2nd IP Fellows Seminar
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How Mathematical Concepts Get Their Bodies

Plan:

1) Kantian generalities about "ideal element"
   - conceptualisation and de-idealisation
2) Some brief examples:
   - history of geometry from Euclid to mid-
   - n-fold product of magnitudes in Arnauld
   - notion of ideal from Kummer via Dedekind to Bourbaki
3) Set-theoretic intuition and its limits
4) Cohen's forcing
5) Forcing in a topos-theoretic setting (intuition strikes back!)
6) General conclusion on (Kantian-like) mathematical intuitionism.

1. Kantian generalities

(also mathematical) (presupposition)

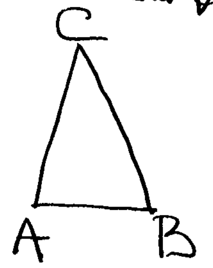
Thinking = conception + intuition
Cognition

Kant: Gedanken — Konzept / Begriff — Anschauung / Intuition

"Gedanken ohne Inhalt sind leer,
Anschauungen ohne Begriffe sind blind"

Ex. Euclid Elem. 1.5

concept	πρότασις	For isosceles triangles the angles at the base are equal
intuition	ἔκθεσις	Let ABC be isosceles
	διορισμός	I say that angle ABC is equal to ACB



But... (further features):

1) „Real" mathematics also deals with
  1a Poorly conceptualised intuitions
  1b Poorly intuited concepts

(before coming to examples...)

more on 1b:

A poorly intuited concept thought of as an individual thing (in individuo, cf. ἔκθεσις) is called (by the abuse of a Kantian term) an ideal element of mathematical thinking.

Cf. in Kant:

"[A] merely transcendental idea is such of which we have no conception... [A] conception of an object that is adequate to the idea given by reason is impossible. For such an object must be capable of being presented and intuited in a possible experience" (CPR, On the dialectical procedure..)

"Still further removed than the idea from objective reality is the ideal, by which term I understand the idea [..] in individuo — as an individual thing determinable or determined by the idea alone."

(ibid, Of the Ideal in general)

sic!
"He [=Plato] certainly extended the application of his conception [of idea] to speculative cognitions also, provided they were given pure and apriori, nay, even to mathematics, although this science cannot possess an object otherwise than in possible experience. I cannot follow him in this"

(ibid, On Ideas in general, footnote)

[2] (Second further feature)
Neither intuitions nor concepts are rigid (an anti-Kantian presupposition)
History of mathematics provides multiple examples of

2a Progressive conceptualisation of early acquired intuitions

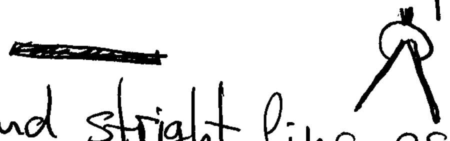
2b Progressive intuiting of earlier acquired concepts (=de-idealisation)

Remark: For historical (or other) reason 2b is less respected than 2a.

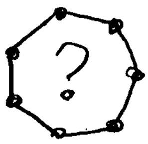
2. Brief examples

2.1 Conceptualisation of geometrical intuition from Euclid to Lobachevsky

Euclid: ruler and compass as tools of conceptualisation

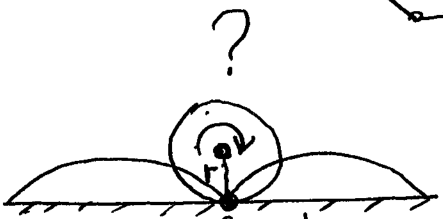


(circle and stright line as generic objects)



regular septagon

too much intuitions!



cycloïd

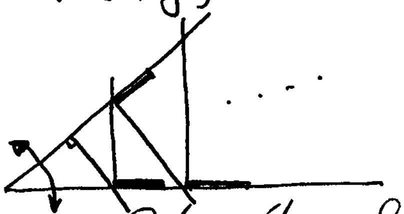
$$\begin{cases} x = r(t - \sin t) \\ y = r(1 - \cos t) \end{cases} \quad 0 \leq t \leq 2\pi$$

mid 19th c.

Descartes: (1637)

cartesian compass as a tool of conceptualisation (produces all algebraic curves)

$P(x, y) = 0$



(see „Géométrie, livre 2)

too much concepts!

