Knowing-How and the Deduction Theorem

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Abstract:

In his seminal address delivered in 1945 to the Royal Society Gilbert Ryle considers a special case of knowing-how, viz., knowing how to reason according to logical rules. He argues that knowing how to use logical rules is not an instance of propositional knowledge. We evaluate this argument in the context of two different types of formal systems capable to represent knowledge and support logical reasoning: Hilbert-style systems, which mainly rely on axioms, and Gentzen-style systems, which mainly rely on rules. We build a canonical syntactic translation between classes of such systems and demonstrate the crucial role of Deduction Theorem in this construction. We show how the standard model-theoretic conception of logical consequence supports a reduction of knowing-how to knowing-that but argue that such a reduction is untenable because this conception of consequence is not appropriate in epistemological contexts. Finally we extend our analysis to the case of extra-logical knowledge-how and discuss a number of open questions, which concern translations between knowledge-how and knowledge-that in this more general semantic setting.

Keywords: Knowledge-How Anti-Intellectualism Axiomatic Styles Deduction Property Carroll's Paradox Proof-theoretic Semantics

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1 Knowing-How without Anti-Intellectualism

1.1 Intellectualism and Anti-Intellectualism

In November 1945 Gilbert Ryle gave his Presidential Address to the Aristotelian Society [24], which produced a wide epistemological debate about the concept of knowledge-how. This continuing debate [3] has been recently summarized in the following words:

> There are two main camps in the debate about the constituent concepts of knowledge-how. One camp, intellectualism, argues that knowledge-how involves propositional knowledge [...], whereas the competing camp argues that knowledge-how does not involve propositional knowledge - a view called anti- intellectualism. According to anti-intellectualists, whereas propositional knowledge is a certain type of belief, knowledge-how consists in abilities, skills, or dispositions [...]. [9, p. 2930]

Even if the titles of intellectualism and anti-intellectualism are often used in this debate as mere labels pointing to certain accounts of propositional and non-propositional knowledge, the choice of these words is not arbitrary. Ryle calls the "intellectualist legend" an epistemological thesis that all knowledge is knowledge-that, i.e., a knowledge of certain proposition or class of propositions. He argues that this view of knowledge leaves aside a sort of knowledge needed for various actions such as riding a bicycle or making logical inferences (more on this last example below), i.e., knowledge-how. Ryle's use of word "intellectualist" has a clear pejorative connotation: intellectualists are people who know a lot but are incapable to undertake an action.

1.2 Rules and Sentences.

As we shall now argue this Ryle's terminological decision is not simply unfortunate but reflects a genuine conceptual confusion, which continues to affect the epistemological debate on knowing-how up to the present.

One issue, which is central in this debate, is the distinction between knowing a proposition and knowing how to act. Another issue, which is widely discussed as a part of the same debate, is the distinction between tacit and explicit knowledge. The popular example of knowing how to ride a bicycle instantiates both these features: it is a knowledge-how and it is tacit because even an experienced rider usually cannot explain in words how she rides a bicycle and cannot transfer this knowledge to another person by linguistic means. Nevertheless it is wrong, we claim, to generalize upon this and similar examples of tacit knowing-how. The two aforementioned distinctions should be analyzed separately. The idea behind the term "anti-intellectualism" according to which the knowledge-how, generally, has an intrinsic tacit character (unless it is represented in a propositional form) is misleading. To see this consider another example of knowing-how that Ryle uses in the same paper, viz., the case of one's knowing how to reason logically. Ryle introduces this example via the following imaginary dialogue:

> [T]he intelligent reasoner is knowing rules of inference whenever he reasons intelligently'. Yes, of course he is, but knowing such a rule is not a case of knowing an extra fact or truth ; it is knowing how to move from acknowledging some facts to acknowledging others. Knowing a rule of inference is not possessing a bit of extra information but being able to perform an intelligent operation. *Knowing a rule is knowing how.* (The emphasis is added by the authors.) [24, p. 7].

Just like a proposition a rule of logical inference allows for an explicit linguistic expression — either in a natural language in the form of imperative sentence or in a symbolic logical calculi in the form of syntactic rule. A competent reasoner who makes an inference according to certain rule R (say, the rule of *modus ponens*), and is aware about this fact, instantiates a case of *explicit* knowing-how. Rule R expressed linguistically or symbolically *represents* one's knowledge of how to act according to this rule in the same sense in which a sentence may represent one's knowledge of the proposition that this sentence expresses. Clearly, logic is not the only domain where explicit rules play a role. Explicit rules are abundant in games, social and political life, technology and in many other domains. Representation of knowledge-how in the form of rules and instructions is a way of making this knowledge explicit. Learning how to act according to linguistically expressed formal rules and instructions is, at least partly, an intellectual activity. This is why, in our view, the title of "anti-intellectualism", which refers to the allegedly tacit character of knowledge-how, is not an appropriate name for the view according to which knowledge-how is epistemically significant and not reducible to propositional knowledge.

