Kant on Objecthood and Objectivity
Kant and New Mathematics
Revival of Transcendental Aesthetics through Category theory
Concluding Remarks

OBJECTS ARE MAPS (v2)

Andrei Rodin

PARIS-DIDEROT

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Three key points of Kant's philosophy of mathematics:

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Three key points of Kant's philosophy of mathematics:

- ▶ (K1) Mathematical reasoning (as distinguished from a philosophical speculation) is, generally, a reasoning *in concreto*, i.e. a reasoning with individual imaginary *objects*. The cognitive capacity to represent general concepts by individual objects is called *intuition*.

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- ▶ (K2) Intuition underlies human sensual perception. Mathematical objects are, generally, objects of *possible* physical experience.

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▶ (K2) Intuition underlies human sensual perception. Mathematical objects are, generally, objects of *possible* physical experience.

▶ (K3) *Objectivity* of mathematics and natural science amounts to the fact that object-construction is a subject to certain universal Principles (*Principles of the Pure Understanding*).

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Kantian vocabulary

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Kantian vocabulary

► K1 and K3 belong to the subject-matter of *Transcendental Logic* . Transcendental Logic is distinguished by Kant from Formal Logic as the logic of "objectual" reasoning from the logic of speculative reasoning with bare concepts.
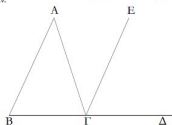
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- ▶ K2 is a subject-matter of *Transcendental Aesthetics*

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Explanation of K1-K2: Euclidean scheme

λβ´.

Παντὸς τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ἴση ἐστίν, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.

proposition — For any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.

"Έστω τρίγωνον τὸ ΑΒΓ, καὶ προσεκβεβλήσθω αὐτοῦ μία πλευρὰ ἡ ΒΓ ἐπὶ τὸ Δ· λέγω, ὅτι ἡ ἐκτὸς γωνία ἡ ὑπὸ ΑΓΔ ἴση ἐστὶ δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ταῖς ὑπὸ ΓΑΒ, ΑΒΓ, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΓΑ, ΓΑΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν.

"Ήχθω γὰρ διὰ τοῦ Γ σημείου τῇ ΑΒ εὐθείᾳ παράλληλος ἡ ΓΕ.

Καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΕ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΑΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΒΑΓ, ΑΓΕ ἴσαι ἀλλήλαις εἰσίν. πάλιν, ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΕ, καὶ εἰς αὐτὴς ἐμπέπτωκεν εὐθεῖα ἡ ΒΔ, ἡ ἐκτὸς γωνία ἡ ὑπὸ ΕΓΔ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΑΒΓ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΑΓΕ τῇ ὑπὸ ΒΑΓ ἴση· ὅλη ἄρα ἡ ὑπὸ ΑΓΔ γωνία ἴση ἐστὶ δυσὶ ταῖς ἐντὸς

καὶ ἀπεναντίον ταῖς ὑπὸ ΒΑΓ, ΑΒΓ.

Κοινὴ προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΔ, ΑΓΒ τρισὶ ταῖς ὑπὸ ΑΒΓ, ΒΓΑ, ΓΑΒ ἴσαι εἰσίν. ἀλλ' αἱ ὑπὸ ΑΓΔ, ΑΓΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΑΓΒ, ΓΒΑ, ΓΑΒ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Παντὸς ἄρα τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ἴση ἐστί, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν· ὅπερ ἔδει δεῖξαι.

Proposition 32

For any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.
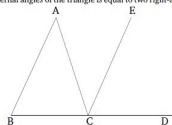
Let ABC be a triangle, and let one of its sides BC — exposition have been produced to D. I say that the external angle — specification ACD is equal to the (sum of the) two internal and opposite angles CAB and ABC, and the (sum of the) three internal angles of the triangle—ABC, BCA, and CAB— is equal to two right-angles.

For let CE have been drawn through point C parallel — construction to the straight-line AB [Prop. 1.31].

