system, based on Martin-Löf's dependent type theory whose *syntax is tightly wedded to the intended semantical interpretation* in the world of everyday mathematics. In particular, it allows the direct formalization of the world of homotopy types; indeed, these are the basic entities dealt with by the system. (Voevodsky 2011)

Naive stuff

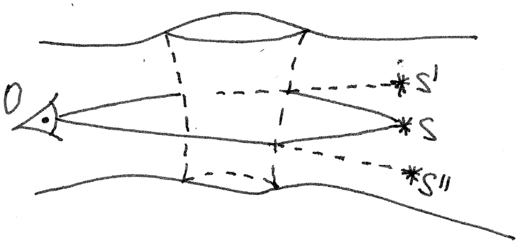
Identity through time

Naive stuff



Gravitational lensing

Naive stuff



Wormhole lensing

Serious stuff: Neo-Classicism in Physics

Topos Physics:
A. Döring, Ch. Isham: 'What is a Thing?': Topos Theory in the Foundations of Physics (2008): http://arxiv.org/abs/0803.0417
Univalent Physics:
Urs Schreiber: Quantization via Linear homotopy types (Feb. 2014) http://arxiv.org/abs/1402.7041

Voevodsky&Coquand

COQ; 4-color problem
Univalent Foundations: making COQ into a universal tool for checking all mathematical routines.

Object-Orientation

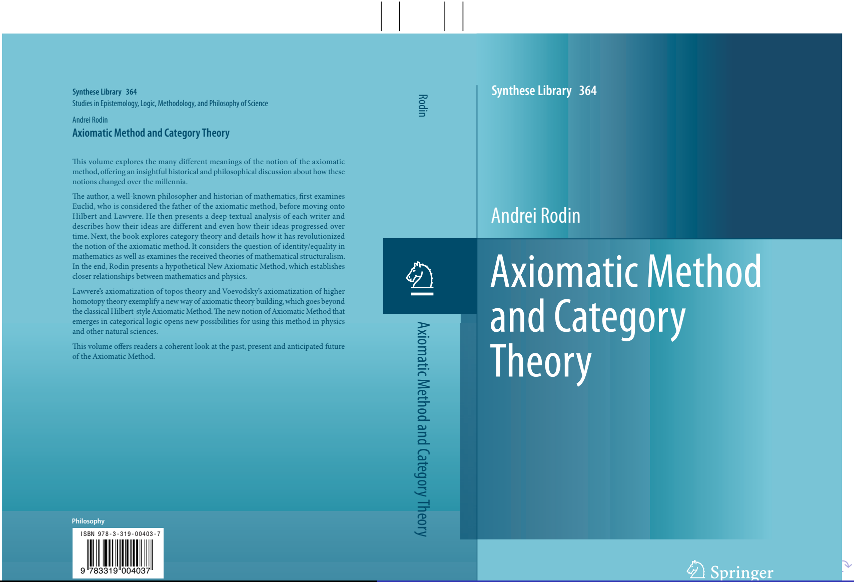
"Procedural languages are generally well understood; their constructs are by now standard, and their formal underpinnings are solid. The fundamental features of these languages have been distilled into formalisms that prove useful in identifying and explaining issues of implementation, static analysis, semantics, and verification. An analogous understanding has not yet emerged for object-oriented languages. There is no widespread agreement on a collection of basic constructs and on their properties. Consequently, practical object-oriented languages often support disparate features and programming techniques with little concern for orthogonality. This situation might improve if we had a better understanding of the foundations of object-oriented languages." (Abadi&Cardelli, A Theory of Objects, Springer 1996)

Knowledge Representation

Why base working ontologies on Classical FOL (in the form of so-called Descriptive Logics) - even if this approach leads to serious computational difficulties? Arguably this choice is based on a metaphysical prejudice (Russellian ontologies based on Russell's theory of descriptions; Barry Smith & Co.). I conjecture that genetic approaches supported by Categorical Logic (in the form of Topos theory or HoTT or else) may perform much better in Knowledge Representation.

Conclusions

Objects in Logic in Mathematics are, generally, not syntactic objects (contra Hilbert). Categorical Logic provides means for treating the (traditional) issue of objecthood more generally. This may have important advantages for applications in the Natural Science and Computer Science.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Andrei Rodin
Axiomatic Method and Category Theory
This volume explores the many different meanings of the notion of the axiomatic method, offering an insightful historical and philosophical discussion about how these notions changed over the millennia.
The author, a well-known philosopher and historian of mathematics, first examines Euclid, who is considered the father of the axiomatic method, before moving onto Hilbert and Lawvere. He then presents a deep textual analysis of each writer and describes how their ideas are different and even how their ideas progressed over time. Next, the book explores category theory and details how it has revolutionized the notion of the axiomatic method. It considers the question of identity/equality in mathematics as well as examines the received theories of mathematical structuralism. In the end, Rodin presents a hypothetical New Axiomatic Method, which establishes closer relationships between mathematics and physics.
Lawvere's axiomatization of topos theory and Voevodsky's axiomatization of higher homotopy theory exemplify a new way of axiomatic theory building, which goes beyond the classical Hilbert-style Axiomatic Method. The new notion of Axiomatic Method that emerges in categorical logic opens new possibilities for using this method in physics and other natural sciences.
This volume offers readers a coherent look at the past, present and anticipated future of the Axiomatic Method.
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