It is generally agreed that knowing a proposition involves such a propositional attitude as *belief.* What sort of attitude or relation an epistemic agent needs to hold to a given rule in order to qualify as a competent knower of this rule? One the one hand "[a] silly pupil may know by heart a great number of logicians' formulae without being good at arguing." [24, p. 7]. It is clear that knowing a rule by hart doesn't imply its knowledge — just like knowing a sentence by hart doesn't imply knowledge. One's understanding of a rule and willingness to implement it doesn't imply one's knowledge of this rule in Ryle's intended sense either. On the other hand, "[t]he sharp pupil may argue well who has never heard of formal logic." (*ibid.*) just like a bird may fly well without knowing aeromechanics. In this latter case, which Ryle qualifies as a proper instance of knowledge-how, the relation between a rule and a knower doesn't involve the knower's awareness (or at least linguistic awareness) of the rule. If such a relation between the rule and the agent also counts as the agent's *knowledge of the rule* (which is controversial), such sort of knowledge-how should be distinguished from one, which requires that the agent is aware of the rule and implements it intentionally.

Thus the answer to the above question hardly has a single answer: there are different possible epistemic relations and attitudes to rules, which give rise to different sorts of knowing-how. In this paper we shall not further explore such different relations and attitudes and shall not try to represent them formally. Instead we shall study the mutual roles of rules and sentences in a large class of formal symbolic calculi and, assuming that rules and sentences may represent the corresponding two sorts of knowledge, viz., knowledge-how and knowledge-that, we make on this basis some epistemological conclusions. In particular, in this way we shed some new light on the widely discussed question of whether or not knowing-how in some sense reduces to knowing-that [26], [3].

2 Two Styles of Axiomatic Thought

In this Section we introduce the standard informal distinction between the so-called Hilbert-style and Gentzen-style symbolic calculi and then explain its relevance to the epistemological debate on knowing-how and knowing-that.

2.1 Hilbert-Style

The standard modern notion of axiomatic theory stems from David Hilbert's seminal work in foundations of geometry [10]. The idea here is to generate the intended theory T from a list of *axioms* A_i by inferring from the axioms further propositions called *theorems*. More precisely, axioms in this setting are *propositional forms*, which become full-fledged propositions under an *interpretation*, which is an assignment to non-logical terms of A_i certain semantic values borrowed from other theories or, less formally, simply from the "world out there" . In his later joint work with Ackermann Hilbert applies the same axiomatic approach to logic itself and presents it in a form of symbolic calculus via a list of axioms (which in this case are tautologies) and syntactic rules, which generate from the axioms all other tautologies [11].

Hilbert never explicitly elaborated on the concept of logical inference but it is plausible that at least in his most influential [10] he had in view a prototype of the model-theoretic truth-conditional semantical concept of logical consequence later made explicit by Alfred Tarski [27]:

Definition 1 *Propositional form B is a logical consequence of propositional forms $A_1, \dots, A_n$ iff every interpretation I of the given language, which makes $A_1, \dots, A_n$ into true propositions $A_1^I, \dots, A_n^I$ makes B into true proposition B^I, in symbols $A_1, \dots, A_n \models B$.*

Notice that this conception of logical consequence does *not* involve that of rule.

2.2 Gentzen-Style

In 1935 Hilbert's associate Gerhard Gentzen published a paper [6] where he argued that

> The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. [6, p. 68]

and proposed an alternative approach to syntactic presentation of deductive systems, which involved relatively complex systems of rules and didn't use logical tautologies. In [6] Gentzen builds in this way two formal calculi known as Natural Deduction (ND) and Sequent Calculus (SC).

Gentzen further remarks that

> The introductions [i.e. introduction rules] represent, as it were, the 'definitions' of the symbol concerned. [6, p. 80]

This remark is seen today by some authors as an origin of an alternative conception of logical consequence and alternative logical semantics more generally, which has been developed in a mature form only in late 1999-ies or early 2000-ies and is known today under the name of *proof-theoretic* semantics (PTS). It is instructive to compare Gentzen's idea to use syntactic rules as a form of implicit definitions with Hilbert's use of axioms as implicit definitions. The two approaches may appear to be very similar but in fact they are not. Think of the usual axioms of Group Theory. These axioms serve as a definition of the group concept in the following sense: any structure, which *satisfy* the axioms, i.e., is their model, is a group. The relevant concepts of satisfaction and model have been made precise by Tarski and are now standard. But what kind of entity X can possibly "satisfy" a rule or a system of rules, so one could claim that the rules "define" X in some reasonable sense? How the satisfaction relation (if it can be used here at all) needs to be construed in this case?

PTS provides some answers to these and other related questions [25], [4], [20]; here we only highlight some key features of PTS, which are important for our following discussion.

- Proof Theory referred to in PTS is *not* the proof theory in Hilbert's sense of the word [12] where a proof is identified with a formal derivation and then made into an object of a meta-mathematical study, but the *General Proof Theory* (GPT) due to Dag Prawitz [21], [22]. In a recent paper Prawitz describes the difference between the proof-theoretic and the standard truth-conditional approaches to semantics as follows:

  > [I]n contrast to a truth-conditional meaning theory, [in PTS] one should explain the meaning of a sentence in terms of what it is to *know* that the sentence is true, which in mathematics amounts to having a proof of the sentence. [20, p. 5-6].

This quote points to a strong conceptual link between PTS, on the one hand, and intuitionisitc and constructivist approaches in logic and foundations of mathematics, on the other hand. We shall explore this link in Section 5.

- PTS is motivated by a broad philosophical view on meaning (and hence on semantics), which is conventionally called “meaning-as-use”. This view on meaning goes back to Wittgenstein and more recently has been defended by Robert Brandom [1] under the name of *inferentialism*. Since PTS is a *formal* semantical approach the reference to “use” amounts here to referring to syntactic *rules*, which specify the use of symbols and symbolic expressions in logical calculi.