And since AB is parallel to CE, and AC has fallen — proof across them, the alternate angles BAC and ACE are equal to one another [Prop. 1.29]. Again, since AB is parallel to CE, and the straight-line BD has fallen across them, the external angle ECD is equal to the internal and opposite (angle) ABC [Prop. 1.29]. But ACE was also shown (to be) equal to BAC. Thus, the whole an-

gle ACD is equal to the (sum of the) two internal and opposite (angles) BAC and ABC.

Let ACB have been added to both. Thus, (the sum of) ACD and ACB is equal to the (sum of the) three (angles) ABC, BCA, and CAB. But, (the sum of) and ACB is equal to two right-angles [Prop. 1.13]. Thus, (conclusion (the sum of) ACB, CBA, and CAB is also equal to two of the proof) right-angles.

Thus, for any triangle, (if) one of the sides (is) pro- conclusion (of duced (then) the external angle is equal to the (sum of the theorem) the) two internal and opposite (angles), and the (sum of

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Explanation of: K1-K2-K3

"An *object* is that in the concept of which the manifold of intuition is united. .... Transcendental unity of apperception is that unity through which all the manifold given in an intuition is united in a concept of the object. It is called *objective* on that account." (Kant, *Critique of Pure Reason*)

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Kant and New Mathematics

Does Kant's philosophy of mathematics remain tenable vis-à-vis
the discovery of Non-Euclidean geometries?

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Does Kant's philosophy of mathematics remain tenable vis-à-vis the discovery of Non-Euclidean geometries?

▶ Q1: Is Kant's philosophy compatible with Non-Euclidean geometry?

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Does Kant's philosophy of mathematics remain tenable vis-à-vis
the discovery of Non-Euclidean geometries?

- ▶ Q1: Is Kant's philosophy compatible with Non-Euclidean
  geometry?
- ▶ Q2: Can Kant's philosophy cope with multiple geometries and
  multiple geometrical spaces?

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Q1: Is Kants philosophy compatible with Non-Euclidean geometry?

"I will not count among my principles those of mathematics, but I will include those on which the possibility and objective a priori validity of the latter are grounded, and which are thus to be regarded as the principle of these principles, and that proceed from concepts to the intuition and not from the intuition to concepts." (Kant, *ib*)

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The fact that one particular mathematical proposition like Euclid's Fifth Postulate turns to be not *objectively valid* cannot have a significant impact on Kant's philosophy of mathematics. Thus the answer to Q1 is in positive (at least *modulo* an appropriate modification of the *Principles of Pure Understanding*).

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Q2: Can Kant's philosophy cope with <u>multiple</u> geometrical spaces?

"One can only represent a single space, and if one speaks of many spaces, one understands by that only parts of one and the same unique space." (Kant, *ib.*)

Thus the answer to Q2 is clearly in negative.

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Q2: Can Kant's philosophy cope with multiple geometrical spaces?

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Thus the answer to Q2 is clearly in negative. The uniqueness of (physical) space is assumed by Newtonian Physics. Given K2 the uniqueness of physical space implies the uniqueness of geometrical space.

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On Kant's account the uniqueness of space is a necessary condition of the *transcendental unity of apperception*. So this feature of Kant's philosophy cannot be modified easily. (However in the next part of this talk I shall suggest a way to do this !)

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Various reactions to the conflict between Kant's philosophy and the new mathematics

A: Replacement of the transcendental philosophy by a new philosophy of mathematics (Frege, Russell and the mainstream Analytic philosophy). Systematic arguments against the transcendental philosophy are not provided. It is simply assumed that the transcendental philosophy is inadequate to the modern mathematics and a replacement is suggested.

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On this account the term "object" becomes interchangeable with the term "individual" (as this latter term is used in logic). Kantian distinction between speculative and mathematical reasoning is blurred.

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As it is so often happens in history this new philosophy revived some traditional approaches, which had been abandoned earlier....

