Categorical Model Theory and Knowledge How

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Tarski-style Model theory

$Model \models Theory$
in words: The Model *satisfies* (= makes <u>true</u>) the corresponding Theory.

$Interpretation : Signature \rightarrow Structure$

Assumptions 1-2

- A theory is a system of formal sentences, which are satisfied in a model;
- Semantics of *logical* terms is rigidly fixed: interpretation concerns only *non-logical* terms.

Two distinct points of a straight line completely determine that line

If different points A,B belong to straight line a and to straight line b then a is identical to b.

Assumption 3

Structures are *set-theoretic* structures.

Tarski 1941

"For precision it may be added, that the considerations which we sketched here are applicable to any deductive theory in whose construction logic is presupposed, but their application to logic itself brings about certain complications which we would rather not discuss here."

Compare Tarski's topological semantics for Classical and Intuitionistic propositional calculi (1935)

Lawvere 1963: Functorial Semantics of Algebraic Theories

Idea: use categories instead of signatures (thus blurring the distinction btw. logical and non-logical terms)

Algebraic (Lawvere) theory: category LT with finite products and distinguished object X s.th. every object A in C is isomorphic to X^n for some finite number n.

Model: $LT \to SET$ that preserves finite limits.
Generalized Models: $LT \to C$ where C has finite limits.

Sketches

Observation: Even if (small) category C does not have (co)limits the presheaf category $\hat{C} = [C, SET]$ does. This allows for using *sketches* "instead of" theories.

Theories in the Categorical perspective (after Awodey & Bauer)

$$\textit{Theory} \rightarrow \textit{Category}$$

- cartesian theories (only $\wedge$ and $\top$)
- regular theories (only $\wedge$ and $\top$ and $\exists$)
  (*regular* category: finite completeness plus image factorization stable under pullbacks)
- coherent theories (plus $\vee$ and $\bot$)
  (*coherent* category: regularity plus unions stable under base change)
- geometric theories (plus infinitary $\bigvee$)
  (*geometric* category: infinitary coherent)

Syntactic aka Classifying aka Walking Categories

Idea: a category "freely generated from the syntax"

- Lawvere's theory

- contextual category (contexts as objects and substitutions as morphisms)

Generic Models

Universal property: $Synt(T)$ is initial in $Mod(T) = [T, C]$

Internal Language

$$Categories \underset{Synt}{\overset{Lang}{\rightleftarrows}} Theories$$

$$Model : T \rightarrow Lang(C)$$

(in Theories)

Problem:

It is not clear whether Tarski's notion of model based on the satisfaction relation and his T-schema covers the functorial notion(s) of model in all cases. Categorical model theory may need an independent philosophical underpinning.

Claim:

Existing models of Homotopy Type theory are <u>not</u> Tarskian models and cannot be described in terms of the satisfaction relation and the T-schema.

MLTT: Syntax

- 4 basic forms of judgement:
  (i) $A : TYPE$;
  (ii) $A \equiv_{TYPE} B$;
  (iii) $a : A$;
  (iv) $a \equiv_A a'$
- Context : $\Gamma \vdash$ judgement (of one of the above forms)
- no axioms (!)
- rules for contextual judgements; Ex.: dependent product :
  If $\Gamma, x : X \vdash A(x) : TYPE$, then $\Gamma \vdash (\Pi x : X)A(x) : TYPE$

Martin-Löf 1983

"Classical" notion of proposition as truth-value is rejected and replaced by the "intuitionistic" one:

"A proposition is defined by laying down what counts as a proof of the proposition."

"A proposition is true if it has a proof, that is , if a proof of it can be given."

MLTT: Semantics of $t : T$ (Martin-Löf 1983)

- t is an element of set T
- t is a proof (construction) of proposition T
- t is a method of fulfilling (realizing) the intention (expectation) T
- t is a method of solving the problem (doing the task) T

MLTT: Proposition (M-L 1983)

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"A proposition is defined by laying down what counts as a proof of the proposition."

"A proposition is true if it has a proof, that is , if a proof of it can be given."

MLTT: Definitional aka judgmental equality/identity

$x, y : A$ (in words: x, y are of type A)

$x \equiv_A y$ (in words: x is y by definition)

MLTT: Propositional equality/identity

$p : x =_A y$ (in words: x, y are (propositionally) equal as this is evidenced by proof p)

Definitional eq. entails Propositional eq.

$$\frac{x \equiv_A y}{p : x =_A y}$$

where $p \equiv_{x =_A y} refl_x$ is built canonically

Equality Reflection Rule (ER)

$$\frac{p : x =_A y}{x \equiv_A y}$$

ER is <u>not</u> a theorem in the (intensional) MLTT (Streicher 1993l).

Extension and Intension in MLTT

- MLTT + ER is called *extensional* MLTT
- MLTT w/out ER is called *intensional*
  (notice that according to this definition intensionality is a negative property!)

Higher Identity Types

- $x', y' : x =_A y$
- $x'', y'' : x' =_{x =_A y} y'$
- $\ldots$

HoTT

"The central new idea in homotopy type theory is that types can be regarded as spaces in homotopy theory, or higher-dimensional groupoids in category theory." (HoTT Book 2013).

One more item to the above list of interpretations? NOT just that.

