Logic for Natural Sciences: A Categorical Perspective

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Historical Overview

Formal and Genetic Axiomatic Method

Constructive Proof theory and Curry-Howard Isomorphism

Categorical Logic
- CCC
- Topos theory
- Homotopy Type theory

Conclusion

Russell contra Kant

It seemed plain that mathematics consists of deductions, and yet the orthodox accounts of deduction were largely or wholly inapplicable to existing mathematics. [..] In this fact lay the strength of the Kantian view, which asserted that mathematical reasoning is not strictly formal, but always uses intuitions, i.e. the a priori knowledge of space and time. Thanks to the progress of Symbolic Logic, especially as treated by Professor Peano, this part of the Kantian philosophy is now capable of a final and irrevocable refutation. (Russell 1903: *Principles of Mathematics*)

Cassirer contra Russell

Dass unsere Begriffe sich auf Anschauungen zu beziehen haben, bedeutet daher, dass sie sich auf die mathematische Physik zu beziehen und in ihrer Gestaltung fruchtbar zu erweisen haben. Die logischen und mathematischen Begriffe sollen nicht l?nger die Werkzeuge bilden, mit denen wir eine metaphysische "Gedankenwelt"- aufbauen: sie haben ihre Funktion und ihre berechtigte Anwendung lediglich innerhalb der Erfahrungs Wissenschaft selbst.(Cassirer 1907: *Kant und die moderne Mathematik*)

Cassirer contra Russell

The principle according to which our concepts should be sourced in intuitions means that they should be sourced in the mathematical physics and should prove effective in this field. Logical and mathematical concepts must no longer produce instruments for building a metaphysical “world of thought”: their proper function and their proper application is only within the empirical science itself. (Cassirer 1907: *Kant and Modern Mathematics*)

Strong Cassirer Principle

The logical structure of a physical theory should be determined by physical principles. It should not come from a linguistic analysis and/or be supported by metaphysical arguments. (Metaphysical are those arguments which are neutral w.r.t. any possible experience).

Russell on Metaphysics

As I have attempted to prove in *The Principles of Mathematics*, when we analyse mathematics we bring it all back to logic. [..] In the present lectures, I shall try to set forth [..] a certain kind of logical doctrine, and on the basis of this a certain kind of metaphysic. (Russell 1918: *The Philosophy of Logical Atomism*)

Remark

An essential part of Russell argument is not purely speculative but also mathematical: he shows how his logical approach allows for laying out the Principles of the contemporary mathematics - while the (Neo-) Kantian approach defended by Cassirer and other people doesn't help them to complete the task.

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- Logic greatly profited from using symbolic mathematical methods.

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- New developments in Logic had a very little impact on the mainstream Mathematics (with a very few possible - and yet controversial - exceptions including Set theory). E.g. the recent proof of the Poincaré Conjecture involves ideas and techniques stemming from Physics (GR) but no symbolic logical techniques.
- New developments in Logic had <u>no</u> impact on the mainstream Physics (leave alone other natural sciences)!
- Within the standard foundational setting (ZF) of Mathematics its effectiveness in Natural Sciences appears wholly "unreasonable" (Wigner 1960). Having no other choice scientists have to apply Mathematics without using any logical guide. Arguably this makes such applications less effective.

Claim:

Advances in Categorical Logic, i.e., logical methods using mathematical Category theory (since late 1960-ies), and more recently (since 2008) in Homotopy Type theory allow for taking Cassirer's view on logic and mathematics seriously. Developing Logic along Cassirer's lines will make Logic into a proper tool for Mathematics and Natural Sciences.

Hilbert&Bernays 1934

The term axiomatic will be used partly in a broader and partly in a narrower sense.We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics.

Hilbert&Bernays 1934

[F]or axiomatics in the narrowest sense, the *existential form* comes in as an additional factor. This marks the difference between the *axiomatic method* and the *constructive* or *genetic* method of grounding a theory. While the constructive method introduces the objects of a theory [..], an axiomatic theory [in the narrow sense of "axiomatic"] refers to a fixed system of things (or several such systems) [i.e. to one or several models].[..] This is an idealizing assumption that properly augments [?] the assumptions formulated in the axioms.

