Space of Possibilities, its Topology and its Internal Language

Andrei Rodin

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IS A "SPACE OF POSSIBILITIES" A MERE METAPHOR?

S4

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- $\Box\phi \vdash \Box\Box\phi$
- $\Box\phi \wedge \Box\psi \vdash \Box(\phi \wedge \psi)$
- $\top \vdash \Box\top$
- $$\frac{\phi \vdash \psi}{\Box\phi \vdash \Box\psi}$$

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- $$\frac{\phi \vdash \psi}{\Box\phi \vdash \Box\psi}$$
- Def.: $\Diamond\phi = \neg\Box\neg\phi$

Topological space

$(X, O(X))$ where $O(X) \subseteq \wp(X)$

- $X, \emptyset \in O(X)$
- If $U, V \in O(X)$ then $U \cap V \; O(X)$
- If $U_i \in O(X)$ then $\bigcup_{i \in I} U_i \in O(X)$ for any index set I

Interior Operator

$A \subseteq X$
Def: $int(A) = \bigcup U$ where $U \in O(X)$ and $U \subseteq X$
Def.: $cl(A) = \overline{int(\bar{A})}$ where $\bar{A}$ is $X - A$

Interior Operator (continued)

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- $int(A) \subseteq A$
- $int(int(A)) = int(A)$
- $int(A \cap B) = int(A) \cap int(B)$
- $X \subseteq int(X)$
- $A \subseteq B \Rightarrow int(A) \subseteq int(B)$

Topological Semantics for Propositional S4

Replace $\Box$ by *int* ..

- $\|\neg\phi\| = X - \phi$
- $\|\phi \wedge \psi\| = \|\phi\| \cap \|\psi\|$
- $\|\phi \vee \psi\| = \|\phi\| \cup \|\psi\|$
- $\|\top\| = X$
- $\|\bot\| = \emptyset$
- $\|\Box\phi\| = int(\|\phi\|)$

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- $\|\top\| = X$
- $\|\bot\| = \emptyset$
- $\|\Box\phi\| = int(\|\phi\|)$
- $(X, \| \cdot \|) \models \phi \Leftrightarrow \|\phi\| = X$

Topological Semantics for Propositional S4 (continued)

- SOUNDNESS: if $\phi \wedge \psi$ is provable in S4 then $\|\phi\| \subseteq \|\psi\|$in any topological interpretation $(X, \|\cdot\|)$

Topological Semantics for Propositional S4 (continued)

- SOUNDNESS: if $\phi \wedge \psi$ is provable in S4 then $\|\phi\| \subseteq \|\psi\|$in any topological interpretation $(X, \| \cdot \|)$
- COMPLETENESS (McKinsey-Tarski, 1944): For any consistent theory T containing S4 there exists topological space X and interpretation $(X, \| \cdot \|)$ such that $\phi \wedge \psi$ is provable in S4 $\Leftrightarrow \|\phi\| \subseteq \|\psi\|$

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- Objective mathematical knowledge about formal systems (meta-theorems) has other grounds than formal axiomatic theories (like ZF) themselves; Hilbert's hope to restrict meta-mathematics to finitary mathematics proved unrealistic in the light of Gödel's Incompleteness results; iteration of meta-levels leads to nowhere.
- Topological semantics of S4 (along with that of Classical and Intuitionistic Propositional logic) suggests considering a different epistemic relationship between logic and geometry.

Philosophical Claim

Geometry (and topology as its core part) provides an objective epistemic ground for formal logical calculi; this ground does not depend on syntactic peculiarities of symbolic presentations of these calculi. Cf. Hegel's notion of Objective Logic

First-order and Higher-order generalization: Grothendieck Topos

- Sheaves instead of opens (ex: sheaf of continuous functions from opens to $\Re$);
- Quantifiers as adjoints to substitution functor;
- Grothendieck pointless topology instead of classical point-based topology: sites instead of base spaces; .
- Grothendieck topos instead of topological space; Grothendieck topology is given by a modal operator on truth-values; sites encode theories;
- Internal language instead of external semantics;
- Geometric logic instead of Classical logic (except the topos of sets); semantic completeness of Classical fragment.

Historical Remark

Toposes first appeared in geometry (Grothendiek); their logical significance was understood later (by Lawvere) in the course of attempts to treat topos theory axiomatically (elementary aka Lawvere topos).

Lawvere on Logic and Geometry

[..] Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for "the" category $\underline{S}$ of abstract sets to an arbitrary topos . We first sum up the principle contradictions of the Grothendieck-Giraud-Verdier theory of topos in terms of four or five adjoint functors [..] enabling one to claim that in a sense logic is a special case of geometry. (*Quantifiers and Sheaves*,1970)

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- double interpretation of types: "sets" and propositions (Curry-Howard isomorphism);
- double interpretation of terms: elements of sets and proofs of propositions;
- higher orders: dependent types (sums and products of families of sets);
- MLTT is the internal language of LCCC (Seely 1983);

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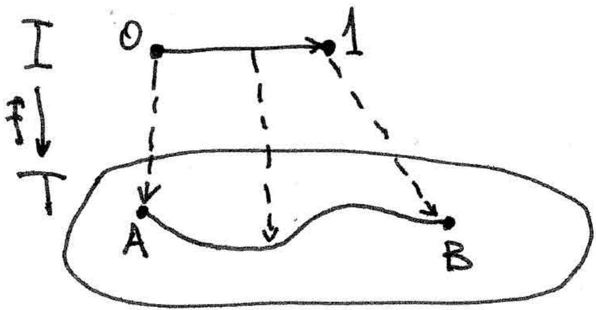
- Definitional identity of terms (of the same type) and of types: $x = y : A$; $A = B : type$ (substitutivity)
- Propositional identity of terms x, y of (definitionally) the same type A:
  $Id_A(x, y) : type$;
  Remark: propositional identity is a (dependent) type on its own.
- Extensionality Principle: propositional identity implies definitional identity.