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A: Frege on geometry

"The truths of geometry govern all that is spatially intuitable, whether actual or product of our fancy. The wildest visions of delirium, the boldest inventions of legend and poetry, ... all these remain, so long as they remain intuitable, still subject to the axioms of geometry." (*ib.*)

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A: Frege on geometry

Kant would disagree. According to Kant the whole difference between the "wildest visions of delirium" and the geometrical intuition is that the latter unlike the former is guided by certain principles, which Kant calls *Principles of Understanding*.

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A: Frege on geometry

"Conceptual thought alone can after a fashion shake off this yoke, when it assumes, say, a space of four dimensions or positive curvature. To study such conceptions ... is to leave the ground of intuition enirely behind. If we do make use of intuition even here as an aid, it is still the same old intuition of Euclidean space, the only space of which we have any picture. Only then the intuition is not taken at its face value, but as symbolic of something else." (*ib.*)

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A: Frege

"It is in this way that I understand *objective* to mean what is independent of our sensation, intuition and imagination, and of all construction of mental pictures out of memories or earlier sensations, but not what is independent of the reason." (*ib.*)

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A: Frege: Remarks

A straightforward Platonism? By Platonism I mean here: making a self-explanotary distinction between "lower" cognitive capacities like sensation, intuition and imagination, on the one hand, and the <u>reason</u> as the "higher" cognitive capacity, on the other hand.

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A: Frege: Remarks

A straightforward Platonism? By Platonism I mean here: making a self-explanotary distinction between "lower" cognitive capacities like sensation, intuition and imagination, on the one hand, and the reason as the "higher" cognitive capacity, on the other hand. Return to the pre-critical thinking? A critical approach amounts to a study of cognitive capacities involved in doing science and mathematics rather than simply to separating "higher" capacities from "lower" ones.

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A: Parsons:

"There is something absurd about inquiring, with complete generality, what an object is. .... The usable general characterization of the notion of object comes from logic. We speak of particular objects by referring to them by singular terms: names, demonstrative and descriptions." (2008)

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B: Development of a transcendental theory of *objects* independent of mathematics (Cassirer and Meinong).

"With this arises a problem, which lies completely outside the scope of logistics ...Worrying about the rules that govern the world of objects is completely left to direct observation, which is the only one that can teach us ... whether we can find here certain regularities or a pure chaos. Logic and mathematics deal only with the order of concepts; they don't contest the order or the disorder of objects and they don't need to confuse themselves with this issue." (Cassirer, *Kant and Modern Mathematics*, reacting on Russell and Couturat)

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B: Remark

Remarkably Cassirer shares with Frege the view according to which in the end of 19th century mathematics has profoundly changed its nature and transformed itself into a pure conceptual speculation. This is a reason why Cassirer unlike Kant doesn't see any specific link between mathematics and the transcendental study of objecthood.

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Euclidean scheme: a modern example

Arguably the modern mathematics did NOT change its traditional nature that far. It still widely uses the Euclidean Scheme. Instantiation of general concepts by individual objects (K1) remains a distinctive feature of mathematical thinking.

Proposition 1. *A category* **C** *with finite limits and small Hom-sets has a subobject classifier if and only if there is an object* Ω *and an isomorphism*

$$\theta_X \colon \mathrm{Sub}_{\mathbf{C}}(X) \cong \mathrm{Hom}_{\mathbf{C}}(X, \Omega), \qquad (4)$$

natural for $X \in \mathbf{C}$. *When this holds,* **C** *is well-powered.*

Proof: Given a subobject classifier as in (3), the correspondence θ_X sending the equivalence class of each monic $S \rightarrowtail X$ to its (unique) "characteristic function" $\phi \colon X \to \Omega$ is a bijection for each X, as required for (4). Now $\mathrm{Sub}_{\mathbf{C}}(X)$ is a (contravariant) functor of X by pullback (= inverse image); so to prove this bijection natural, we must show that pullback along $f \colon Y \to X$ in $\mathrm{Sub}_{\mathbf{C}}(-)$ corresponds to composition with f in $\mathrm{Hom}_{\mathbf{C}}(-, \Omega)$. This is immediate by the elementary fact that two pullback squares placed side by side, as in

$$S' \longrightarrow S \longrightarrow 1$$

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C: Formalisation of mathematics (after Hilbert and Bernays in *Foundations of Mathematics* of 1934)

This project aims at providing the modern mathematics with a clear intuitive basis and thus making it compatible with the original Kantian approach. This project aims at changing mathematics itself rather than its philosophy. The philosophy remains Kantian.