Hilbert&Bernays 1934

When we now approach the task of such an impossibility proof [= proof of consistency], we have to be aware of the fact that we cannot again execute this proof with the method of axiomatic-existential inference. Rather, we may only apply modes of inference that are free from idealizing existence assumptions.

Hilbert&Bernays 1934

Yet, as a result of this deliberation, the following idea suggests itself right away: If we can conduct the impossibility proof without making any axiomatic-existential assumptions, should it then not be possible to provide a grounding for the whole of arithmetic directly in this way, whereby that impossibility proof would become entirely superfluous?

Hilbert's answer is in negative because of his worries about infinities in Set theory and elsewhere in mathematics.

Comment 1

Genetic object-building is not wholly suppressed in the Hilbert-style Formal Mathematics but is

- limited to syntactic constructions
- isolated in a special area of Mathematics called *Meta*mathematics.

Comment 2

This “official” view poorly describes what mathematicians do in practice (cf. Group Theory). However just saying that in practice mathematicians work *informally* does not solve the problem!

Comment 3

A non-syntactic object-building in Mathematics is essential for the mathematical modeling of physical experiments, i.e. for the *experimental design* (van Fraassen).

Alternative Approach: Idea

A proof is, generally, not a chain of propositions (and hence a merely semantic construction) but an appropriate semantic construction (as a mental action and as the result of this action), which makes the proved proposition evident. E.g. Euclid's geometrical proofs.

Constructive Proof theory

"[P]roof and knowledge are the same. Thus, if proof theory is construed not in Hilbert's sense, as metamathematics, but simply as a study of proofs in the original sense of the word, then proof theory as the same as theory of knowledge, which, in turn, is the same as logic in the original sense of the word, as the study of reasoning, or proof, not as metamathematics." (Martin-Löf 1983)

Remark

constructive = genetic

A reconciliation of the two views

Curry-Howard Isomorphism

Curry-Howard: Simply typed lambda calculus

Variable: $\dfrac{}{\Gamma, x : T \vdash x : T}$

Product: $\dfrac{\Gamma \vdash t : T \quad \Gamma \vdash u : U}{\Gamma \vdash \langle t, u \rangle : T \times U}$

$\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_1 v : T}$ $\dfrac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_2 v : U}$

Function: $\dfrac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x.t : U \to T}$

$\dfrac{\Gamma \vdash t : U \to T \quad \Gamma \vdash u : U}{\Gamma \vdash tu : T}$

Curry-Howard: Natural deduction

Identity: $\overline{\Gamma, A \vdash A}$ (Id)

Conjunction: $\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A\&B}$ (& - intro)

$\dfrac{\Gamma \vdash A\&B}{\Gamma \vdash A}$ (& - elim1); $\dfrac{\Gamma \vdash A\&B}{\Gamma \vdash B}$ (& - elim2)

Implication: $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}$ ($\supset$-intro)

$\dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$ ($\supset$-elim aka *modus ponens*)

Curry-Howard Isomorphism

$$\& \equiv \times$$

$$\supset \equiv \rightarrow$$

Brouwer-Heyting-Kolmogorov (BHK interpretation)

- proof of $A \supset B$ is a procedure that transforms each proof of A into a proof of B;
- proof of $A \& B$ is a pair consisting of a proof of A and a proof of B

Comment 1

Curry-Howard relates mathematical (λ-calculus) and meta-mathematical (natural deduction) concepts. It blurs the distinction between the two sorts of concepts (if one wants to blur this distinction).

Comment 2

Foundational consideration played a crucial role in this story from the outset (Schönfinkel, Curry, Church, Kolmogorov, Lawvere, Lambek). The expression "Curry-Howard isomorphism", which suggests that we have here an unexplained/surprising formal coincidence, is due to Howard 1969. The *true* history (and the true meaning) still waits to be explored.

