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How Mathematical Concepts Get Their Bodies

Plan:

- 1) Kantian generalities about "ideal element"
 - conceptualisation and de-idealisation
- 2) Some brief examples:
 - history of geometry from Euclid to mid-
 - ~~n~~-fold product of magnitudes in Arnaud
 - notion of ideal from Kummer via Dedekind to Bourbaki
- 3) Set-theoretic intuition and its limits
- 4) Cohen's forcing
- 5) Forcing in a topos-theoretic setting
(intuition strikes back!)
- 6) General conclusion on (Kantian-like) mathematical intuitionism.

1. Kantian generalities

②

(also mathematical) (presupposition)

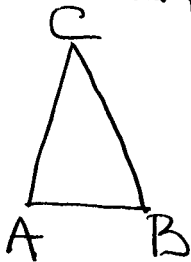
Thinking = conception + intuition
Cognition

Kant: Gedanken	Konzept Begriffe	Anschauung Intuition
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"Gedanken ohne Inhalt sind leer,
Anschauungen ohne Begriffe sind blind."

Ex. Euclid Elem. 1.5

concept	πρότασις	For isosceles triangles the angles at the base are equal
intuition	ἕξις	Let ABC be isosceles
	διόρισμός	I say that angle ABC is equal to ACB



But... (further features):

- 1) "Real" mathematics also deals with
 - 1a Poorly conceptualised intuitions
 - 1b Poorly intuited concepts

(before coming to examples...)

more on 16:

(3)

A poorly intuited concept thought of as an individual thing (in individuo, cf. $\xi\alpha\theta\epsilon\sigma\iota\varsigma$) is ~~is~~ called (by the abuse of a kantian term) an ideal element of mathematical thinking.

cf. in kant:

• [A] merely transcendental idea is smth of which we have no conception... [A] conception of an object that is adequate to the idea given by reason is impossible for such an object must be capable of being presented and intuited in a possible experience" (CPR, ^{On the dialectical procedure.})
"Still further removed than the idea from objective reality is the ideal, by which term I understand the idea [...] in individuo - as an individual thing determinable or determined by the idea alone."

(ibid, Of the Ideal in general)

sic! ④
"He [=Plato] certainly extended the application of his conception [of idea] to speculative cognitions also, provided they were given pure and a priori, nay, even to mathematics, although this science cannot possess an object otherwise than in possible experience. I cannot follow him in this"

(ibid, On Ideas in general, footnote)

② (Second further feature)

Neither intuitions nor concepts are rigid (an anti-Kantian presupposition)

History of mathematics provides multiple examples of

2a Progressive conceptualisation of early acquired intuitions

2b Progressive intuiting of earlier acquired concepts (= de-idealisation)
(or other)

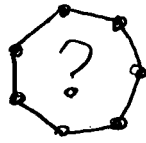
Remark: For historical reason 2b is less respected than 2a.

2. Brief examples

⑤

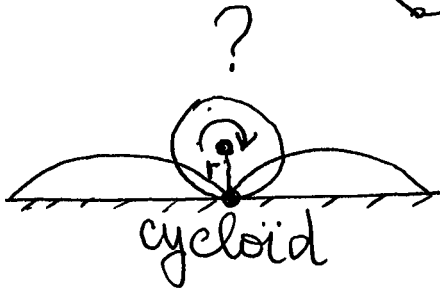
2.1 Conceptualisation of geometrical intuition from Euclid to Lobachevsky

Euclid: ruler and compass as tools
 of conceptualisation
 (circle and straight line as generic objects)



regular septagon

too much intuitions!



cycloid

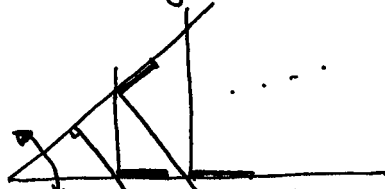
$$\begin{cases} x = r(t - \sin t) \\ y = r(1 - \cos t) \end{cases} \\ 0 \leq t \leq 2\pi$$

mid 19th c.

too much concepts!

