Categorical Model Theory and the Semantic View of Theories

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Categorical Model theory

Today's Categorical Model theory (CMT) stems from the functorial semantics of algebraic theories proposed by Lawvere in his thesis back in 1963 [1]. This theory uses a family of concepts of model none of which can be called today fairly standard. This fact is evidenced by the continuing discussion in the Homotopy Type theory (HoTT) [2] where presently there is no full agreement among the researchers in the field as to what counts as a model of this theory and what does not.

One approch relies on the concept of *classifying category* T freely generated from the syntax of the given theory. Then a model M is a functor $T \to C$ into the category of sets ($C = Set$) or another appropriate category. This functorial setting has an important universal property: up to the categorical equivalence T can be identified with the initial object in the functor category of T-models. This property allows one to think of a theory in this setting as being a "generic model" (Lawvere).

Voevodsky [3] pursues a different approach, which involves the concept of *contextual category* (more recently - in a modified form of C-system) earlier proposed by Cartmell [4]. The idea behind the concept of contextual category is that of a category, which fully encodes all relevant algebraic features of the given syntax. In this case the initiality property of the syntactic category $S(T)$ is not implied by any general theorem. The initiality conjecture for HoTT still stands open.

Finally, there is an approach in CMT, which involves the concept of *internal language* of a given category. It has been proposed to think of internal languages and syntactic categories in terms of adjoint functors between a category of theories and a category of categories:

$$Categories \underset{Synt}{\overset{Lang}{\rightleftarrows}} Theories$$

Then a model of given theory T in a certain ground category C is a functor of the form

$$M : T \to Lang(C)$$

which expresses the idea of representation of a given theory in the language of some other theory .

Modeling HoTT

The classical Tarskian notion of model based on the T-schema and the satisfaction relation does not fully support the model theory of HoTT in its existing form. HoTT involves a semi-formal interpretation of its syntax in the Homotopy theory: types are interpreted as spaces and terms are interpreted as points of these spaces. This interpretation helped to reveal a feature of MLTT's syntax, which earlier remained hidden: types in MLTT are stratified into the *homotopy levels*. This stratification necessitates a revision of the popular "propositions-as-types paradigm" : only types of certain homotopic level (namely, of level (- 1) as defined in [2]) can be identified with propositions while the higher types should be interpreted differently. This fact implies that HoTT cannot be coherently interpreted as a system of propositions or sentences; correspondingly, the Tarskian notion of model applies only to the propositional fragment of HoTT but not to this theory as a whole.

Semantic View of theories: a constructive perspective

P. Suppes [5] argued that a typical scientific theory should be identified not with any particular class of statements (formal or contentual) but rather with a certain class of models. On this basis Suppes and his followers designed a Bourbaki-style format of formal presentation where a scientific theory is presented through an appropriate class of itsset-theoretic models.Albeit such a Bourbaki-style presentation can be useful for purposes of logical and structural analysis, it appears to be useless as a practical tool, which may help working scientists to formulate and develop their theories in a formal setting [6].

HoTT and its model theory provides novel notions of theory and its model, which involve a higher-order non-propositional structure. They better fit the colloquial counterparts of these notions in the scientific practice than the standard Tarskian notions because the *procedural* content of a typical scientific theory does not reduce to the procedures of logical inference (if by the logical inference one understands a procedure which inputs and outputs sentences) but also comprises procedures of many different sorts.

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