Lawvere and Lambek 1969

The structure behind the Curry-Howard isomorphism is precisely captured by the notion of *Cartesian closed category* (CCC), which is an (abstract) category with the terminal object, products and exponentials.
Examples: Sets, Boolean algebras
Simply typed lambda-calculus / natural deduction is the *internal language* of CCC.

- Objects: types / propositions
- Morphisms: terms / proofs

Lawvere's philosophical motivation

- objective invariant structures vs. its subjective syntactical presentations
- objective logic vs. subjective logic (Hegel)

ETCS (1964)

- composition of functions instead of membership relation $\in$: a change of primitive terms;
- category of sets instead of Universe of sets;
- Internalization of (Classical) Logic (including quantifiers and truth-values);
- discovery of the CCC concept

Lawvere on logic and geometry

The unity of opposites in the title [Quantifiers and Sheaves] is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. [..] [A] Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for "the" category $\underline{S}$ of abstract sets to an arbitrary topos . We first sum up the principle contradictions of the Grothendieck-Giraud-Verdier theory of topos [..] enabling one to claim that in a sense logic is a special case of geometry. (Lawvere 1970)

Lawvere's axioms for topos

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(Elementary) topos is a category which

- has finite limits
- is CCC
- has a subobject classifier

Remark

Lawvere's axioms for topos (including the topos of sets) do not "describe defining properties" of a topos with some pre-established logical framework - but (re)construct a topos genetically with its specific internal logic. So logic and geometry turn to be two complimentary aspects of the same object, viz. a topos. Features of the internal language (internal logic) of a given topos reflect (geometrical) features of this given topos.

Remark

IF geometrical features of the given topos are physically meaningful then so does the internal logic. Cf. Cassirer.

Topos Physics

: A. Döring, Ch. Isham: 'What is a Thing?': Topos Theory in the Foundations of Physics (2008): http://arxiv.org/abs/0803.0417

MLTT (Martin-Löf 1980): key features

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- double interpretation of types: "sets" and propositions
- double interpretation of terms: elements of sets and proofs of propositions
- higher orders: dependent types (sums and products of families of sets)
- MLTT is the internal language of LCCC (Seely 1983)

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- Definitional identity of terms (of the same type) and of types: $x = y : A$; $A = B : type$ (substitutivity)
- Propositional identity of terms x, y of (definitionally) the same type A:
  $Id_A(x, y) : type$;
  Remark: propositional identity of given terms is a (dependent) type on its own.

MLTT: Higher Identity Types

- $x', y' : Id_A(x, y)$
- $Id_{Id_A}(x', y') : type$
- and so on

MLTT: extensional and intensional versions

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- Definitional identity implies propositional identity

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- Definitional identity implies propositional identity
- Extensionality: Propositional identity implies definitional identity

HoTT: the Idea

The central new idea in homotopy type theory is that types can be regarded as spaces in homotopy theory, or higher-dimensional groupoids in category theory. (HoTT Book 2013)

HoTT: the History

- Groupoid model of MLTT (Streicher 1993): basic types are groupoids, terms are their elements, dependent types are fibrations of groupoids (families of groupoids indexed by groupoids - rather than families of sets indexed by sets). "Extensionality one dimension up".
- Higher (homotopical) groupoids model higher identity types (Voevodsky circa 2008). Intensionality all way up .

Fundamental group

Fundamental group G^0_T of a topological space T:

- a base point P;
- loops through P (loops are circular paths $l : I \to T$);
- composition of the loops (up to homotopy only! - see below);
- identification of homotopic loops;
- independence of the choice of the base point.