(1637)
 Descartes:
 cartesian compass
 as a tool of conceptualisation
 (produces all algebraic curves)

$$P(x, y) = 0$$



(see "Géométrie, livre 2")

Pythagorean theorem in
 Hyperbolic (Lobachevskian) space:

$$\begin{aligned} \cosh^2(x) - \sinh^2(x) &= 1 & \text{Cf: } \cos^2(x) + \sin^2(x) &= 1 \\ \sinh(x) &= \frac{e^x - e^{-x}}{2} & \frac{1}{\cos(x)} &= \sec(x) \\ \cosh(x) &= \frac{e^x + e^{-x}}{2} & \frac{1}{\cos(x)} &= \sec(x) \end{aligned}$$

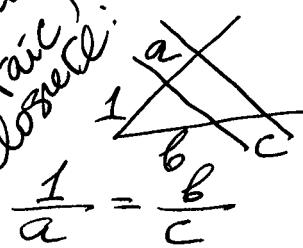
2.2 (De-)idealisation of n-fold products (6)
 ($n > 3$) in Arnould (Nouveaux Éléments
 de Géométrie, 1667)

"[C]e qui ne ~~sont pas~~ se peut multiplier par la nature se peut multiplier par une fiction d'esprit [...]. On multiplie aussi par la même fiction d'esprit des surfaces par des surfaces, quoy que cela donne pour produit une estendue de 4 dimensions qui ne peut estre dans la nature. [...] Je sçay bien qu'on dit que c'est parce que ces produit imaginaires se peuvent reduire en lignes [...] Mais il n'y a guerre d'apparence que la vérité de ces sortes de preuves dépendent de ces lignes qui sont visiblement étrangères à ces démonstrations" (p.38)

$$\frac{a \text{ times } b}{b} = a \boxed{}$$

$$\boxed{} \text{ times } \boxed{} = ? \text{ 4D}$$

Cartesian
 (algebraic)
 closure:



$$\frac{1}{a} = \frac{b}{c}$$

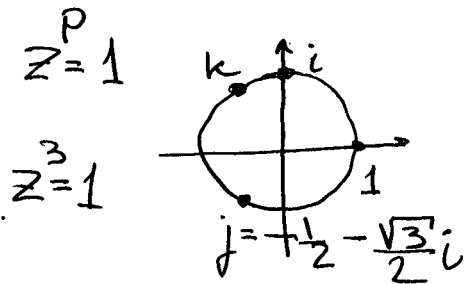
$$a \cdot b = c$$

"Formal (symbolic) intuition is not sufficient?"
 Lines as general magnitudes?
 got de-idealised

2.3. De-idealisation of ideals

(7)

Kummer 1847: "Über die Zerlegung der aus Wurzeln der Einheit gebildeten complexen Zahlen in ihre Primfactoren". (integers)



cyclotomic numbers:

$$l + mj + nk$$

$$l, m, n \in \mathbb{Z}$$

With $p=23$ the unique prime factorisation breaks:

(a different example): $3 \times 7 = (4 + i\sqrt{5})(4 - i\sqrt{5}) = 21$

Dedekind 1871, 1879, 1894 | Ideal numbers fix this!

Supplements to Dirichlet's Vorlesungen über

R -ring (Dedekind: of complex-based integers) Zahlentheorie

$I \subset R$ is left ideal $(R, +)$ -its additive group if

- $(I, +)$ is a subgroup of $(R, +)$

- for all $x \in I$ and all $r \in R$ $x * r \in I$

| right } ideal
| two-side }

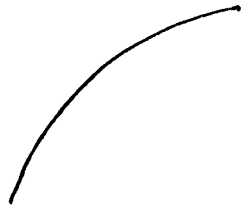
Dedekind:
non-trivial
identification
with Kummer's
"ideal numbers"

3. Set-theoretic intuition and its limits (8)

Two steps of building set-th intuition:
1) intuiting infinite collections of previously intuited objects

ex. $\mathbb{N}: \{1, 2, 3, \dots\}$ (an infinite series)

ex.



