HoTT and Univalent Foundations

Andrei Rodin

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Martin-Löf's Constructive Type theory (MLTT)

(Higher) Homotopies

Homotopy Type theory

Univalent Foundations

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- double interpretation of terms: elements of sets and proofs of propsitions
- higher orders: dependent types (sums and products of families of sets)
- MLTT is the internal language of LCCC (Seely 1983)

products

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- If $\Gamma \vdash d : A \times B$, then $\Gamma \vdash \pi_0(d) : A$ and $\Gamma \vdash \pi_1(d) : B$

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- If $\Gamma \vdash f : (\Pi x : X)A(x) and \Delta \vdash t : X$, then $\Gamma, \Delta \vdash apply(f, t) : A(t)$

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- If $\Gamma \vdash c : A \times B$, then $\Gamma \vdash \pi_0(c) : A$ and $\Gamma \vdash \pi_1(c) : B$

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- Definitional identity of terms (of the same type) and of types: $x = y : A$; $A = B : type$ (substitutivity)
- Propositional identity of terms x, y of (definitionally) the same type A:
  $Id_A(x, y) : type$;
  Remark: propositional identity is a (dependent) type on its own.

MLTT: Higher Identity Types

- $x', y' : Id_A(x, y)$
- $Id_{Id_A}(x', y') : type$
- and so on

Fundamental group

Fundamental group G^0_T of a topological space T:

- a base point P;
- loops through P (loops are circular paths $l : I \rightarrow T$);
- composition of the loops (up to homotopy only! - see below);
- identification of homotopic loops;
- independence of the choice of the base point.

Fundamental (1-) groupoid

G^1_T:

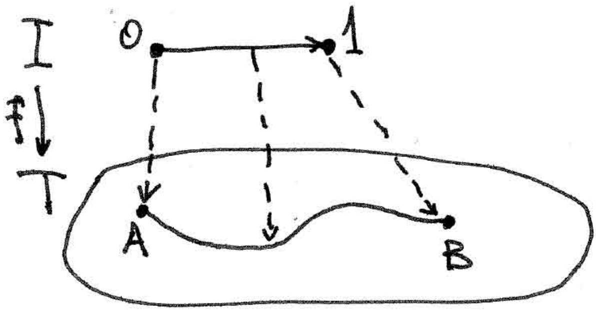
- all points of T (no arbitrary choice);
- paths between the points (embeddings $s : I \to T$);
- composition of the *consecutive* paths (up to homotopy only! - see below);
- identification of homotopic paths;

Since not all paths are consecutive G^1_T contains more information about T than G^0_T!

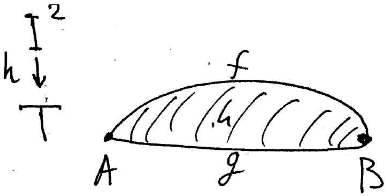
Path Homotopy and Higher Homotopies

$s : I \rightarrow T$, $p : I \rightarrow T$ where $I = [0, 1]$: paths in T
$h : I \times I \rightarrow T$: homotopy of paths s, t if $h(0 \times I) = s$, $h(1 \times I) = t$
$h^n : I \times I^{n-1} \rightarrow T$: n-homotopy of $n-1$-homotopies h_0^{n-1}, h_1^{n-1} if $h^n(0 \times I^{n-1}) = h_0^{n-1}$, $h^n(1 \times I^{n-1}) = h_1^{n-1}$;
Remark: Paths are zero-homotopies

Path Homotopy and Higher Homotopies



Homotopy categorically and Categories homotopically



$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

Higher Groupoids and Omega-Groupoids (Grothendieck 1983)

- all points of T (no arbitrary choice);
- paths between the points ;
- homotopies of paths
- homotopies of homotopies (2-homotopies)
- higher homotopies up to n-homotopies
- higher homotopies ad infinitum

G_T^n contains more information about T than G_T^{n-1}!

Composition of Paths

Concatenation of paths produces a map of the form $2I \to T$ but not of the form $I \to T$, i.e., not a path. We have the whole space of paths $I \to 2I$ to play with! But all those paths are homotopical. Similarly for higher homotopies (but beware that n-homotopies are composed in n different ways!)
On each level when we say that $a \oplus b = c$ the sign $=$ hides an infinite-dimensional topological structure!