- PTS is not denotational: it does not assign entities to symbols. It assigns to symbols their *meaning*, which is not construed in this case as an entity. The procedure of such an assignment is called after Martin-Löf the *meaning explanation* and consists, roughly, of the explication of computational content of logical constructions in terms of their building blocks, which are presented in a self-explanatory canonical form. Martin-Löf compares a meaning explanation with a program compiler, which translate a program written in some higher-level programming language into a lower-level command language [17].

- The general PTS conception of logical consequence is as follows: B is a logical consequence of $A_1, \ldots, A_n$ iff there exists a proof of B from assumptions $A_1, \ldots, A_n$. Further details depend on what exactly qualifies as a proof. The standard approach here is to identify proofs with derivations (and hence the logical consequence with the derivability) in a suitable deductive system such as ND [18]. However a more nuanced approach has been recently offered where a PST-based conception of logical consequence is construed in more abstract terms and is distinguished from the derivability [5].

2.3 Comparison of the Two Styles

The difference between Hilbert-style and Gentzen-style formal systems is usually described in the recent literature by saying that Hilbert-style systems are typically presented by long lists of axioms or axiom schemes and only few (typically one) rules, while Gentzen-style systems are presented by a small (possibly empty) sets of axioms and long lists of rules . This is a very loose and informal description — as are the ways in which these titles are actually used in logicians' professional parlance. In order to be in a position to describe the two axiomatic "styles" more rigorously we introduce in the next Section the concept of (propositional) *Hilbertian theory* (Def. 11) which is more narrow than what people may call a Hilbert-style propositional theory. We shall not provide a complementary syntactic definition of Gentzen-style theory because in what follows we use in its stead a general syntactic of symbolic calculus (Def. 2). Since any axiom A can be straightforwardly read as a rule of form $\vdash A$ with the empty set of premises, at the syntactic level Gentzen's approach is more general than Hilbert's. So we shall study the place of Hilbert-style theories in this more general syntactic setting.

Tarski's truth-conditional semantics and PTS do not depend directly on syntactic details. Nevertheless both for historical and conceptual reasons it is natural to associate the truth-conditional semantics with Hilbert-style and PST with Gentzen-style. Such semantic assumptions, once again, are stronger that the current use of titles "Hilbert-style" and "Gentzen-style" may suggest. Authors often use these titles referring only to the syntax without any semantic commitment. Our formal definitions in the next Section are also purely syntactic, so in the formal part of the paper we don't go against the common language. However in the following epistemological discussion the semantic aspects of the two axiomatic styles turn to be crucial.

2.4 Two Axiomatic Styles and the Debate on Knowing-How and Knowing-That

We assume that axiomatic theory T represents a piece of knowledge; in other words, we assume that T can be known by an epistemic agent. Since T, generally, comprises propositions (axioms and theorems) and rules of inference, we further assume that one's knowledge of T splits accordingly into a propositional (knowledge-that) and a procedural (knowledge-how) parts. Since every non-trivial axiomatic theory comprises at least one rule of inference one's knowledge of a theory always comprises a procedural part. But in Hilbert- and Gentzen-style theories the procedural and the propositional knowledge are distributed in different ways. A study of syntactic translations between Hilbert- and Gentzen-style presentations sheds a light on the issue of translatability of procedural knowledge into a propositional form and vise versa. The question of whether or not the procedural knowledge in some reasonable sense reduces in a formal axiomatic setting to the propositional knowledge is treated in Section 4.

3 Translation between the two axiomatic Styles and the Deduction Theorem

In this Section we study the syntactic translatability between Hilbert-style and Gentzen-style systems and show the role of Deduction Theorem. This material is by and large standard but in the view of our epistemological purpose we present it here in a more general form than usual.

3.1 Hilbertian Theories

Definition 2 *Symbolic calculus* comprises:

- alphabet of symbols;
- a set of words w_i built with the alphabet;
- a set of rules r_i of form $w_1, \dots, w_k \vdash w$, which derive word w from given words $w_1, \dots, w_k$;
- set A (possibly empty) of axioms which are rules of special form $\vdash w$.

Definition 3 *Propositional language* is a calculus with a distinguished finite set of symbols called *connectives*, which includes connective "$\rightarrow$"; other symbols are called *propositional variables*.

Definition 4 *Propositional theory* is a set T of formulae closed under application of the standard *modus ponens* (MP) (other rules are allowed but not required). Elements of T are called *theorems* of the given theory. The theory is called *axiomatic* when it comprises a distinguished subset $A \subset T$ of *axioms* such that all theorems of T are derivable from the axioms via applications of MP. The notion of derivation from a set Γ of hypotheses (denoted $\Gamma \vdash_T F$ or $\Gamma \vdash F$ when there is no risk of confusion) is standard.

Definition 5 An axiomatic theory is called *Hilbertian* when it comprises as theorems all formulae of the form $K_{A,B}$ and $S_{A,B,C}$ where

$$\begin{aligned} K_{A,B} &\doteq \quad A \rightarrow (B \rightarrow A) \\ S_{A,B,C} &\doteq \quad (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \end{aligned}$$

and has exactly one rule, namely MP.

3.2 Deduction Property

Definition 6 Theory T is said to have the *Deduction Property* (DP for short) if $\Gamma, F \vdash G$ entails $\Gamma \vdash F \rightarrow G$ for all Γ, F and G.

