Constructive Identities for Physics

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Axiomatization of Physics
- History of Axiomatization according to Schreiber
- Hilbert and Kant
- Lawvere and Hegel
- Dialectics in Axiomatic Method

Identity and Objecthood
- Frege and Venus
- Identity in MLTT and HoTT
- HoTT Identity in Physics
  - Classical case
  - Relativistic case
  - Quantum case

Conclusions

Step 1 : Hilbert

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Corry 2004 : "From all the problems in the list, the sixth is the only one that continually engaged [Hilbert's] efforts over a very long period, at least between 1894 and 1932."

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(2) build the foundations of physics *synthetically* inside topos theory by
(a) imposing properties on a topos which ensure that the objects have the structure of differential geometric spaces (1998)
(b) formalizing classical mechanics on this basis by universal constructions ("Toposes of laws of motion" 1997)"

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(2) Modern physics needs to refine classical mechanics to quantum mechanics and quantum field theory at small length/high energy scales."

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"[R]efine Lawvere's synthetic approach on Hilberts sixth problem from classical physics formalized in synthetic differential geometry axiomatized in topos theory to high energy physics formalized in higher differential geometry axiomatized in higher topos theory. Specifically, the task is to add to (univalent) homotopy type theory axioms that make the homotopy types have the interpretation of differential geometric homotopy types in a way that admits a formalization of high energy physics."

Claim :

Hilbert's and Lawvere's understanding of axiomatization (including the axiomatization of physics) are significantly different from an epistemological viewpoint.
It is essential to realize this difference for making a progress in Hilbert-Lawvere-Schreiber's project.

Hilbert 1

"Finally we could describe our task as a logical analysis of our intuitive capacities (Anschauungsvermögens). The question if our space intuition has a-priori or empirical origins remains nevertheless beyond our discussion." (1898-99)

Hilbert 1

“[I]f we want to erect a system of axioms for geometry, the starting point must be given to us by the intuitive facts of geometry and these must be made to correspond with the network that must be constructed. The concepts obtained in this way, however, must be considered as completely detached from both experience and intuition.” (1905)

Hilbert 1

"The general basic principles and the leading questions of the Kantian theory of knowledge preserve in this way their full significance. But the boundaries between what we a-priori possess and logically conclude, on the one hand, and that for which experience is necessary, on the other hand, we must trace differently than Kant." (1922-23).

The a-priori status of logic remains here beyond any doubt. Thus Hilbert's view during this (earlier) period is a revised Kantianism leaning towards the Logical Empiricism.

Lawvere on Hegelian dialectics

It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. In turn the explicit attention by mathematicians to such philosophical questions is necessary to achieve the goal of making mathematics (and hence other sciences) more widely learnable and useable. Of course this will require that philosophers learn mathematics and that mathematicians learn philosophy. (1992)

Lawvere &Rosebrugh on subjective presentation vs. objective content

Presentations of algebraic structures for the purpose of calculation are always needed, but it is a serious mistake to confuse the arbitrary formulations of such presentations with the objective structure itself or to arbitrarily enshrine one choice of presentation as the notion of logical theory, thereby obscuring even the existence of the invariant mathematical content. In the long run it is best to try to bring the form of the subjective presentation paradigm as much as possible into harmony with the objective content of the objects to be presented ; with the help of the categorical method we will be able to approach that goal. (2003)

Lawvere on objective and subjective logic

[C]ategory theory has developed such notions as adjoint functor , topos , fibration , closed category, 2-category, etc. in order to provide
(i) a guide to the complex, but very non-arbitrary constructions of the concepts and their interactions which grow out of the study of space and quantity. [..]
If we replace "space and quantity" in (i) above by "any serious object of study", then (i) becomes my working definition of *objective logic*. [..] Category theory has also objectified as a special case
(ii) the subjective logic of inference between statements. Here statements are of interest only for their potential to describe the objects which concretize the concepts. (1994)

Hegel on objective logic

The objective logic, then, takes the place rather of the former metaphysics which was intended to be the scientific construction of the world in terms of thoughts alone. [..] *It is first and immediately ontology whose place is taken by objective logic.* [..] But further, objective logic also comprises the rest of metaphysics in so far as this attempted to comprehend with the forms of pure thought particular substrata taken primarily from figurate conception, namely the soul, the world and God [..] Former metaphysics [..] incurred the just reproach of having employed these forms uncritically [..]. Objective logic is therefore the genuine critique of them - a critique which *does not consider them as contrasted under the abstract forms of the a priori and the a posteriori*, but considers the determinations themselves according to their specific content. (1812)

Lawvere's axioms for Topos

The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. At the same time, in the present joint work with Myles Tierney there are important influences in the other direction : a Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for "the" category $\underline{S}$ of abstract sets to an arbitrary topos. (1970)

Two Ways of Logical Analysis

- Hilbert : arranging physical and mathematical concepts distilled from their intuitive and empirical contents with some pre-given logical means. Logical semantics is fixed, non-logical semantics is variable (multiplicity of models) ;
- Lawvere : providing mathematical (and ideally also physical) concepts with a logical semantics. In the axiomatic order logical contents emerge along with mathematical and physical contents (internalization of logic).