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C: Formalization

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- ▶ Objects of Elementary Arithmetic are strings of 1s

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C: Formalization

- ▶ Objects of Elementary Arithmetic are strings of 1s
- ▶ Objects of Elementary Algebra (Algebra of Polynomials) are strings of algebraic symbols (letters, brackets, $+, -, \bullet$, etc.)

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C: Formalization

- ▶ Objects of Elementary Arithmetic are strings of 1s
- ▶ Objects of Elementary Algebra (Algebra of Polynomials) are strings of algebraic symbols (letters, brackets, $+, -, \bullet$, etc.)
- ▶ Objects of a formalized mathematical theory are strings of logical symbols, which are similar to algebraic symbols. "Then, in principle, we have the same situation as in our treatment of the Elementary Arithmetic."

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C: Remarks

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▶ (Symbolic) Logic is used here as a *means* helping to provide the whole of mathematics with a simple and uniform "figural" intuitive support.

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C: Remarks

- ▶ (Symbolic) Logic is used here as a *means* helping to provide the whole of mathematics with a simple and uniform "figural" intuitive support.

- ▶ These foundations are "liberal" in the sense that they don't rule out a possibility of providing mathematics (or a part of mathematics) with a different intuitive basis. Hilbert and Bernays suggest the reader *a* way in which mathematics *can* be done; they do NOT argue that this is *the* way in which mathematics *must* be done.

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D: Hintikka

Kant's Transcendental Logic is right and but his Transcendental Aesthetics is wrong. Hintikka's modern version of Transcendental Logic is a version of modern first-order logic, namely, IF-logic provided with Game-Theoretic semantics.

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D: Hintikka

"Kant thought that the general character of the mathematical method amounts to using particular representatives of general concepts. In modern language this is tantamount to a systematic use of rules of exemplification. Since those rules make part of the modern first-order logic Kants theory applies to the logic of quantification (the first-order logic) rather than mathematics." (*Transcendental argumentation revived*)

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D: Hintikka

"Since according to Kant the last word of the mathematical
method is the treatment of particular objects, those objects are, for
him, what modern logicians call "individual". But where such
individuals come from? Kants answer is attractive but false. He
thinks that those objects "are given to us through the sensual
perception, which alone is capable of providing us with the
intuition" and he defines such intuitions as representation of
particulars." (*ib.*)

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Revival of Transcendental Aesthetics through Category theory

I'm sympathetic to Jaakko's enthusiasm about Transcendental Logic but I'm not sympathetic to his skepticism about Transcendental Aesthetics. I believe that Kant's Transcendental Aesthetics also can and should be modernized! The following is an attempt to do this.

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Transcendental Aesthetics matters!

Transcendental Aesthetics is not only about the sensual perception. It is also about how mathematical objects are *generated* in the human imagination.

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Transcendental Aesthetics matters!

Transcendental Aesthetics is not only about the sensual perception. It is also about how mathematical objects are *generated* in the human imagination.

"We can think no line without drawing it in thought, no circle without describing it." (Kant, *ib.*)

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Euclid's Postulates

Not to be confused with axioms! Postulates 1-3 are NOT
propositions.

Αἰτήματα.

α΄. Ἡιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

β΄. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ᾽ εὐθείας ἐκβαλεῖν.