DP allows one to represent a rule $A \vdash B$ by the implication $A \rightarrow B$, which is a proposition. We assume that one's knowledge *how* to derive B from A is represented in this case, accordingly, by the knowledge *that* A implies B. Our next Lemma shows that the concept of Hilbertian theory and that of theory with Deduction Property are co-extensional:

Lemma 7 *An axiomatic propositional theory is Hilbertian if and only if it has the Deduction Property.*

Proof:

"$\Rightarrow$" (the "only if" part). The standard proof of the Deduction Theorem [13].

"$\Leftarrow$" (the "if" part). By the definition of derivation in propositional theories we have $A, B \vdash A$. Using the deduction property twice we get from the former formula $\vdash A \to (B \to A)$. Similarly, by using twice the deduction property from $A \to (B \to C)$, $A \to B$, $A \vdash C$ we get $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$. $\lhd$

Lemma 7 says that the Deductive Property is a proper feature of Hilbertian theories (Definition 11), which other propositional theories do not possess. Among popular logical calculi without Deduction Property are many versions of Quantum Logic [16]; another family of useful calculi without DP has been more recently introduced in Computer Science [7], [8], [15].

3.3 Translating between the Two Styles

DP and Lemma 7 bear onto the distribution of propositional and procedural knowledge *within* a Hilbert-style axiomatic setting. Our more general goal is to understand how the procedural knowledge represented with a Gentzen-style system may (or may not) translate into a Hilbert-style system and vice versa. At the time of writing we don't have the full answer to this question in the form of necessary and sufficient conditions. Below we present some partial results: we show that Hilbertian theories allow for a canonical translation into a sequential form (Lemma 9 below) and then specify a class of Gentzen-style systems which canonically translate into Hilbertian theories (Lemma 10). We need a preliminary lemma:

Lemma 8 *All axiomatic propositional theories have the following property (rule* $(\to\vdash)$*): if* $\Gamma \vdash F$ *and* $\Gamma, G \vdash H$ *then* $\Gamma, F \to G \vdash H$*.*

Proof:

Given the two above derivations form the following sequence of formulae: $\Gamma \vdash F$, $F \to G$, $\Gamma, G \vdash H$. The sequence qualifies as a derivation $\Gamma, F \to G \vdash H$. (Some formulae may enter into this sequence more than once but the definition of derivation does not rule this possibility out.) $\lhd$

From a Hilbertian theory to its sequential presentation:

Let T be a Hilbertian theory. Consider the following sequential calculus T_G. Sequences in T_G are of form $\Gamma \Rightarrow F$. Rules of T_G comprise all structural rules (axioms of form $F \Rightarrow F$, contraction, weakening and the cut rule). For each axiom A of T there is the corresponding sequence $\Rightarrow A$ in T_G. Finally T_G has the usual rules for implication, namely:

$$\frac{\Gamma \Rightarrow F \quad \Gamma, G \Rightarrow H}{\Gamma, F \to G \Rightarrow H}\,(\to\Rightarrow) \qquad \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \to G}\,(\Rightarrow\to)$$

Lemma 9 *$\Gamma \vdash F$ in theory T if and only if sequence $\Gamma \Rightarrow F$ is derivable in T_G.*

Proof:
"$\Rightarrow$" (the "only if" part): Induction by $\Gamma \vdash F$. If F is an axiom of T or member of Γ then $\Gamma \Rightarrow F$ is derivable using structural rules. Given the cut MP is admissible:

$$\dfrac{\Gamma \Rightarrow A \qquad \dfrac{\dfrac{\Gamma \Rightarrow A \to B}{\Gamma, A \Rightarrow A \to B} \qquad \dfrac{\Gamma, A \Rightarrow A \quad \Gamma, A, B \Rightarrow B}{\Gamma, A, A \to B \Rightarrow B}\,(\to\Rightarrow)}{\Gamma, A \Rightarrow B}\,(Cut)}{\Gamma \Rightarrow B}\,(Cut)$$

"$\Leftarrow$" (the "if" part): Translation that replaces all entries of $\Rightarrow$ by $\vdash$ is sound with respect to all rules of T_G. See Lemmas 7,8. $\lhd$

From a sequent calculus to a Hilbertian theory:

Let sequent calculus T_G contain all structural rules and rules $(\to\Rightarrow)$, $(\Rightarrow\to)$ are admissible in T_G. T_G may also contain other rules. Consider set

$$T = \{F \mid \text{sequence} \Rightarrow F \text{ is derivable in } T_G\}$$

Lemma 10 *T is a propositional Hilbertian theory*

Proof: Axioms are all formulae of T. Since MP is admissible in T_G, T is closed with respect to MP. Formulae of forms $(K_{A,B})$ and $(S_{A,B,C})$ are elements of T because sequences $\Rightarrow K_{A,B}$ and $\Rightarrow S_{A,B,C}$ are derivable using rule ($\Rightarrow\rightarrow$) and the structural rules. $\triangleleft$

Lemma 9 tells us that a Hilbertian theory admits a translation into a sequential Gentzen-style form, which preserves and reflects its deductive properties. Lemma 10 says that a sufficiently strong sequent calculus admits a translation into a Hilbertian theory.