Pythagorean theorem in Hyperbolic (Lobachevskian) space:

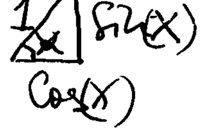
$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

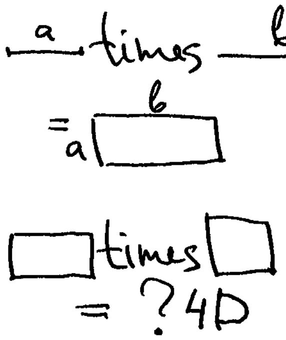
?

Cf.: $\cos^2(x) + \sin^2(x) = 1$

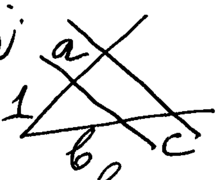


2.2 (De)-idealisation of n-fold products ($n > 3$) in Arnauld (Nouveaux Éléments de Géométrie, 1667)

"[C]e qui ne ~~sont pas~~ se peut multiplier par la nature se peut multiplier par une fiction d'esprit [...]. On multiplie aussi par la même fiction d'esprit des surfaces par des surfaces, quoy que cela donne pour produit une estendue de 4 dimentions qui ne peut estre dans la nature. [...] Je sçay bien qu'on dit que c'est parce que ces produit imaginaires se peuvent reduire en lignes [...] Mais il n'y a guerre d'apparence que la verite de ces sortes de preuves dépendent de ces lignes qui sont visiblement étrangères à ces démonstrations" (p.38)



cartesian (algebraic closure):



$$\frac{1}{a} = \frac{b}{c}$$

$$a \times b = c$$

"Formal (symbolic) intuition is not sufficient?

Lines or general magnitudes?

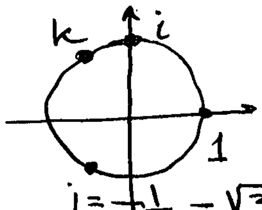
got de-idealised

2.3. De-idealisation of ideals

Kummer 1847: "Über die Zerlegung der aus Wurzeln der Einheit gebildeten complexen Zahlen in ihre Primefactoren". | (integers) cyclotomic numbers:

$z^p = 1$

$z^3 = 1$



$j = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$l + mj + nk$

$l, m, n \in \mathbb{Z}$

With $p = 23$ the unique prime factorisation brakes:

(a different example): $3 \times 7 = (4 + i\sqrt{5})(4 - i\sqrt{5}) = 21$

Dedekind 1871, 1879, 1894 | Ideal numbers fix this!

Supplements to Direchlêt's Vorlesungen über Zahlentheorie

R-ring (Dedekind: of complex-based integers)

$I \subset R$ is left ideal if $(R, +)$ - its additive group

- $(I, +)$ is a subgroup of $(R, +)$
- for all $x \in I$ and all $r \in R$ $x \times r \in I$

righ
two-side } ideal

Dedekind: non-trivial identification with Kummer's "ideal numbers"

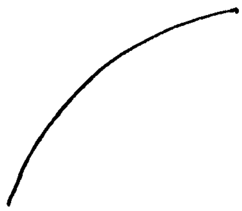
3. Set-theoretic intuition and its limits

Two steps of building set-th intuition:

1) intuiting infinite collections of previously intuited objects

ex. $\mathbb{N}: \{1, 2, 3, \dots\}$ (an infinite series)

ex.



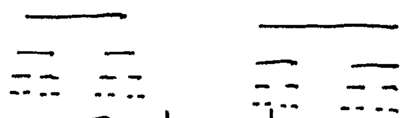
How many points you can see on this picture?

ex. fractals

Koch 1904

Mandelbrot 1967
How long is coast of Britain?

Mandelbrot 1982
The Fractal Geometry of Nature.



Cantor dust
(Smith 1875)

2) Conceiving of abstract (point-?) sets as a building material for all mathematics

Bourbaki: a mathematical construction =
= set + structure

Set theory as universal semantic tool (9)

(which also provides semantics for itself)

" Let us consider three distinct systems of things..." (Hilbert, 1899)

ZF: "systems" are sets

"things" also are sets

Set-theoretic intuition provides a support for <u>any</u> sound mathematical concept (<u>Cf. Bourbaki</u>).

<u>No ideal elements any longer?</u>

? " Ordinary maths like Fermat's last theorem uses no sets larger than continuum [$=\aleph_1$]" (from a recent posting on blog "Maths and physics")

Some hints why this is WRONG

↓

Universal set (the set of all sets) U

Cantor's and Russell's paradoxes

Transcendental Idea?