Fundamental (1-) groupoid

G^1_T:

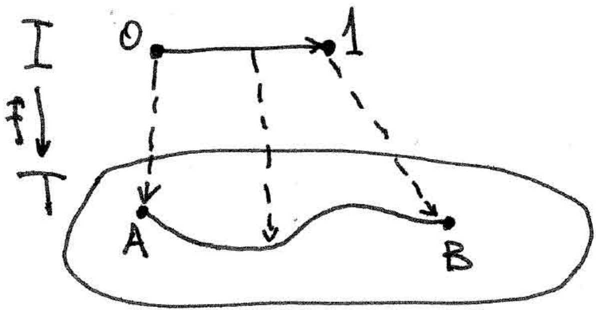
- all points of T (no arbitrary choice);
- paths between the points (embeddings $s : I \longrightarrow T$);
- composition of the *consecutive* paths (up to homotopy only! - see below);
- identification of homotopic paths;

Since not all paths are consecutive G^1_T contains more information about T than G^0_T!

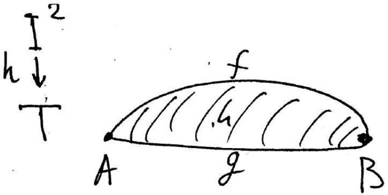
Path Homotopy and Higher Homotopies

$s : I \longrightarrow T$, $p : I \longrightarrow T$ where $I = [0, 1]$: paths in T
$h : I \times I \longrightarrow T$: homotopy of paths s, t if $h(0 \times I) = s$, $h(1 \times I) = t$
$h^n : I \times I^{n-1} \longrightarrow T$: n-homotopy of $n-1$-homotopies h_0^{n-1}, h_1^{n-1} if
$h^n(0 \times I^{n-1}) = h_0^{n-1}$, $h^n(1 \times I^{n-1}) = h_1^{n-1}$;
Remark: Paths are zero-homotopies

Path Homotopy and Higher Homotopies



Homotopy categorically and Categories homotopically



$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

Higher Groupoids and Omega-Groupoids (Grothendieck 1983)

- all points of T (no arbitrary choice);
- paths between the points ;
- homotopies of paths
- homotopies of homotopies (2-homotopies)
- higher homotopies up to n-homotopies
- higher homotopies ad infinitum

G_T^n contains more information about T than G_T^{n-1}!

Composition of Paths

Concatenation of paths produces a map of the form $2I \to T$ but not of the form $I \to T$, i.e., not a path. We have the whole space of paths $I \to 2I$ to play with! But all those paths are homotopical. Similarly for higher homotopies (but beware that n-homotopies are composed in n different ways!)
On each level when we say that $a \oplus b = c$ the sign $=$ hides an infinite-dimensional topological structure!

Grothendieck Conjecture:

G_T^ω contains all relevant information about T; an omega-groupoid is a complete algebraic presentation of a topological space.

Voevodsky on Univalent Foundations

Whilst it is possible to encode all of mathematics into Zermelo-Fraenkel set theory, the manner in which this is done is frequently ugly; worse, when one does so, there remain many statements of ZF which are mathematically meaningless. [..]

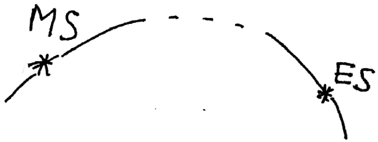
Voevodsky on Univalent Foundations (continued)

Univalent foundations seeks to improve on this situation by providing a system, based on Martin-Löf's dependent type theory whose *syntax is tightly wedded to the intended semantical interpretation* in the world of everyday mathematics. In particular, it allows the direct formalization of the world of homotopy types; indeed, these are the basic entities dealt with by the system. (Voevodsky 2011)

Prospective Applications in Mathematics and Computer Science

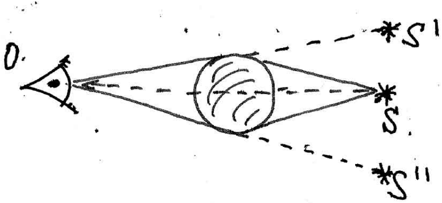
Coquand: COQ; 4-color problem
Voevodsky: Univalent Foundations: making COQ into a universal tool for checking mathematical routines.