γ΄. Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

δ΄. Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

ε΄. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ᾽ ἄπειρον συμπίπτειν, ἐφ᾽ ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

Postulates

1. Let it have been postulated to draw a straight-line from any point to any point.

2. And to produce a finite straight-line continuously in a straight-line.

3. And to draw a circle with any center and radius.

4. And that all right-angles are equal to one another.

5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).[†]

[†] This postulate effectively specifies that we are dealing with the geometry of *flat*, rather than curved, space.

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Hilbert and Bernays on Postulates

"The intuitive meaning of the figures is not ignored in Euclid. Furthermore, its axioms are not in existential form either: Euclid does not presuppose that points or lines constitute any fixed domain of individuals. Therefore, he does not state any existence axioms either, but only construction postulates." (Hilbert and Bernays, *ib.*)

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Constructive Postulates

- In a formalized axiomatic theory the role of non-propositional constructive postulates is played by *formation rules*.

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Constructive Postulates

- ▶ In a formalized axiomatic theory the role of non-propositional constructive postulates is played by *formation rules*.
- ▶ Formation rules make part of foundations of a given formalized theory along with axioms of this theory.

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Constructive Postulates

- ▶ In a formalized axiomatic theory the role of non-propositional constructive postulates is played by *formation rules*.
- ▶ Formation rules make part of foundations of a given formalized theory along with axioms of this theory.
- ▶ In my view such constructive postulates is an indispensable element of any mathematical theory; such postulates make part of *foundations* of a given theory along with axioms of this theory.

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Towards a new Transcendental Aesthetics

However Kant didn't take these (as well as many other) mathematical developments into his account.

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Representations in mathematics and philosophy

The issue of *representation*, which is central for Kant's Transcendental philosophy, was a hot issue in mathematics itself already in Kant's times, think of Projective geometry. Projective geometry originates from the *study of perspective*, i.e. the study of maps (projections) between the Euclidean space and the Euclidean plane. This study involved different geometrical spaces from the very outset.

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Representations in mathematics and philosophy

Since according to Kant "the" space is essentially unique the study of perspective is in odds with Kant's Transcendental Aesthetics. However the study of perspective *is* supported by the spatial intuition and *is* relevant to the human sensual perception. A representation of a 3D object by its plane picture *is* intuitively conceivable. This shows that Kant's Transcendental Aesthetics doesn't wholly cover its own subject.

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Examples

The following two geometrical examples will help me to introduce
a general notion of (mathematical) object.

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Ex.1: Euclidean plane

Let's distinguish between two notions of Euclidean plane:

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Ex.1: Euclidean plane

Let's distinguish between two notions of Euclidean plane:

► the universe of Euclidean Planimetry (EPLANE) and

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Ex.1: Euclidean plane

Let's distinguish between two notions of Euclidean plane:

- ▶ the universe of Euclidean Planimetry (EPLANE) and
- ▶ an object of Euclidean stereometry (an eplane in ESPACE)

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Ex.1: Euclidean plane

Let's distinguish between two notions of Euclidean plane:

- ▶ the universe of Euclidean Planimetry (EPLANE) and
- ▶ an object of Euclidean stereometry (an eplane in ESPACE)

Than an eplane can be described as an injective map
eplane: EPLANE→ ESPACE

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Ex.1: Euclidean plane

The EPLANE cannot be (and needs not to be) visualized within the Planimetry. However one gets an intuitive *image* of EPLANE by mapping it into ESPACE (i.e., into a different space). While EPLANE is unique its images in ESPACE (i.e., eplanes) are many.

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Ex.1: Euclidean plane

Every eplane "carries with it" the whole world of Planimetry into the ESPACE.

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Ex.1: Euclidean plane

Every eplane "carries with it" the whole world of Planimetry into the ESPACE. The capacity of switching between the plane intuition and the stereometric intuition is required for doing the traditional Euclid-style geometry. Such an intuitive capacity is not accounted for by Kant's Transcendental Aesthetics.

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Ex.2: horosphere

Like a number of his predecessors Lobachevsky first developed
Hyperbolic geometry by combining Euclid-style intuitive reasoning
with some analytic tools (in the sense of Analytic geometry of 18th
century). His main achievement was fixing the *analytic* part of this
new theory.