The homotopical semantics of MLTT, which is used in HoTT, is <u>not</u> compatible with the informal semantics of MLTT proposed by Martin-Löf in 1983!

h-stratification in MLTT

- (i) Given space A is called *contractible* (aka space of h-level -2) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- (ii) We say that A is a space of h-level $n+1$ if for all its points x, y path spaces $paths_A(x, y)$ are of h-level n.

h-hierarchy

(-2) single point pt;

(-1) the empty space $\emptyset$ and the point pt : truth values aka *classical* or "mere" propositions

(0) sets aka *intuitionistic* propositions aka theorems

(1) (flat) groupoids

(2) 2-groupoids

-
-

(n) n-groupoids

- ...

(ω) ω-groupoids

The above stratification of types is a robust mathematical structure in MLTT discovered via the homotopic interpretation of MLTT syntax. MLTT intended semantic does not take this structure into account. HoTT semantics does.

HoTT semantics (or the version thereof that I defend) does not license the idea that *every* type is a proposition. It recovers within

the MLTT syntax the classical notion of proposition as well as the intuitionistic notion of proposition-as-set (under a different name) and determines the precise place of both in the hierarchy of types. These semantic decisions are not arbitrary but based on the robust mathematical structure of *h*-stratification of types. *h*-stratification should be reflected semantically. Logical rules are specializations of more general constructive rules.

HoTT semantics for $t : T$ for (-1)-types

propositions and truth-values

HoTT semantics for $t : T$ for (0)-types

theorems and their proofs / sets and their elements

HoTT semantics for $t : T$ for higher -types

(also valid for lower types):

spaces and points, which support higher-order structures from elements of some other spaces (viz. map spaces);

objects are points;
constructions are points provided with additional higher-order structures: paths, surfaces (homotopies), etc.

Competing approaches to modeling HoTT

- Awodey: classifying categories, natural models
- Voevodsky: contextual categories (Cartmell), C - systems, Initiality Conjecture (open)

Since HoTT is *not* a system of sentences (propositions) Tarski's notion of model may account at most for the propositional fragment (level) of HoTT/MLTT

Theories, which are *not* systems of sentences are less exotic than one could think. Gentzen's Natural Deduction and the geometrical theory of Euclid's *Elements*, Books 1-4 are other examples.

Euclid's *Common Notions* and *Postulates* are rules rather than axioms in the modern sense of the term.

Arguably a typical scientific theory is not a system of proposition either (The "Non-Statement View of Theories" of P. Suppes, B. van Fraassen et al.).

Ryle 1945

"[I]ntelligent reasoner is knowing rules of inference whenever he reasons intelligently [. . .] [K]nowing such a rule is not a case of knowing an extra fact or truth ; it is knowing how to move from acknowledging some facts to acknowledging others. Knowing a rule of inference is not possessing a bit of extra information but being able to perform an intelligent operation. Knowing a rule is knowing how."

Carrol Paradox 1895

What the Tortoise Said to Achilles?

$A : X = Y$, $B : X = Z$, C: Things equal to the same thing are also equal to one another

$A, B, C, D, \ldots$

The standard concept of knowledge as a justified true belief (in a proposition) is *local*: it cannot account for knowing *theories* (even if theories are thought of in the usual way as systems of sentences related by the logical inference).

"Mute" knowledge?

"Principles of inference are not extra premisses and knowing these principles exhibits itself not in the recitation of formulas but in the execution of valid inferences and in the avoidance, detection and correction of fallacies, etc. "

A good experimentalist exercises his skill not in reciting maxims of technology but in making experiment.

Gellner

“ [T]here is a tendency . . . to make knowing how do what “intuitions” used to do.”

Toulmin on tacit knowledge; the example of riding a bicycle, etc.

Rules do allow for linguistic representation along with propositions. Ex. Euclid's Common Notions (Axioms) and Postulates; logical rules.

Stanley and Williamson 2001

"We believe that any successful account of natural language must postulate entities such as *ways*. But we shall not have much more of substance to say about the metaphysics of ways in this paper. "

However from a formal point of view rules are more fundamental than axioms: while axioms are dispensable, rules are not.

Conclusion 1

A theory, generally, may comprise "non-logical" rules, which apply to non-propositional objects.
Examples : Euclid, HoTT , Newton (?).

One's knowledge of a theory involves, generally, the knowledge of rules of logical inference along with rules for building non-propositional objects of various types. As the example of HoTT clearly demonstrates the knowledge-how can be well expressed formally and explicitly. The idea that the knowledge-how is essentially tacit is a misconception.

Conclusion 2

A *model* of a given theory is an implementation of its rules in a certain background. A "low-level" (material) model of HoTT (say, in an engineering or physical application) can be thought of as a system of (demonstratively realizable) instructions of how to perform certain set of operations that produce constructions, which make true the related propositions obtained via (-1)-truncation of these constructions (as in designing experiments).

The generalized satisfaction relation between fragments of the theory and their models is expressed in terms of *rule-following*. I does not reduce to the standard Tarskian notion of satisfaction for sentences at the pain of Carrol paradox.

Conclusion 3

Categorical Model Theory (based on the Functorial Semantics) is a mathematical tool, which accounts for the notion of rule-following in this context.

Conversely, the proposed constructive epistemological explanation of Functorial Semantics helps one to orientate among multiple technical concepts of model, which arise in HoTT and elsewhere.

THANK YOU!