3.4 Richer Systems

The minimal setting where the Deduction Theorem holds (i.e., which has the Deduction Property) is the *minimal* logic ML, which comprises only one connective $\rightarrow$, all axioms $K_{A,B}; S_{A,B,C}$ and one rule MP (compare Def. 11 above). So the question whether or not in a given theory T has DP is the question of whether or not T interprets ML. Hilbertian theories interpret ML fully and faithfully in the sense that every derivation in T has a pre-image in ML. But if one adds to MP some further rules then DP, generally, fails to hold as it happens, for example, in various systems of modal logic. In some such cases DP can be forced by an appropriate correction of additional rules. Thus the Deduction Theorem can be proved for usual First-Order Logic if one uses a convention according to which the usual rule of universal generalization with hypotheses

$$\frac{\Gamma \vdash P(x)}{\Gamma \vdash \forall x.P(x)}$$

applies only if Γ does not contain variable x in the free form. Without this additional requirement DP fails to hold.

For these reasons the core content DP can be fully understood and studied at the propositional level. Richer systems may have this property when they are used in a restricted form, which is essentially a way to emulate the propositional reasoning in such systems.

4 An Attempted Reduction of Knowing-How to Knowing-That

In the last Section we have seen that Hilbertian theories allow for a smooth passage from Hilbert-style presentation to Gentzen-style and vice versa. Does this property of Hilbertian theories allow for a full "reduction" of knowing-how to knowing-that?

Any axiomatic theory deserving the name comprises at least one rule of inference (usually MP). Recall, however, that the Tarskian semantic concept of logical consequence does not involve the concept of rule. This allows one to think of rule $A \vdash B$ (granting soundness of the given theory with respect to its semantics) as a mere symbolic representation of *relation* $A \models B$, which is a fact of the matter fully explained in terms of truth-conditions. Under this reading formula $A \models B$ stands for meta-theoretical *proposition*:

($SC^{A \models B}$): All models of A are models of B,

Accordingly, so the argument goes, knowledge of rule $A \vdash B$ reduces to knowledge of $A \models B$, which is propositional knowledge. Now we shall discuss some details of this argument, reply to some possible objections and finally present our own objection showing that the argument is invalid.

4.1 Carroll Paradox

Notice that an attempted replacement of rule $A \vdash B$ by proposition $A \to B$ with a help of DP does not go through because it leads to an infinite regress known as Carroll Paradox [2]:

$A \vdash B$ if and only if $\vdash A \to B$
$A, A \to B \vdash B$ if and only if $A \vdash (A \to B) \to B$
$A, (A \to B) \to B \vdash B$ if and only if $A \vdash ((A \to B) \to B) \to B$
....

where each application of DP increases the number of implication signs in each formula by one [1]. Since our proposed reduction does not rely on DP it is immune to Carroll Paradox in this straightforward form. Indeed, $SC^{A \models B}$ is not a theorem of the same theory T where $A \vdash B$ belongs; $SC^{A \models B}$ belongs to the model theory M_T of T. But one may argue that in order to make use of $SC^{A \models B}$ one needs to apply certain logical rules at the meta-theoretical level, which once again opens Carroll's infinite regress in a new form. To block this objection it is sufficient to remark that one doesn't need to think of M_T as a

[1]Here we follow [24, p. 6-7]

theory on equal footing with T; in fact the above argument doesn't require that M_T has any deduction structure at all. We can describe the proposed propositional reduction in a clearer form by replacing the usual syntactic notion of theory T by a semantic conception of theory T_S which extends the syntax of T with a single symbol $\models$ and rule

$$(\vdash\models): \qquad \frac{A \vdash B}{\vdash A \models B};$$

Expressions of form $A \models B$ stand in T_S for propositions $SC^{A\models B}$. Such expressions are sterile in the sense that they cannot be used in derivations along with usual formulas (the use of syntactic rules in T_S is restricted accordingly); their sole role is to make explicit the model-theoretic semantics of derivations. We allow T_S to comprise true sentences of form $\Gamma \models B$ (and, in particular, $\models B$: Gödel sentences) when T does not provide corresponding derivations $A \vdash B$ ($\vdash B$). So we get a theory and its rudimentary model theory in one pocket. Since the rudimentary theory in question is deductively sterile such a combination doesn't produce an inconsistency. Now the reduction of knowing-how (i.e., knowing rules) to knowing-that along the above lines proceeds wholly within a single theory T_S.

4.2 Logical Consequence and Logical Inference

The model-theoretic conception of logical consequence has a number of features, which makes it vulnerable to an epistemological critique. Prawitz argues that it involves a form of circularity and is uninformative [22, p. 67-68)]. A part of the problem is that the extension of expression "all models of theory T" is not precisely defined. Should one think here only about the "real world models" developed in natural sciences, models borrowed from other parts of mathematics or models built with a logically informed metaphysical speculation? If one determines some domain D of all possible models of T using another theory S then the above semantic construction of T_S is no longer self-sustained because now its semantic part is essentially determined by theory S (which, generally, cannot be incorporated into T_S as above on pain of inconsistency). In that case a critic arguing that the model-theoretic conception of logical consequence opens a regress will be right. There are two ways of preventing this regress from being infinite. One option, is to give theory S an exceptional epistemic status of being *the* universal theory of most general features of the world (or even of all possible worlds). The traditional name of a theory, which may fit this description, is metaphysics. Another option is to diversify the concept (but not the general conception, which remains model-theoretic in all such versions) of logical consequence by making it dependent on S. In this way one can conceive of, for example, of Quantum Logic as an interpreted logical calculus, which represents symbolically the corresponding special semantic notion of logical consequence that draws on Quantum Physics.