How many points you can see on this picture?

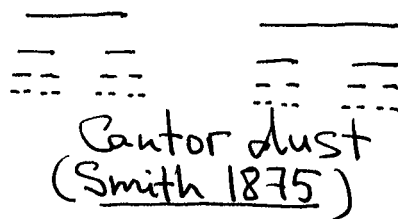
ex. fractals

Mandelbrot 1967

How long is coast of Britain?

Mandelbrot 1982

The Fractal Geometry of Nature.



2) Conceiving of abstract (point-?) sets as a building material for all mathematics

Bourbaki: a mathematical construction =
= set + structure

Set theory as universal semantic (9)
tool

(which also provides semantics for itself)

"Let us consider three distinct systems of things..." (Hilbert, 1899)

ZF: "systems" are sets

"things" also are sets

Set-theoretic intuition provides a support for any sound mathematical concept.
(cf. Bourbaki).

→ No ideal elements any longer?

? "Ordinary maths like Fermat's last theorem uses no sets larger than continuum [= \aleph_1]" (from a recent posting on blog "Maths and physics")

Some hints why this
is WRONG



Logic in a topos is generally intuitionistic: $77A \neq A$

Universal set (the set of all sets) U (10)

Cantor's and Russell's paradoxes

Transcendental Idea?

Proper classes are (nearly) sterile!

Category of Sets: a de-idealisation of U

idea: take ~~to~~ the notion of function rather than membership \in as primitive. Then two steps:

- 1) describe an abstract category: objects with composable morphisms
- 2) add conditions making an abstract category into the category of sets (Lawvere 1963)

Category of sets is better used in further constructions.

Ex. $\text{Set}^{\text{Top}} = (\text{Top}, \text{Set})$: category of sheaves over a topological space T .

Ex. $\text{Set}^{\text{C}^{\text{op}}}$ with any small $\text{C} \rightarrow \hat{\text{C}}$ "continuously variable sets"

$\text{C} \hookrightarrow \hat{\text{C}}$: Yoneda embedding: C gets "limits"

Set^{Top} and $\text{Set}^{\text{C}^{\text{op}}}$ are toposes.

Th. For any topos \mathcal{E} and small C , \mathcal{E}^{C} is a topos.

4. Cohen's forcing

(11)

History: Cantor circa 1890: $\aleph_1 = 2^{\aleph_0}$ (CH)

Gödel 1939: consistency of CH and ZFC
"Constructible universe L"

Cohen 1963: consistency of \neg CH with ZFC
forcing models

Desired construction:

Let M be standard model
(the same \in and well-founded)

take $F: \aleph_0 \times \aleph_k \rightarrow \{0, 1\}$, $k \geq 2$

$F = \{F: \aleph_0 \rightarrow \{0, 1\}: \text{continuum is } \aleph_k!\}$

Problem: Such F is not constructible

idea 0: Start with minimal (countable and constructible) standard model M and then adjoin F . How?

option: From a larger model

problem: This confuses cardinalities

another option: forcing!

(12)

Idea 1: Describe F piece-by-piece by
an infinite chain of forcing conditions
(formulae of ZF = sets)

$$P_0 \leq P_1 \leq \dots \leq P_i \leq P_j \leq \dots$$

Analogy: rational approximation of
irrational numbers

Idea 2: Adjoin F as generic set formal symbol and
consider ZF-formulae with F
as elements of a new model N .

Analogy: (quoting Cohen):

"The situation is analogous to the construction of a field k formed by adjoining the root α of an irreducible equation $f(x)=0$. The elements of the extension field are all of the form $p(\alpha)$ where p is a polynomial and α is taken as a formal symbol" (Set Theory and CH).