Proper classes are (nearly) sterile!

$\boxed{\text{Category of Sets: a de-idealisation of } U}$

<u>idea</u>: take the notion of function rather than membership $\in$ as primitive. Then two steps:

1) describe an abstract category: objects with composable morphisms
2) add conditions making an abstract category into <u>the</u> <u>category of sets</u> (Lawvere 1963)

Category of sets is better used in further constructions.

Ex. $Set^{T^{op}} = (T^{op}, Set)$: category of sheaves over a topological space T

Ex. $Set^{C^{op}}$ with any small C $= \hat{C}$ „continuously variable sets"

$C \hookrightarrow \hat{C}$: Yoneda embedding: C „gets limits"

$Set^{T^{op}}$ and $Set^{C^{op}}$ are <u>toposes</u>.

<u>Th.</u> For any topos $\mathcal{E}$ and small C, $\mathcal{E}^{C}$ is a topos.

Logic <u>in</u> a topos is generally intuitionistic: $\neg\neg A \neq A$

4. Cohen's forcing

History: Cantor circa 1890: $\aleph_1 = 2^{\aleph_0}$ (CH)

Gödel 1939: consistency of CH and ZFC

"Constructible universe L"

Cohen 1963: consistency of $\neg$CH with ZFC

forcing models

Desired construction:

Let M be standard model (the same $\in$ and well-founded)

take $F: \aleph_0 \times \aleph_k \to \{0,1\}$, $k \geq 2$

$F = \{F_{\aleph_k}: \aleph_0 \to \{0,1\}\}$: continuum is $\aleph_k$!

Problem: Such F is not constructible

idea 0: Start with minimal (countable and constructible) standard model M and then adjoin F. How?

option: From a larger model

problem: This confuses cardinalities

another option: forcing!

Idea 1: Describe F piece-by-piece by an infinite chain of forcing conditions (formulae of ZF = sets)

$$P_0 \le P_1 \le \dots \le P_i \le P_j \le \dots$$

Analogy: rational approximation of irrational numbers

Idea 2: Adjoin F as formal symbol [generic set] and consider ZF-formulae with F as elements of a new model N.

Analogy: (quoting Cohen):

"The situation is analogous to the construction of a field k formed by adjoining the root α of an irreducible equation $f(x)=0$. The elements of the extension field are all of the form $p(\alpha)$ where p is a polynomial and α is taken as a formal symbol" (Set Theory and CH).

"The chief point is that we do not wish (a) to contain „special" information about M, which can only be seen from the outside [such that the „real" cardinality of M]" (ibid.)

More precisely...

<u>Def.</u> "A "labeling" is a mapping defined in ZF, which assignes to each ordinal $0<\alpha<\alpha_0$ a set S_α, the "label space", and functions φ_α defined in S_α such that the sets S_α are disjoint and if $c \in S_\alpha$, $\varphi_\alpha(c)$ is a formula $A(x)$, <u>which may have elements of S_β with $\beta<\alpha$ appearing as constants</u>. [...]
The set S_0 is defined as $\omega \cup \{a\}$ where a is a formal symbol. We write $S = \bigcup_\alpha S_\alpha$."

Overcoming a Fregean "metaphysical barrier": variables of formulae range over formulae of the same type!

Cf. in algebra $p(x) = x^2+1$, ~~x=t~~

$q(x) = x^2-1$

$p(q(x)) = (x^2-1)^2+1 = x^4-2x^2-2$

Take $G \subseteq S = \bigcup_\alpha S_\alpha$: "generic sets": such that $M(G) = N$ is a <u>model</u>

"All our sets and variables are actually functions of the generic set (a). So in analogy with field theory, we are actually dealing with the space of all (rational) functions of (a), not actual sets" (ibid)

"Ideal character" of Cohen's construction:

"How could one decide whether a statement about ⓐ is true before we have ⓐ? In a somewhat exaggerated sense, it seemed that I would have to examine the very meaning of truth and think about it in a new way" (The Discovery of forcing, 2002)

Cf. Kummer's case: Cohen's is stronger

Cf. "I think most mathematicians, as ~~distinct~~ distinct from philosophers, will not find much interest in the various polemical publications of even prominent mathematicians. My personal opinion is that this is a kind of "religious debate." (ibid.)