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Hyperbolic geometry by combining Euclid-style intuitive reasoning
with some analytic tools (in the sense of Analytic geometry of 18th
century). His main achievement was fixing the *analytic* part of this
new theory. For this end Lobachevsky considered a special surface
in the hyperbolic 3-space (HSPACE), which he called a *horosphere*.
Lobachevsky observed that the intrinsic geometry of a horosphere
is the same as that of an eplane. Then by applying the usual plane
trigonometric identities to triangles on a horosphere he figured out
the corresponding identities for triangles on a tangent hyperbolic
plane.

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Ex.2: horosphere

A horosphere can be described as an injective map
horosphere: EPLANE$\rightarrow$ HSPACE
it is an object living in the Hyperbolic 3-space.

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Objects are maps!

Generalizing upon these and similar examples I shall call a *geometrical object* any map (not necessarily injective) of geometrical spaces. I shall try to think similarly about *all* mathematical objects (through replacing geometrical spaces by representation schemes of other sorts).

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A suggestive terminology

It is appropriate to call the source-space of a given object by the term "type" reserving the word "space" for its target-space. Think again about eplane: EPLANE→ ESPACE. Actually we think about the EPLANE as a *space* only when we study objects of the form $x : X \rightarrow$ EPLANE living *in* the EPLANE (i.e., plane figures). Given an eplane we rather think about the EPLANE as a *type* shared by all eplanes.

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A duality between types and spaces

Being a *type* and being a *space* (in my sense) are relational
properties. Studying objects of the form $X \rightarrow Y$ with fixed Y and
variable X one describes Y as a *space*. This provides an *intrinsic*
view on Y.

Studying objects of the form $Y \rightarrow X$ with fixed Y and variable X
(once again) one describes the same Y as a *type*. This provides an
extrinsic view on Y.

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A comparison with Hilbert's (nowadays standard) view

According to the standard view an eplane and a horosphere are two different intuitive images (representations, realizations, models) of the same plane Euclidean *structure*; this structure is supposed to catch the notion of EPLANE in purely conceptual terms. Such an approach privileges the *intrinsic view* on a given object against the *extrinsic view*. I can see no epistemic reason for making this preference. The fact that eplanes live in the ESPACE and that horospheres live in the HSPACE is just as "essential" as the fact that these objects are images of one and the same EPLANE!

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A comparison with Kant's view: the Right/Left Hands puzzle

Here is a modern solution. Right and left hands are *intrinsically* the same, i.e. such things belong to the same *type* HAND. The difference between right and left hands is an *extrinsic* difference that is due to the fact that hands live in the ESPACE. Left and right hands are maps of *different sorts* albeit all these maps have the same form *hand*: HAND→ ESPACE. The fact that certain maps into ESPACE can be classified in this way is an *intrinsic* property of ESPACE (but not of HAND!), which is called *orientability*.

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A non-orientable 3-space (TO SKIP IN CASE OF SHORTAGE OF TIME)

A simple example of non-orientable 3-space is MÖXI, which is a product (X) of Möbius strip MÖ and an interval I. *Hands* are at home in the MÖXI just like in the ESPACE but here one cannot classify them into right and left ones!

(Any trouble to imagine this? Try first with Γ-shapes on the usual Möbius strip and try to forget that this strip lives in the ESPACE. This latter piece of information is in fact wholly irrelevant to what you are after. Try to think of a Möbius strip as a *space* rather than an object in the ESPACE. Imagine yourself being an ant living on the Möbius strip. After this painful exercise you'll feel in MÖXI perfectly comfortable: you'll be happy about the more convenient dimension number as well as about the fact that you don't need any longer to care about the orientation!)