"The chief point is that we do not wish
(a) to contain "special" information about
 M , which can only be seen from the outside
[such that the "real" cardinality of M]" (ibid.)

More precisely...

(13)

Def. "A "labeling" is a mapping defined in ZF , which assigns to each ordinal $0 < \alpha < \omega_0$ a set S_α , the "label space", and functions φ_α defined in S_α such that the sets S_α are disjoint and if $c \in S_\alpha$, $\varphi_\alpha(c)$ is a formula $A(x)$, which may have elements of S_β with $\beta < \alpha$ appearing as constants. [...]

The set S_0 is defined as $\omega \cup \{a\}$ where a is a formal symbol. We write $S = \bigcup_\alpha S_\alpha$."

Overcoming a ^{Fregean} "metaphysical barrier":
variables of formulae range over formulae
of the same type!

Cf. in algebra

$$p(x) = x^2 + 1, \quad x \neq i$$

$$q(x) = x^2 - 1$$

$$p(q(x)) = (x^2 - 1)^2 + 1 = x^4 - 2x^2 + 2$$

Take $G \subseteq S = \bigcup_\alpha S_\alpha$: "generic sets": such that

"All our sets and variables

$M(G) = \mathcal{N}$
is a model

are actually functions of the generic set \mathcal{G} .
So in analogy with field theory, we are
actually dealing with the space of all (rational)
functions of \mathcal{G} , not actual sets" (ibid)

"Ideal character" of Cohen's construction: (14)

"How could one decide whether a statement about ω is true before we have ω ? In a somewhat exaggerated sense, it seemed that I would have to examine the very meaning of truth and think about it in a new way" (*The Discovery of forcing*, 2002)

Cf. Kunen's case: Cohen's is stronger

Cf. "I think most mathematicians, as ~~distinct~~ distinct from philosophers, will not find much interest in the various polemical publications of even prominent mathematicians. My personal opinion is that this is a kind of "religious debate." (ibid.)

Back to syntax: forcing relation \Vdash

Forcing condition in M \Vdash truth in $M[G]$
(for fixed G)
"generic"

e.g.

③ P forces $\sim B$ iff for all $Q \geq P$
 Q does not force B think of intuit. negation, and it's Kripke's semantics

5. Forcing in a topos-theoretic setting: (15)
geometrical intuition strikes back.

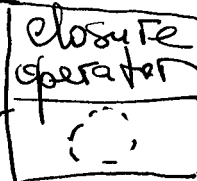
Scott & Solovay 1965 (!): Reformulation
of Cohen's construction with Boolean-valued
models of ZF.

- ① Set P of forcing conditions is made
into a Boolean algebra.
- ② The obtained Boolean-valued model is
collapsed into a binary-valued model
along a ultrafilter (maximal filter)

Tierney 1972 : reformulation of Cohen's
(joint work with Lawvere) construction in a
topos-theoretic setting.

My presentation: Mackaue & Meerdijk 1992
• Sheaves in Geometry and Logic

Th1 Topos is Boolean iff $\neg: \Omega \rightarrow \Omega$ satisfies $\neg\neg = \text{id}_\Omega$



Th2 $\neg\neg$ is a (Lawvere-Tierney) topology.

It is discrete topology: every subset is clopen
(is this still a topology? YES!)

Morphisms are defined by their points.