However all these versions of the model-theoretic conception of logical consequence leave the idea of an epistemic *act* (such as acknowledgement, rejection, verification, falsification or questioning a proposition) outside of logic proper and place it into the disciplinary domains of Psychology, Sociology and other disciplines, which study the "context of discovery". This is why one who believes that the concept of epistemic act is fundamental and regards logic as a normative discipline that tells one how to perform such acts properly, cannot accept the idea that the model-theoretic conception of logical consequence provides a complete account of logical inference [2]. For example, to the best of today's mathematical knowledge the relation of model-theoretic logical consequence $ZFC + U \models FLT$ where $ZFC + U$ is a strengthened version of Zermelo-Fraenkel Set theory [19] and FLT is Fermat Last Theorem, holds. However this model-theoretic relation by itself does not constitute a *proof* of FLT and does not validate the inference from $ZFC + U$ to FLT. Recall that under the PTS-based conception of logical consequence FLT counts as a consequence of $ZFC + U$ only when there exists a proof of FLT from axioms of $ZFC + U$ used as assumptions.

The two conceptions of logical consequence represent the two opposite sides of a genuine philosophical controversy about the nature and the scope of logic. The model-theoretic conception is associated with a view on logic as a tool for managing truths and falsities disregarding the question of how these things are known. The proof-theoretic conception is associated with a view on logic as an epistemic tool providing norms and techniques of proof. However in order to evaluate the argument given in the beginning of this Section we don't even need to go deep into this controversy. This argument

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[2]

> [K]nowing ... a rule [of logical inference] is not a case of knowing an extra fact or truth ; it is knowing how to move from acknowledging some facts to acknowledging others. [24, p. 7]

supports the claim according to which knowledge-how is a special case of knowledge-that and thus is epistemological in its character. At the same time it essentially uses a formal logical semantics, viz., the model-theoretic semantics, which deliberately leaves epistemological issues aside. As soon as this semantics is used in an epistemological argument it *must* be evaluated from an epistemological viewpoint in its turn. But when one considers the relation of logical consequence from an epistemological viewpoint one comes up with a different logical semantics, viz., the proof-theoretic semantics, which no longer supports the argument. This shows that the purported reduction of knowledge-how to knowledge-that does *not* go through. Knowing a logical rule does not reduce to knowing a proposition via the model-theoretic semantic account of logical consequence because this account of logical consequence is not appropriate for answering epistemological questions. Saying this we do not claim that this standard notion of logical consequence is incoherent and cannot be used for some other purposes.

5 Constructive Theories

The explicit form of knowing-how, i.e., knowing how to follow certain formal rules, is not limited to logic. In this Section we extend our analysis beyond the "logical" knowing-how, i.e., knowing how to make logical inferences. Notice that in the preceding part of the paper we did not apply any formal criterion of logicality. The syntactic part of our analysis did not involve anything (except some traditional names), which made it specific to logic. Now we shall consider some non-logical interpretations of the same or similar syntactic constructions. For a suggestive example, which demonstrates this approach, think of Kolmogorov's *calculus of problems* **CP** [14]. Syntactically **CP** is identical to the standard intuitionistic propositional logic but has a different intended semantics known as BHK semantics. This semantics is not logical or at least not logical in a narrow sense of the word: formulae represent here problems rather than propositions. Following [23] we call hereafter formal theories, which comprise rules for non-propositional objects, *constructive* theories.

5.1 Constructive Deduction Theorem

Let T be a Hilbertian theory. We associate now with T a typed sequential calculus CT, which is more apt to standard PTS-style constructive interpretations than the sequential calculus T_G from 3.3 above. We prove the deductive equivalence between T and CT (Lemma 12) and, finally, prove for CT a "constructive version" of Deduction Theorem (Theorem 13), which gives us some insights about extra-logical forms of knowing-how.

Definition 11 CT comprises:

- Types of CT are all formulae of T;
- With each axiom A of T associate constant c^A, which we interpret as the trivial derivation of A in T. In the cases of axioms $(K_{A,B})$ and $(S_{A,B})$ we use the established notation and denote the corresponding constants as $k^{A\to(B\to A)}$ and $s^{(A\to(B\to C))\to((A\to B)\to(A\to C))}$ omitting the upper index when this cannot cause a confusion.
- Terms of CT correspond to derivations in T; these terms are built from variables and constants with a single binary operation (multiplication), which is an application of rule MP. Each such term determines a unique binary tree such that its internal nodes are marked by MP and its leaves (??) correspond either to T-derivations of axioms or to variables. Rules of CT specify when this tree is the correct tree of derivation from hypotheses in T.
- Sequences of CT are expressions of form

  $$x_1{:}\,F_1, \ldots, x_n{:}\,F_n \vdash t{:}\,F,$$

  where $x_1, \ldots x_n$ are mutually different variables, $F_1, \ldots F_n, F$ are types (formulae) and t is a term. Sequences determine the same trees but comprise an additional markup: they put label F to the root and attach mark F_i to each leave x_i , which signifies that x_i is a variable over derivations of formula F_i). The obtained tree can get new isolated nodes marked by variables, which are not elements of term t; leaves, which are not in the list $x_1, \ldots x_n$ may remain unmarked.
- Axioms and rules of CT:

– $x_1{:}F_1,\ldots,x_n{:}F_n \vdash c^A{:}A$, where A is an axiom of T,

– $x_1{:}F_1,\ldots,x_n{:}F_n \vdash x_i : F_i$,

– $$\frac{x_1{:}F_1,\ldots,x_n{:}F_n \vdash u{:}(F \to G) \quad x_1{:}F_1,\ldots,x_n{:}F_n \vdash v{:}F}{x_1{:}F_1,\ldots,x_n{:}F_n \vdash (u \cdot v){:}G}.$$

Lemma 12 *Every derivable sequence $x_1 : F_1,\ldots,x_n : F_n \vdash t : F$ in CT corresponds to a unique derivation $F_1,\ldots F_n \vdash F$ in* **T**. *Each derivation $F_1,\ldots F_n \vdash F$ in* **T** *corresponds to a unique term t such that its associated sequence $x_1{:}F_1,\ldots,x_n{:}F_n \vdash t{:}F$ is derivable in CT.*

Proof: Induction by derivations.
Remark: When variable x_i is not an element of t the introduction and the elimination of declaration $x_i : F_i$ to/from the given context does not affect the derivability of the sequence. These operations correspond to introduction and elimination of hypothesis, which is not used in the derivation.

Theorem 13 (*"Constructive" Deduction Theorem* or CDP) If sequence $x_1{:}F_1,\ldots,x_n{:}F_n, x{:}F \vdash t : G$ is derivable in CT, then there exists term u such that sequence $x_1{:}F_1,\ldots,x_n{:}F_n \vdash u{:}(F \to G)$ is also derivable.

Proof: This follows immediately from Lemma 12 and the standard Deduction Theorem. In Appendix 1 we provide a direct proof, which is instructive because it makes explicit the computational content of Theorem 13.

The standard PTS-style constructive reading of Theorem 13 is as follows: if in the given context one is in a position to produce from a given token x of type F a new token t of type G then one is also in a position to produce in the same context a token u of type $F \to G$, i.e., a *method* of producing tokens of G from tokens of F. In this framework "method" u is an object on equal footing with tokens of other types such as F and G. Here is a dummy example: if one knows how to produce porridge from oat one also knows how to produce a method of cooking porridge from oat - say, in the form of written recipe. This property of Hilbertian systems can be very useful in applications but at the same it would be unreasonable to expect that everyone who knows how to cook porridge also knows how to write cooking books!

5.2 Tacit Knowledge Revisited

The constructive Deduction Property sheds some light on the issue of the allegedly "tacit" character of knowing-how in many practical examples such as that of riding a bicycle. So far we called knowledge-how explicit when it involved knowing certain explicitly written rules. Let us now change this vocabulary and assume for the sake of the argument that in a given symbolic calculus, which is supposed to represent some bulk of knowledge, all syntactic rules are hidden from view while all its formulae (words) are observable. Let us now call one's knowing of rule

$$\Gamma, v : V \vdash w : W$$

explicit only if it translates into the the form

$$u : (V \to W)$$

(in the sense that the given calculus is Hilbertian and hence $\Gamma, v : V \vdash w : W$ entails $\Gamma \vdash u : (V \to W)$; otherwise we call this knowledge *tacit* . Using these terms we shall now call tacit one's knowledge how to cook porridge if this person is unable to write a recipe, and call the same knowledge-how explicit if this person also has this extra capacity. At the syntactic level the difference between the two forms of knowing-how corresponds to the difference between the calculi, which do have and do not have the Deduction Property. As it has been already shown in Section 3 there is no good reason to expect that all symbolic calculi, which represent certain useful knowledge-how in the form of system of rules, have the Deduction Property.

5.3 Which Rules are Logical?

Let C be a sequential calculus interpreted in some constructive terms, which satisfies conditions of Lemma 10 from 3.3. According to this Lemma C admits a syntactic translation into Hilbertian form C_H that comprises the single rule MP. As far as one wants to interpret MP in the usual logical sense of *modus ponens* one has to interpret formulas in C_H as propositions. This points to a possibility of translating constructive theories into Hilbert-style axiomatic theories which comprise no extra-logical rules. A historical example of such a translation is Hilbert's semi-formal axiomatization of Euclidean geometry [10]. It translates Euclid's *Postulates*, which are extra-logical geometrical rules (of how to produce a straight segment from given two points and other) into a convenient propositional form [23]. There is a general consensus that Hilbert's axiomatic presentation of Euclidean geometry, which gets rid of Euclid's extra-logical rules and in this way makes Euclid's geometrical proofs "purely logical", somehow makes this mathematical theory more rigorous. However formal details of this procedure and the total score of related epistemic gains and losses remains rather unclear. Since such a translation essentially involves logical and extra-logical semantics it cannot be fully analyzed in syntactic terms. We leave a study of semantic aspects of such translations for our further research.

6 Conclusion

Following Ryle's remark that "Knowing a rule is knowing how" [24] we argued, on the contrary to a popular opinion, that knowing-how does not have an intrinsically tacit character but in many cases allows for an explicit representation in the form of formal rules. This holds both for natural languages, which allow one to formulate rules and related deontic expressions, and formal languages where rules are represented symbolically and play an important role in the architecture of formal calculi. Leaving natural languages aside we reviewed two "styles" of building formal systems one of which employs few rules and an many axioms (the Hilbert-style) while the other employs complex systems of rules and may use no axiom (the Gentzen-style). Using some historical indications we construed the difference between the two styles not only syntactically but also semantically by associating Tarski's model-theoretic semantics with the Hilbert-style and the proof-theoretic semantics with the Gentzen-style.