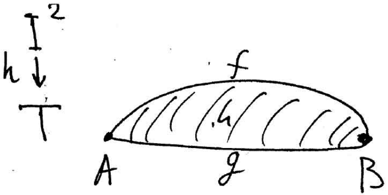
MLTT: Higher Identity Types

- $x', y' : Id_A(x, y)$
- $Id_{Id_A}(x', y') : type$
- and so on

Path Homotopy and Higher Homotopies



Homotopies categorically and Categories homotopically



$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

Groupids, Higher Groupoids and Omega-Groupoids (Grothendieck 1983)

- all points of T (no arbitrary choice);
- paths between the points (taken up to homotopy produce the fundamental groupoid of T
- homotopies of paths
- homotopies of homotopies (2-homotopies)
- higher homotopies up to n-homotopies
- higher homotopies ad infinitum

G^n_T contains more information about T than G^{n-1}_T!

Grothendieck Conjecture:

G_T^ω contains all relevant information about T; an omega-groupoid is a complete algebraic presentation of a topological space.

Homotopy Type theory

- Groupoid model of MLTT: basic types are groupoids, terms are their elements, dependent types are fibrations of groupoids (families of groupoids indexed by groupoids). Extensionality one dimension up. (Streicher 1993).
- Higher (homotopical) groupoids model higher identity types. Intensionality all way up (Voevodsky circa 2008).

h-levels (Voevodsky)

- (i) Given space is called A *contractible* (aka space of h-level 0) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- (ii) We say that A is a space of h-level $n+1$ if for all its points x, y path spaces $paths_A(x, y)$ are of h-level n.

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- Level 0: up to homotopy equivalence there is just one contractible space that we call "point" and denote pt;
- Level 1: up to homotopy equivalence there are two spaces here: the empty space $\emptyset$ and the point pt. (For $\emptyset$ condition (ii) is satisfied vacuously; for pt (ii) is satisfied because in pt there exists only one path, which consists of this very point.) We call $\emptyset, pt$ *truth values*; we also refer to types of this level as *properties* and *propositions*. Notice that h-level n corresponds to the logical level $n-1$: the propositional logic (i.e., the propositional segment of our type theory) lives at h-level 1.

h-universe

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- Level 2: Types of this level are characterized by the following property: their path spaces are either empty or contractible. So such types are disjoint unions of contractible components (points), or in other words *sets* of points. This will be our working notion of set available in this framework.

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- Level 4: 2-groupoids

h-universe

- ..
- Level n+2: *n*-groupoids
- ..
- ω-groupoids
- ω-groupoids $(\omega + 1 = \omega)$

Key Feature

Proofs are (geometrical, not just syntactic) constructions (= objects, not meta-objects). Constructive mathematics is not metamathematics! (Harper) The schizophrenia of "meta" mathematics is effectively avoided.

Linguistic Conclusion

A space of possibilities, if properly construed, is an objective geometrical representation rather than a mere metaphor.

Epistemological Conclusion

Logic is a part of mathematics, rather than mathematics is an application of logic (Brouwer) [while mathematics is a part of (mathematized) physics rather than physics is an application of mathematics. (Arnold)].

Methodological Conclusion

Building a formal language for managing an experimental setup or empirical situation is a dependent part of mathematical design (or mathematical modeling) of this setup (situation). Geometric modeling grounds logical modeling rather than the other way round. In order to design effective logical tools for decision making and similar practical tasks one should start with non-logical mathematical modeling of the given subject-matter rather than linguistic intuitions about this subject-matter.

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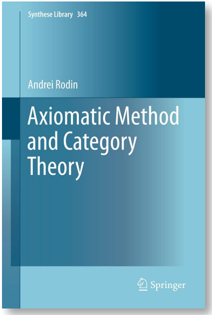

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A. Rodin, State University of Saint-Petersburg, Saint-Petersburg, Russian Federation

Axiomatic Method and Category Theory

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- **Offers readers a coherent look at the past, present and anticipated future of the Axiomatic Method**
- **Provides a deep textual analysis of Euclid, Hilbert, and Lawvere that describes how their ideas are different and how their ideas progressed over time**
- **Presents a hypothetical New Axiomatic Method, which establishes closer relationships between mathematics and physics**

This volume explores the many different meanings of the notion of the axiomatic method, offering an insightful historical and philosophical discussion about how these notions changed over the millennia.

The author, a well-known philosopher and historian of mathematics, first examines Euclid, who is considered the father of the axiomatic method, before moving onto Hilbert and Lawvere. He then presents a deep textual analysis of each writer and describes how their ideas are different and even how their ideas progressed over time. Next, the book explores category theory and details how it has revolutionized the notion of the axiomatic method. It considers the question of identity/equality in mathematics as well as examines the received theories of mathematical structuralism. In the end, Rodin presents a hypothetical New Axiomatic Method, which establishes closer relationships between mathematics and physics.

Lawvere's axiomatization of topos theory and Voevodsky's axiomatization of higher http://philomatica.org/

THE END