In order to violate CH in a topos (of sets) (16)
 we look for (set) B such that

$$N \xrightarrow{h} B \xrightarrow{g} 2^N \text{ properly, i.e. monos } h, g \text{ are not isos.}$$

In a given topos Set take B
 with $\text{Card}(B) > 2^N$, so there is
 (no) mono $B \hookrightarrow 2^N$

Def.:
 monos are right-cancellative:
 for all f, g, S
 $fg = sg \Rightarrow f = s$

Consider poset P of forcing conditions ...

$$(B, N, 2) \Rightarrow (\tilde{B}, \tilde{N}, \Omega) \xrightleftharpoons{a: \text{right adjoint}} (\tilde{B}, \tilde{N}, \Omega_{\mathcal{T}})$$

in Set

in topos of presheaves
 $\text{Set}^{P^{\text{op}}}$

in topos

$\text{Sh}(P, \mathcal{T})$

(Cohen's boolean topos of double-negation sheaves)

$$? \tilde{g}: \tilde{B} \hookrightarrow \Omega^{\tilde{N}}$$

$p \in P$: a "stage of knowledge about \tilde{g} "

$$\tilde{g}(p): \tilde{B}(p) \hookrightarrow \Omega(p) = \text{Nat}(y(p) \times \tilde{N}, \Omega)$$

$$\tilde{B} \cong$$

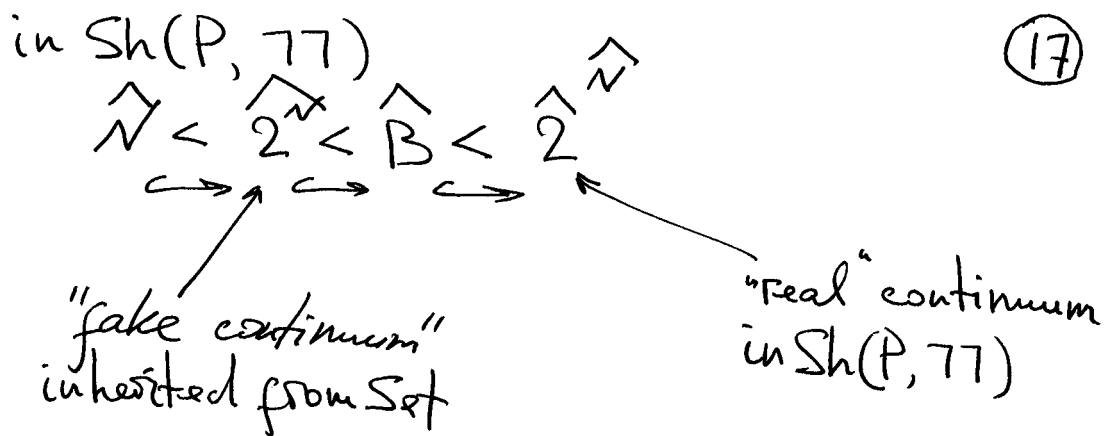
$$\text{take } y(p) = \text{Hom}(-, p)$$

the set of incoming morphisms
 of p : mind the reversal of arrows!

Def.: $C = A^B$ iff

$$\begin{array}{ccc} C \times B & \rightarrow & A \\ \uparrow & & \uparrow \\ C' \times B & & \end{array}$$

If topology P is dense then $y(p)$ is in $\Omega_{\mathcal{T}}$
 Intuitive: the chain of P 's converges!



"Sheaves for the dense topology on the poset of 'finite stages of knowledge' about the desired impossible monomorphism $[B \rightarrow \hat{2} \text{ in Set}]$ form the new model of sets in which that mon is really there!" (MacLane & Moerdijk)

Cohen's syntactic notion of forcing is a method to describe truth in $\text{Sh}(P, \tau)$

Kripke-Joyal Semantics

for posets
of "stages"

for any small
category of
"stages"

6. Conclusion

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6.1 Topos-theoretic rendering of Cohen's forcing provides it with a precise and intuitively appealing geometrical background. (This intuition requires an appropriate training.)

These remain nothing "ideal" in the topos-theoretic forcing construction.

6.2 A reason of failure of intuitionistic programs in mathematics of 20th century is that these programs were mostly reactionary: they tried to forbid certain ways of ("classical") reasoning. (while the adversary Set-theoretic programs offered something really new...)
CT program ~~might~~ performs better and gives mathematical intuitionism (in a broad Kantian sense) a new chance.