In this context we introduced a syntactic definition of Hilbertian theory (Def. 11), which reflects and narrows the informal idea of "Hilbert-style axiomatic theory", and proved a lemma (Lemma 7) that says that the concept of Hilbertian theory is co-extensional with that of theory having Deduction Property, i.e., a theory for which the Deduction Theorem holds. The Deduction Property is of interest in the context of epistemological discussion on knowing-how and knowing-that because there is a sense (which has been made precise in the paper) in which it represents rule $A \vdash B$ (and, as we assume, the associated knowledge of how to follow this rule) in the form of (knowledge of) proposition $A \rightarrow B$. We also studied how Hilbertian theories can be syntactically translated into sequent calculi (i.e., in the Gentzen-style systems) and vice versa (Lemmas 9 and 10)

These preparatory steps allowed us to attack the question of whether or not knowing-how in a reasonable sense reduces to knowing-that. First, we considered the special case of logical knowing-how, i.e., knowing how to make logical inferences. Our conclusion here is negative: while the model-theoretic conception of logical consequence allows for seeing syntactic rules as mere symbolic representations of the meta-theoretical propositions, in terms of which the logical consequence is defined in this case, this conception of logical consequence is inappropriate for solving epistemological problems and thus cannot justify the wanted reduction. The proof-theoretic conception of logical consequence, which is appropriate in an epistemological discussion, does not allow for such a reduction.

Finally we considered a more general case of rule-based knowledge-how, which includes knowledge-how outside logic. For this purpose we proposed a canonical translation of Hilbertian theories into typed sequential calculi, which are apt for extra-logical constructive interpretations. For such calculi we proved the "constructive version" of Deduction Theorem (Theorem 13) and proposed its informal interpretation in terms of knowing-how. In this context we pointed to the open problem of semantic translations between constructive and standard Hilbert-style axiomatic theories.

As a final remark we would like to stress that formal systems, which admit proof-theoretic semantics, are natural candidates for the role of representational tools for the procedural knowledge. Given the importance of procedural knowledge in the Society one may expect that such systems can have more applications in Knowledge Representation than they presently have. There is apparently a general bias towards the Hilbert-style approach in thinking about knowledge and reasoning in many areas from Philosophy to Information Engineering. However philosophical questions about the nature of knowing-how are answered there is no good reason to understate this type of knowledge in the development of Knowledge Representation Systems and other relevant applications. Moreover there is no reason to picture this type of knowledge as somewhat "anti-intellectual".

Appendix: Direct Proof of Theorem 13

We construct term u using the induction by steps of derivation of the sequence

$$x_1{:}F_1, \ldots, x_n{:}F_n, x{:}F \vdash t{:}G.$$

Case 1: Axiom of the form $(t : G = c^A : A)$. Then $u = k^{A\to(F\to A)} \cdot c^A$. Observe that term c^A contains no variable. Hence the sequence

$$x_1 {:} F_1, \ldots, x_n {:} F_n \vdash c^A {:} A$$

is also derivable. Then apply the rule

$$\frac{x_1 {:} F_1, \ldots, x_n {:} F_n \vdash k^{A\to(F\to A)} {:} (A \to (F \to A)) \quad x_1 {:} F_1, \ldots, x_n {:} F_n \vdash c^A {:} A}{x_1 {:} F_1, \ldots, x_n {:} F_n \vdash (k^{A\to(F\to A)} \cdot c^A) {:} (F \to A)}$$

Case 2: Axiom of the form $(t {:} G = x_i {:} F_i$, where x is not one of the $x_i)$. Proceed as in Case 1; $u = k^{F_i\to(F\to F_i)} \cdot x_i$.

Case 3: $(t {:} G = x {:} F)$. In this case declaration $x {:} F$ cannot be eliminated from the context because t contains x. But in this case $F = G$, and so the wanted term u is a Hilbertian proof of formula $F \to F$:

$(F \to ((F \to F) \to F)) \to ((F \to (F \to F)) \to (F \to F))$	scheme($S_{\ldots}$)
$F \to ((F \to F) \to F)$	scheme($K_{\ldots}$)
$(F \to (F \to F)) \to (F \to F)$	(MP)
$F \to (F \to F)$	scheme($K_{\ldots}$)
$F \to F$	(MP)

Thus we obtain the wanted term $u = (s^{(\ldots)} \cdot k^{(\ldots)}) \cdot k^{(\ldots)}$ where the upper indexes are the first, the second and the forth lines of the Hilbertian derivation.

Application of the rule: By the inductive hypotheses the following sequences are derivable:

$$x_1 {:} F_1, \ldots, x_n {:} F_n \vdash v {:} (F \to (X \to G)),$$
$$x_1 {:} F_1, \ldots, x_n {:} F_n \vdash w {:} (F \to X).$$

Let $u = (s^{(\ldots)} \cdot v) \cdot w$, where the type of the first factor is $(F \to (X \to G)) \to ((F \to X) \to (F \to G))$. This guaranties that the product is well-typed in the same context: the product $(s^{(\ldots)} \cdot v)$ is of type $(F \to X) \to (F \to G)$ and term $u {:} (F \to G)$ is as required. $\lhd$

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