Hilbert 2 : Dialectics in the Development of Axiomatic Method

No more than any other science can mathematics be founded by logic alone ; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. [..] [W]hat we consider is the concrete signs themselves, whose shape [..] is immediately clear and recognizable. This is the very least that must be presupposed ; no scientific thinker can dispense with it, and therefore everyone must maintain it, consciously or not. (Hilbert 1927)

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(2) recognize the constitutive role of geometric intuition and physical experience in mathematical reasoning as such (rather than only in the non-axiomatic isolated area of metamathematics).

Martin-Löf against the metamathematics

"[P]roof and knowledge are the same. Thus, if proof theory is construed not in Hilbert's sense, as metamathematics, but simply as a study of proofs in the original sense of the word, then proof theory as the same as theory of knowledge, which, in turn, is the same as logic in the original sense of the word, as the study of reasoning, or proof, not as metamathematics." (1983)

Voevodsky on Univalent Foundations

Whilst it is possible to encode all of mathematics into Zermelo-Fraenkel set theory, the manner in which this is done is frequently ugly ; worse, when one does so, there remain many statements of ZF which are mathematically meaningless. [..]

Voevodsky on Univalent Foundations (continued)

Univalent foundations seeks to improve on this situation by providing a system, based on Martin-Löf's dependent type theory whose *syntax is tightly wedded to the intended semantical interpretation* in the world of everyday mathematics. In particular, it allows the direct formalization of the world of homotopy types ; indeed, these are the basic entities dealt with by the system. (Voevodsky 2011)

Frege 1892

"The discovery that the rising Sun is not new every morning, but always the same, was one of the most fertile astronomical discoveries. Even today the identification of a small planet or a comet is not always a matter of course. Now if we were to regard identity as a relation between that which the names a and b designate, it would seem that $a = b$ could not differ from $a = a$ (provided $a = b$ is true)."

Venus Example

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Cf.T. Budavari & A.S. Szalay, *Probabilistic Cross-Identification of Astronomical Sources*, The Astrophysical Journal 679 (2008) 301

Frege's solution

the sense (aka meaning) / reference distinction

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- the alleged "opacity" of intensional contexts : identical objects MUST be known to begin with !
- no account of how empirical or other evidences justify judgement $\vdash a = b$;
- linguistic examples from the everyday talk and a historical narrative (like "Napoleon recognized the danger to his right flank") are used for fixing the notion of identity and the meaning of objecthood in empirical sciences.

Usual formalization

- Introduction rule :
  $\Gamma \vdash t = t$ for any term t
- Elimination rule :
  If $\Gamma_1 \vdash t_1 = t_2$ and $\Gamma_2 \vdash \phi$ then $\Gamma_1, \Gamma_2 \vdash \phi'$ where ϕ' is obtained from ϕ' by replacing zero or more occurrences of t_1 with t_2, provided that no bound variables are replaced, and if t_2 is a variable, then all of its substituted occurrences are free.
- Problem : "opaque" intensional contexts

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- Propositional identity of terms x, y of (definitionally) the same type A :
  $Id_A(x, y) : type$;
  Remark : propositional identity is a (dependent) type on its own.

MLTT : Higher Identity Types

- $x', y' : Id_A(x, y)$
- $Id_{Id_A}(x', y') : type$
- and so on

MLTT : extensional versus intensional

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- Extensionality : Propositional identity implies definitional identity : no higher identity types
- First intensional (albeit 1-extensional) model : Hofmann & Streicher 1994 :
  groupoids instead of sets
  families groupoids indexed by groupoids instead of families of sets indexed by sets

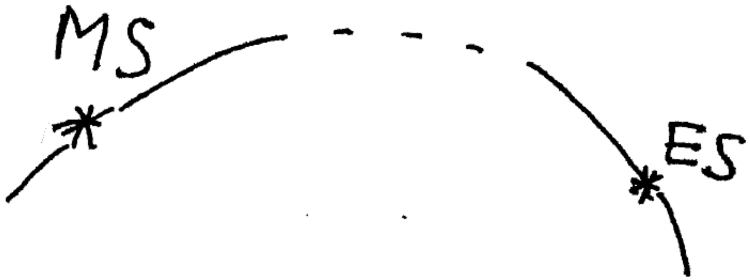
Hofmann & Streicher groupoid model

judgement $\vdash A : type$ - groupoid A
judgement $\vdash x : A)$ - object x of groupoid A type $Id_A(x, y)$ - arrow groupoid $[I, A]_{x,y}$ of groupoid A
(no reason to be empty unless x = y !)

HoTT : the idea

Types are modeled as spaces in homotopy theory, or, equivalently (Grothendieck conjecture) as higher-dimensional groupoids in category theory.

