WHAT IS A GEOMETRICAL OBJECT?

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1. Introduction

Charles Parsons in his recent monograph describes a general notion of object (that he uses as a basis for his following discussion about mathematical objects) as this:

> The usable general characterization of the notion of object comes from *logic*. We speak of particular objects by referring to them by singular terms: names, demonstrative and descriptions.

As Parsons himself remarks, a similar notion of object is found in Quine, Carnap, Russell and Frege. The way, in which these authors use the term "object" makes a sharp contrast with the way, in which this term (or more precisely its German counterparts "Objekt" and "Gegenstand") is used by Kant, Meinong (in particular, in his *Theory of Objects*), Cassirer and other authors following Kants terminology. Saying that the term is used by the two group of authors in different senses is a truism. My claim is that this terminological difference reflects a significant philosophical controversy, which, among other things, has a bearing on the question "What is a mathematical object?".

2. Historical background: a Kant-Frege controversy

In this part of my talk I shall explore the aforementioned controversy. A relevant discussion is found in Frege's *Foundations of Arithmetic* where this author critically revises Kant's views on mathematics. This discussion shows, among other things, that inspite of the fact that the case of mathematical objects is quite specific (with respect to the general notion of object) it nevertheless, historically, played a crucial role in the development of this controversy. I shall pay a special attention to Frege's argument according to which while Kant's notion of mathematical object appears to be overtly inadequate to some contemporary (i.e. developed by 1880-ies) mathematical concepts (like infinite cardinals and heigher-dimensional geometrical spaces), his (Frege's) proposed notion of object allegedly copes with such novel mathematical inventions. In this context I shall also briefly discuss a relevant part of Russell's *Principles of Mathematics*.

3. Objects are maps

Here I shall propose a new solution of the Kant-Frege controversy. Just like the Frege-Russell's approach my proposed approach aims at providing a philosophical account of and a foundation for some significant recent mathematical developments (as a matter of course I also take into account developments occured between 1900 and 2000). However unlike Frege and Russell I am trying to improve on Kant's views rather than get rid of the transcendental problematics altogether. In order to limit the discussion I shall consider only the special case of geometrical objects (but be ready to discuss possible generalizations of my proposal during the question time).

I shall start this part of the talk with a historical remark showing that the issue of representation, which is a central topic of Kant's *transcendental esthetics*, has been a hot issue in mathematics itself already in Kant's time (in Projective geometry). This line of mathematical research brought about an important change in the common understanding of the subject-matter of geometry: today this discipline no longer studies figures and magnitudes represented in a fixed representation space but studies geometrical spaces and their mutual representations (i.e. mutual maps) themselves. The fact that on Kant's view a space of representation is supposed to be rigidly fixed and remain out of the reach of a properly mathematical study (being a subject-matter of critical phylosophy rather than pure geometry), clearly makes Kant's original approach inadequate to these important developments.

Contrary to Frege I shall argue that the capacity of switching between different representation spaces is a basic feature of the human spatial intuition, which plays a major role in geometry since the early days of this discipline. Here is a simple example from Greek geometry. Let me distinguish between three different notions of Euclidean plane, which are often confused in the common mathematical parlance:

- (i) the universe (domain) of Euclidean Planimetry (EPLANE)
- (ii) an object studied in Euclidean Stereometry (eplane)
- (iii) a type of such objects

While the distinction between ii) and iii) is not specific for geometry (its an istance of the usual type/token distinction) the distinction between i) and ii) deserves our special attention. In Euclidean Planimetry (think of the content of Books 1-4 of Euclid's *Elements*) EPLANE doesn't appear as a geometrical object; in this framework one cannot represent or imagine EPLANE and so one cannot study (or make constructions with) this thing along with studying circles, triangles and other plane figures. In this case EPLANE functions as the representation space in the Kantian sense of the term. However in (Euclidean) Stereometry (think of Books 11-13 of the *Elements*) one gets a representation (and in fact multiple representations) of an Euclidean plane, namely one represents it as an eplane.
Unlike EPLANE an eplane is a full-fledged geometrical object, which is represented in the 3-dimensional Euclidean space (hereafter ESPACE). The fact that an eplane, which is a particular geometrical object living in ESPACE, "carries with it" the whole geometrical universe of EPLANE (i.e., the universe of plane Euclidean geometry) is far from being trivial. Switching between the two representation spaces, namely between EPLANE and ESPACE, is a fundamental operation of the traditional Euclidean geometry. The often repeated claim that ESPACE is the only representation space compatible with (or perhaps constituted by) the human spatial intuition doesn't take this feature into account. Notice that the plane Euclidean geometry in its traditional form can not be developed in ESPACE: dimension matters!