Back to syntax: forcing relation $\Vdash$

Forcing condition in M $\Vdash$ truth in $M(G)$ (for fixed G "generic")

e.g.

③ P forces $\sim B$ iff for all $Q \supseteq P$ Q does not force B | think of intuit. negation and its Kripke's semantics

5. Forcing in a topos-theoretic setting: geometrical intuition strikes back.

Scott & Solovay 1965 (!): Reformulation of Cohen's construction with Boolean-valued models of ZF.

1. Set P of forcing conditions is made into a Boolean algebra.
2. The obtained Boolean-valued model is collapsed into a binary-valued model along a ultrafilter (maximal filter)

Tierney 1972 (joint work with Lawvere): reformulation of Cohen's construction in a topos-theoretic setting.

My presentation: MacLane & Moerdijk 1992 "Sheaves in Geometry and Logic"

Th1 Topos is Boolean iff $\neg : \Omega \to \Omega$ satisfies $\neg\neg = id_\Omega$

closure operator

Th2 $\neg\neg$ is a (Lawvere-Tierney) topology.

It is discrete topology: every subset is clopen (is this still a topology? YES!)

Morphisms are defined by their points.

In order to violate CH in a topos (of sets) we look for (set) B such that

$$N \overset{h}{\hookrightarrow} B \overset{g}{\hookrightarrow} 2^N$$ properly, i.e. monos h, g are not isos.

Def.: monos are right-cancellable: for all f, s: $fg = sg \Rightarrow f = s$

In a given topos Set take B with $\mathrm{Card}(B) > 2^N$, so there is no mono $B \hookrightarrow 2^N$

Consider poset P of forcing conditions ...

$$(B, N, 2) \Rightarrow (\tilde{B}, \tilde{N}, \Omega) \underset{a:\text{ right adjoint}}{\rightleftarrows} (\hat{B}, \hat{N}, \Omega_{\neg\neg})$$

in Set; in topos of presheaves $\mathrm{Set}^{P^{op}}$ $(\Omega \approx P)$; in topos $Sh(P, \neg\neg)$ (Cohen's boolean topos of double-negation sheaves)

? $\tilde{g}: \tilde{B} \hookrightarrow \Omega^{\tilde{N}}$

$p \in P$: a „stage of knowledge about $\tilde{g}$"

$$\tilde{g}(p): \tilde{B}(p) \hookrightarrow \Omega^{\tilde{N}}(p) = \mathrm{Nat}(y(p) \times \tilde{N}, \Omega)$$

$\tilde{B}(p) = \tilde{B}$

take $y(p) = \mathrm{Hom}(-, p)$

the set of incoming morphisms of p: mind the reversal of arrows!

Def.: $C = A^B$ iff

$$\begin{array}{ccc} C \times B & \to & A \\ !\uparrow & \nearrow & \\ C' \times B & & \end{array}$$

If topology P is $\boxed{\text{dense}}$ then $y(p)$ is in $\Omega_{\neg\neg}$

Intuitively: the chain of P-s converges!

in $Sh(P, \neg\neg)$

$$\hat{N} < \widehat{2^{N}} < \hat{B} < \hat{2}^{\hat{N}}$$

$$\hat{N} \hookrightarrow \widehat{2^{N}} \hookrightarrow \hat{B} \hookrightarrow \hat{2}^{\hat{N}}$$

"fake continuum" inherited from Set

"real" continuum in $Sh(P, \neg\neg)$

"Sheaves for the dense topology on the poset of "finite stages of knowledge" about the desired impossible monomorphism [$B \hookrightarrow 2^{N}$ in Set] form the new model of sets in which that mono is really there!" (MacLane & Moerdijk)

Cohen's syntactic notion of forcing is a method to describe truth in $Sh(P, \neg\neg)$

Kripke-Joyal Semantics

for posets of "stages"

for any small category of "stages"

6. Conclusion

6.1 Topos-theoretic rendering of Cohen's forcing provides it with a precise and intuitively appealing geometrical background. (This intuition requires an appropriate training.) There remain nothing "ideal" in the topos-theoretic forcing construction.

6.2 A reason of failure of intuitionistic programs in mathematics of 20th century is that these programs were mostly reactionary: they tried to forbid certain ways of ("classical") reasoning (while the adversary set-theoretic programs offered something really new...) CT program ~~might~~ performs better and gives mathematical intuitionism (in a broad Kantian sense) a new chance.