Venus with Homotopy Type theory : Classical case



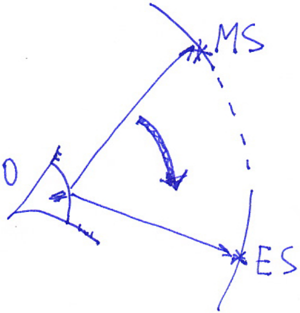
$$o : Id_U(MS, ES)$$

EX1 : “extensionality one dimension up”

no identity types of h-level ≥ 1, more precisely :

$$\frac{\vdash h : Id_{Id_U(MS,ES)}(o_i, o_j)}{\vdash o_i = o_j : Id_U(MS, ES)}$$

"Critical" viewpoint



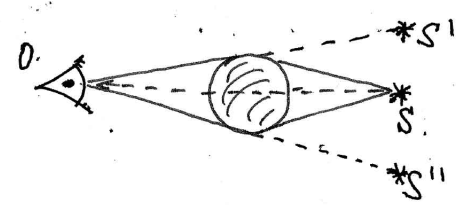
$$\text{vdash } p_i, p_j : Id_U(O, MS/ES)$$
$$\vdash h : Id_{Id_U(O,MS/ES)}(p_i, p_j)$$

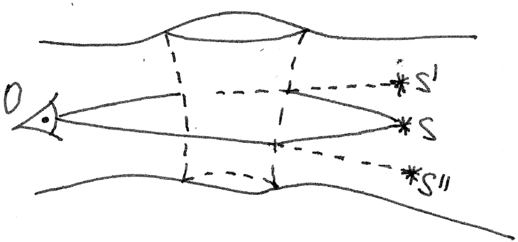
EX2

no identity types of h-level ≥ 2, more precisely :

$$\frac{\vdash s : Id_{Id_{Id_U(O,MS/ES)}(p_i,p_j)}(h_i, h_j)}{\vdash h_i = h_j : Id_{Id_U(O,MS/ES)}(p_i, p_j)}$$

No EX2 in GR ? :





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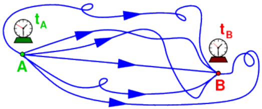
- Does the Block Universe picture is an adequate interpretation of GR ? Probably NOT.

Homotopy theory of path integrals (after Suzuki 2011)



Consider a system of n free spinless indistinguishable particles in space $\mathbb{R}^d$ and its configuration space X : of $x = (x_1, ..x_n) \in X$ with $x_i \in \mathbb{R}^d$.

Theorem (Laidlaw&DeWitt 1971)

Let the configuration space X of a physical system be the topological space. Then the probability amplitude K for a given transition is, up to a phase factor, a linear combination

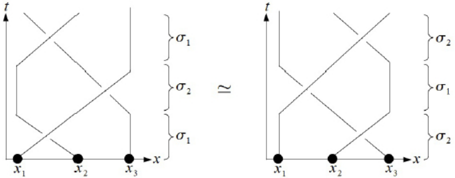
$$\sum_{\alpha \in \pi_1(X)} \chi(\alpha) K^\alpha$$

of partial probability amplitudes K^α obtained by integrating over paths in the same homotopy class in X, where the coefficients $\chi(\alpha)$ form a one-dimensional unitary representation of the fundamental group $\pi_1(X)$.

fundamental group by permutations

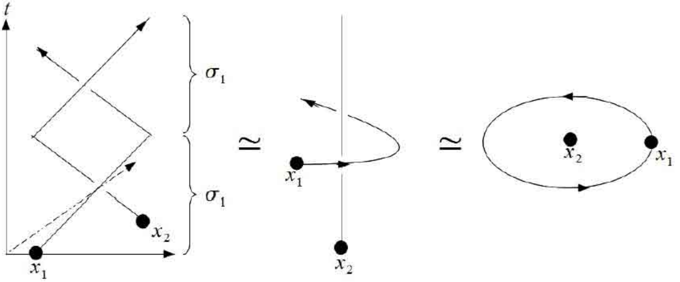
$\sigma_i = s_{i,i+1}$

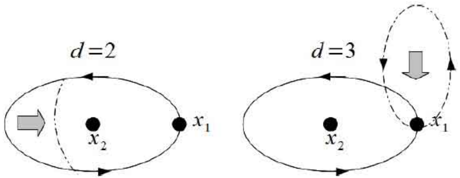
1. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
2. if $|i - j| > 1$ then $\sigma_i \sigma_j = \sigma_j \sigma_i$
3. $\sigma_i^2 = e$



(1) ; (2) is obvious

(3)



(3) 

For $d \geq 3 \pi_1(X) = S_n$; since S_n has two 1D unitary representations we have two cases :
$\chi^B = 1$ for all $\alpha \in S_n$ (bosons) ;

$$\chi^F = \begin{cases} +1, & \text{when } \alpha \text{ is even} \\ -1, & \text{when } \alpha \text{ is odd} \end{cases}$$

(fermions)
For $d = 2 \pi_1(X) = B_n$ (anyons)

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- This identity concept is both *intuitive* and *empirically-based* ; it recovers the traditional notions of physical object and physical process as spatio-temporal continua ;
- It suggests a more general notion of physical object/process construed as an identity groupoid, which involves not just a single trajectory but also multiple trajectories, their homotopies and higher homotopies ; this more general construal of objects/processes applies both in Classical and Quantum cases.

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- A realistic theory physics at small and large scales is possible.

THE END