Grothendieck Conjecture:

G_T^ω contains all relevant information about T; an omega-groupoid is a complete algebraic presentation of a topological space.

Homotopy Type theory

- Groupoid model of MLTT: basic types are groupoids, terms are their elements, dependent types are fibrations of groupoids (families of groupoids indexed by groupoids - rather than families of sets indexed by sets). Extensionality one dimension up. (Streicher 1993).
- Higher (homotopical) groupoids model higher identity types. Intensionality all way up (Voevodsky circa 2008).

Voevodsky 2011

The broad motivation behind univalent foundations is a desire to have a system in which mathematics can be formalized in a manner which is as natural as possible. Whilst it is possible to encode all of mathematics into Zermelo-Fraenkel set theory, the manner in which this is done is frequently ugly; worse, when one does so, there remain many statements of ZF which are mathematically meaningless. This problem becomes particularly pressing in attempting a computer formalization of mathematics; in the standard foundations, to write down in full even the most basic definitions - of isomorphism between sets, or of group structure on a set - requires many pages of symbols.

Voevodsky 2011

Univalent foundations seeks to improve on this situation by providing a system, based on Martin-Löf's dependent type theory whose syntax is tightly wedded to the intended semantical interpretation in the world of everyday mathematics. In particular, it allows the direct formalization of the world of homotopy types; indeed, these are the basic entities dealt with by the system.

New Axiomatic Method

Гомотопическая модель теории типов Мартина-Лефа интерпретирует (геометрически) логические символы наряду с нелогическими. Одни и те же операции интерпретируются и как логические (т.е. как операции над пропозициями), и как геометрические (как операции над геометрическими объектами). Пример: путь и тождество. Поэтому аксиоматический метод Воеводского больше похож на метод Евклида, чем на метод Гильберта.

h-levels

- (i) Given space is called A *contractible* (aka space of h-level 0) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- (ii) We say that A is a space of h-level $n+1$ if for all its points x, y path spaces $paths_A(x, y)$ are of h-level n.

h-universe

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- Level 1: up to homotopy equivalence there are two spaces here: the empty space $\emptyset$ and the point pt. (For $\emptyset$ condition (ii) is satisfied vacuously; for pt (ii) is satisfied because in pt there exists only one path, which consists of this very point.) We call $\emptyset, pt$ *truth values*; we also refer to types of this level as *properties* and *propositions*. Notice that h-level n corresponds to the logical level $n-1$: the propositional logic (i.e., the propositional segment of our type theory) lives at h-level 1.

h-universe

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- Level 2: Types of this level are characterized by the following property: their path spaces are either empty or contractible. So such types are disjoint unions of contractible components (points), or in other words *sets* of points. This will be our working notion of set available in this framework.

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- Level 3: Types of this level are characterized by the following property: their path spaces are sets (up to homotopy equivalence). These are obviously (ordinary flat) *groupoids* (with path spaces hom-sets).
- Level 4: 2-groupoids

h-universe

- ..
- Level n+2: *n*-groupoids
- ..
- ω-groupoids
- ω-groupoids ($\omega + 1 = \omega$)

How it works

Let $isontr(A)$ and $isaprop(A)$ be formally constructed types " A is contractible" and "A is a proposition" (for formal definitions see Voevodsky:2011, p. 8). Then one formally deduces (= further constructs according to the same general rules) types $isaprop(iscontr(A))$ and $isaprop(isaprop(A))$, which are non-empty and thus "hold true" for each type A; informally these latter types tell us that for all A "A is contractible" is a proposition and "A is a proposition" is again a proposition.

How it works

With the same technique one defines in this setting type $weq(A, B)$ of *weak equivalences* (i.e., homotopy equivalences) of given types A, B (as a type of maps $e : A \to B$ of appropriate sort) and formally proves its expected properties. These formal proves involve a *different* type $isweq(A, B)$ of h-level 2, which is a proposition saying that A, B are homotopy equivalent, i.e., that type $weq(A, B)$ is inhabited.)

Axiom of Univalence

Homotopically equivalent types are (propositionally) identical. This means that the universe *TYPE* of homotopy types is construed like a homotopy type (and also modeled by ω-groupoid).
Axiom of Univalence is the only axiom of Univalent Foundations on the top of MLTT.

THE END