In modern mathematical terms a representation of EPLANE in ESPACE can be described as an *injective map* of the form:

eplane: EPLANE $\rightarrow$ ESPACE

Notice that while spaces EPLANE and ESPACE are unique there are as many different maps of the given form as diferent Euclidean planes sharing the same Euclidean 3-dimentional space.

Generalizing upon this and other similar examples I shall identify a geometrical object with a map (not necesserily injective) between some corresponding spaces. Having in mind Kants subtle distinctions between his notions of *Vorstellung*, *Gegenstand* and *Objekt*, one can introduce such further distinctions into my proposed account too. However I shall not try to do this now and - as the first approximation - talk about maps (between spaces), representations and objects interchangeably.

Crucially, my proposed notion of geometrical object works for general Riemannian spaces as well as for pure topological spaces. In all these cases the spatial intuition remains at work. I shall give here just one historical example. In his *Studies in Theory of Parallels* (1840) Lobachevsky first showed that EPLANE can be represented not only in ESAPCE as usual but also in a different (namely, Hyperbolic) 3-dimentional space (HSPACE). This representation can be described as a map of the form

horosphere: EPLANE $\rightarrow$ HSPACE

The corresponding object that Lobachevsky called a *horosphere* didn't "look like" an eplane (actually it looked like a curve surface) but still carried on it the usual plane Euclidean geometry. Unlike Frege Lobachevsky didn't believe that the fact that one and the same EPLANE has such different representations shows that the geometrical intuition becomes unreliable in this case. Indeed if one takes Kant's point that the representation space matters - so what is a given geometrical object (and also how it "looks like") essentially depends on a given representation space - then the fact that an eplane and a horosphere look so differently (being images of the same EPLANE) no longer seems particularly surprising. Let me stress that unlike an eplane and a horosphere EPLANE is not a geometrical object (in my sense) but a geometrical space like ESPACE or HSPACE. If we take it now for granted (after Kant) that a mathematical reasoning always require an object this fact implies that EPLANE cannot be mathematically studied by itself independently of some relevant objects, i.e. without some appropriate maps from and to this given space. (Relevant maps also include the reversible maps of ESPACE into itself: the importance of such maps was first stressed by Klein in his "Erlangen Programme".) Given a map between two geometrical spaces (or a map of a given space to itself) it would be senseless, in my view, to assume that the source space of this object determines this object "essentially" while the target space of this very object is responsible for some sort of "imprecision" of the resulting image. Such a thinking applies an old-fashioned metaphysics in a quite inappropriate context. Geometrical representations (maps, objects) don't "hide" anything behind them; they don't need any ideal - "purely logical" or other - non-spatial prototype. They merely represent some spaces in some other spaces. Although I disagree with Frege, Russell and Parsons that the "usable general characterization of the notion of object comes from logic" I agree with these people that in mathematics in general and in geometry in particular the notion of object is more fundamental than that of space. Given a geometrical object one can make a distinction between its source space and its target space. However there is no sense in which a space can be given independently of some associated objects.

A Kantian may object that unless all thinking subjects share the same representation space geometry cannot be objective and so would reduce to a subjective play of imagination. I shall argue that this argument is wrong. A shared spaced may indeed provide a notion of objectivity but there are other ways to do this. What makes modern geometry objective is not any particular representation space but general rules about representation, i.e., about mapping geometrical spaces onto each other. In todays mathematics such general rules are formulated as postulates of Category theory; these rules belong to foundations of modern geometry. Noticeably these rules have a strong intuitive appeal just like Euclid's Postulates. (I may remind and discuss these rules during the question time.)

4. Conclusion

In the Conclusion of my talk I shall come back to Parsons views about mathematical objects and try to compare his views with mine. Parsons claims that mathematical objects "are distinctive in being abstract" and then explains:

> An object is abstract if it is not located in space and time and does not stand in causal relations.

Since I identify geometrical objects with maps between geometrical spaces there is a sense in which such objects are always "located in space". So they are not abstract in Parsons' sense (if I interpret him correctly). However the space - or more precisely spaces - associated with geometrical objects are, of course, geometrical spaces, which should not be straightforwardly identified with physical spaces. If by a "physical space" one means a notion of space belonging to some well-established physical theory (that may be or be not fundamental) such a notion is necessarily mathematical (geometrical). However the converse, generally, is not true: a mathematical notion of space doesn't authomatically become a part of some physical theory even if it has a potential for it. Having this in mind it is not unreasonable to describe, generally, geometrical spaces and geometrical objects as *abstract* with respect to corresponding physical spaces and physical objects. This latter sense of being abstract is rather traditional (think of Aristotles notion of abstraction) and quite different from Parsons'. Denying that mathematical objects are, generally, abstract in Parsons' sense I endorse the view that they are, generally, abstract in this more traditional sense.