Two Categorical Approaches To Knowledge Representation

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16 septembre 2014

CFOL in KR

Harmelen & Lifschitz & Porter (2008) *Handbook of KR* : Classical First-Order Logic (CFOL) is the logical backbone of KR. In particular CFOL plays this role in Descriptive Logics.

Why ?

An ontological assumptions (commitment) according to which domain D of the relevant knowledge K_D is construed as a set of individuals aka *objects* O_D.

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- Higher-order logical structure : idem.

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- Moral : Changing domains (*contexts*) is a fundamental logical operation. Function $f : A \rightarrow B$ "changes domain B for domain A".
- Lawvere Discovery : Right and left *adjoints* to substitution functor s_f along f are the existential quantifier $\exists_f$ and $\forall_f$ correspondingly.

A constructive principle of ontological anchoring/grounding of logic useful for CS :

Never use logic as if it were god-given. Always ground your logic with an appropriate hand-made ontology.

What is wrong about *SET* with CFOL ?

Problem 1 :
SET is *static*. It is the static topos (Lawvere).
For a simple example of a *dynamic* topos consider $SET^{\circlearrowleft}$ = topos of sets with autho-maps (= functions to itself).

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- (iii) In many cases K_D has a *local* character. In other words it varies over D rather than accounts for D statically. Ex. : Knowledge of today's weather various not only through time but also over the globe.

How to fix it ?

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- (i) and (ii) : $SET^{\cdots}$: stages of knowledge and temporal stages (correspondingly)
- (iii) Topos $Sh(T)$ of shaves over a topological space T or (more generally) a site T (Grothendieck topology)

Unless a given topos E is SET its internal logic L_T is not classical but intuitionistic. It also has specific features that reflects geometrical features of T. If $E = SET^{\cdots}$ then the (category of) truth-values for L_T is $\cdots$. If $E = Sh(T)$ then the (category of) truth-values for L_T is T.

Lawvere 1970

[A] Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions.

Conclusion of the above Argument

It is more useful to construe a domain of knowledge D as a general topos rather than SET. Topos (intuitionistic) logic L_D can better serve for KR than CFOL.

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  - to find an appropriate notion of *transformation* (= categorical morphisms) btw objects of D ; - to define the (partial) composition of morphisms and study its algebra
- (ii) grant that D is a topos by providing D with - the terminal object, pullbacks, etc.
  - exponentials
  - truth-value object

Useful theorem

If D is a small category then $\hat{D} = SET^D$ is a topos in which D plays the role of truth-value object (pre-sheafification). $\hat{D}$ is a categorical *completion* of D (Yoneda embedding).

Corollary 1 : In many contexts one can replace D by $\hat{D}$ (does this work for KR ?) ;

Corollary 2 : When D is construed as a site then one may use $Sh(D)$ instead of $\hat{D}$ (sheafification). This move provides the domain of knowledge in question with a geometrical structure (coherent with its internal logical structure).

Drawback

The construals of $\hat{D}$ and $Sh(D)$ still essentially involve SET !

What is wrong about *SET* with CFOL ?

Problem 2 : CFOL does not reflect how work computers ! *SET* is a purely speculative (and under some interpretations metaphysical) construal, which does not reflect the usual computing environment. $FSET \subset SET$ may indeed reflect some features of primitive human experience like manipulations with pebbles. But *SET* certainly doesn't !

What is wrong about *SET* with CFOL ?

Problem 3 : CFOL-based KR represents only the *propositional* knowledge aka *knowledge-that*. Arguably there is also the non-propositional *knowledge-how* irreducible to the propositional *knowledge-that* (Fantl). My analysis of mathematical knowledge (from Euclid on) justifies this view and moreover shows that the two types of knowledge are intertwined at some very basic level of mathematical reasoning.

How to fix ?

How to build an ontology fairly constructively ? How to represent non-propositional knowledge ? Here's how : use the constructive logic and mathematics for it.

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- terms of a proposition-type are *proves* of this proposition.

Voevodsky circa 2010

MLTT is internal for the (categorically-construed) higher homotopy theory. Under this interpretation MLTT (possibly extended with some further axioms) is known as HoTT. Making a (formal) proof in HoTT is a special case of making a construction. Cf. Euclid.

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- Problem 3 : MLTT first principles are constructive *rules* (including rules of inference for proposition-based judgements), not propositions or propositional forms. In HoTT these rules have a geometrical meaning. HoTT constructions like Euclid's geometrical constructions are basically complex rules build with elementary rules taken for granted. This structure seems me appropriate for representing the *knowledge-how* (without reducing it to certain propositional *knowledge-that*.

Perspectives :

Jakob Lurie's ongoing work points to a convergence and possible synthesis of the two above approaches in what he calls the *HIgher Topos theory* (2009). The idea is to make HoTT into the internal language of a wider higher-order topos-like framework, which would allow for making truth-evaluation for HoTT internally. (Observe that MLTT doesn't provide this option !)

A lot needs to be done in order to implement at least some of the discussed theoretical possibilities in practice. However it worths trying.

THANK YOU !