Prospective Physical Applications: Naive Example



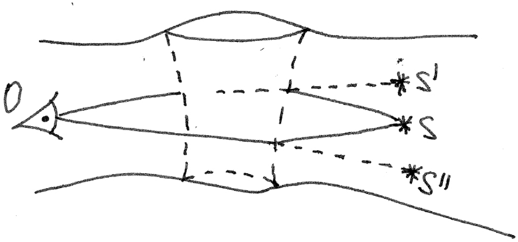
Identity through time

Naive Example



Gravitational lensing

Naive Example



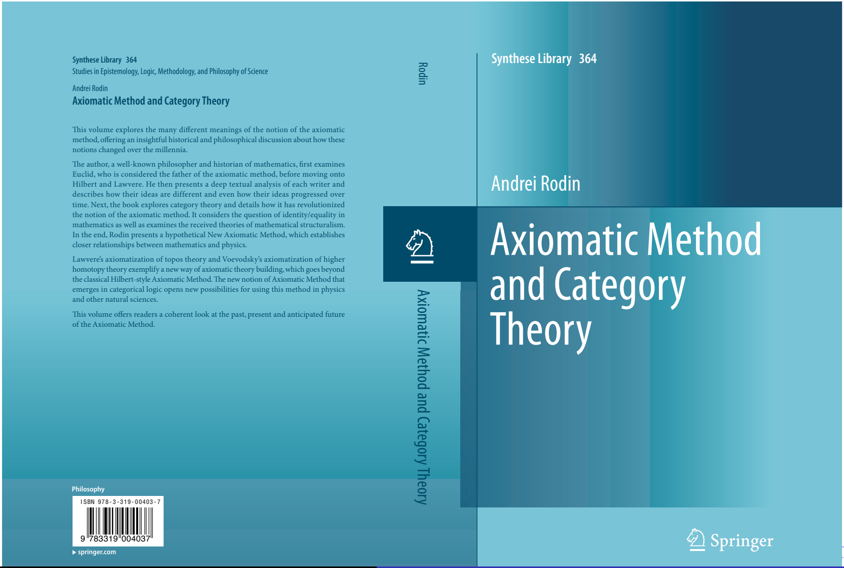
Wormhole lensing

Serious stuff

Using HoTT as a mathematical foundation for QFT ("Univalent Physics"):
Schreiber: Quantization via Linear homotopy types (Feb. 2014)
http://arxiv.org/abs/1402.7041

Conclusion:

The Strong Cassirer Principle can be in principle enforced by means of Categorical logic. This fact can be used for designing research programs aiming at new applications of logical methods in Mathematics, Physics and other Natural Sciences. There are reasons to expect that this approach will allow for more effective applications of Logic in Mathematics, Physics and other natural sciences.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Studies in Epistemology, Logic, Methodology, and Philosophy of Science
Andrei Rodin
Axiomatic Method and Category Theory
This volume explores the many different meanings of the notion of the axiomatic method, offering an insightful historical and philosophical discussion about how these notions changed over the millennia.
The author, a well-known philosopher and historian of mathematics, first examines Euclid, who is considered the father of the axiomatic method, before moving onto Hilbert and Lawvere. He then presents a deep textual analysis of each writer and describes how their ideas are different and even how their ideas progressed over time. Next, the book explores category theory and details how it has revolutionized the notion of the axiomatic method. It considers the question of identity/equality in mathematics as well as examines the received theories of mathematical structuralism. In the end, Rodin presents a hypothetical New Axiomatic Method, which establishes closer relationships between mathematics and physics.
Lawvere's axiomatization of topos theory and Voevodsky's axiomatization of higher homotopy theory exemplify a new way of axiomatic theory building, which goes beyond the classical Hilbert-style Axiomatic Method. The new notion of Axiomatic Method that emerges in categorical logic opens new possibilities for using this method in physics and other natural sciences.
This volume offers readers a coherent look at the past, present and anticipated future of the Axiomatic Method.
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http://arxiv.org/abs/1210.1478

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