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Kant's Right/Left Hands Puzzle: the modern solution

Kant rightly realized that HAND <u>as a type</u> leaves no room for
making the right/left distinction, and that this latter distinction is
related to the way in which this type is *represented* in space. What
Kant couldn't do is to treat HAND and ESPACE on equal footing
as we do this today. The distinction between orientable and
non-orientable spaces is as much "conceptual" as that between
hands and feet. Kant couldn't see this because he didn't know
that ESPACE can (and actually should!) be also thought of as a
<u>type</u> and not only as a space.

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A Transcendental Problem

In Kant the uniqueness of representation space guaranties the *objective* character of what Kant calls the *transcendental unity of apperception* as well as the *objective* character of intuitive mathematics. Now let's assume that at least some non-Euclidean representations (i.e. representations in various non-Euclidean spaces) are as much intuitively appealing as Euclidean representations. Then there arise the following problem: what would provide the transcendental unity of apperception and the objective character of mathematics in a multi-space environment?

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Desiderata:

We need certain principles applicable to all representations in a
multi-space environment. At least some of these principles must be
of constructive character (like Euclid's *Postulates*). The
multi-space environment must by *unified* according to those
principles.

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Solution

The notion of *category* of spaces does the job!

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Solution

The notion of *category* of spaces does the job!

Following Kant I shall not put here the standard axioms of Category theory explicitly but try to provide a *transcendental deduction* of certain principles *behind* these axioms

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Solution

The notion of *category* of spaces does the job!

Following Kant I shall not put here the standard axioms of Category theory explicitly but try to provide a *transcendental deduction* of certain principles *behind* these axioms

WARNING: my terminology in this papers differs from the standard terminology of CT. My "objects", alia "maps", are called in CT "maps" or "morphisms" while my "types/spaces" are called in CT "objects". This terminological change is essential for my purpose, which is to look at CT in an unusual way.

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Transcendental Deduction of Category Theory

Consider after Kant a "manifold of intuition", which is NOT yet "united" and hence does not yet provide us with well-distinguishable *objects*.

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Transcendental Deduction of Category Theory

Consider after Kant a "manifold of intuition", which is NOT yet "united" and hence does not yet provide us with well-distinguishable *objects*. In order to identify an object one needs first of all to distinguish between its *type* (which the given object represents) and its *space* (in which the given object is represented); geometrically speaking one needs to distinguish between *intrinsic* and *extrinsic* features of the given (proto-)object.

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Transcendental Deduction of Category Theory

I consider specification of types/spaces of given (proto-)objects as a *relational* matter: I assume that it cannot be reasonably done for a single isolated (proto-)object. It can be done for a *manifold* of (proto-)objects through its *categorification*, which is a procedure that makes the given manifold of intuition into a *category*.

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Unity through Categorification

Categorification comprises three operations (which are not independent):

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Unity through Categorification

Categorification comprises three operations (which are not independent):

- typing : specification of objects sharing a type;

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Unity through Categorification

Categorification comprises three operations (which are not independent):

▶ typing : specification of objects sharing a type;

▶ spacing : specification of objects sharing a space;

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Unity through Categorification

Categorification comprises three operations (which are not independent):

- ▶ typing : specification of objects sharing a type;
- ▶ spacing : specification of objects sharing a space;
- ▶ combining : specification of pairs of elements such that the type of one given object is also the space of another given object. (Informally speaking this latter condition means that one object provides a space for the other).
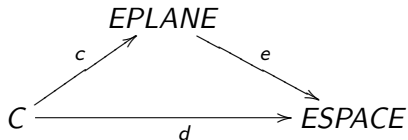
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Typing

Typing is understood here as usual. (I leave aside the old
controversy between nominalistic and realistic interpretations of
typing.)

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Spacing

Think of Euclidean and hyperbolic triangles. They share their type but they do not live in the same space!

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Combining

I assume - as usual - that *combining* of two objects always brings
about a (unique) new object. However in the given context this
notion of combining has a specific sense. Here is an example. By
combining a plane circle $c : C \rightarrow EPLANE$ with an eplane
$e : EPLANE \rightarrow ESPACE$ one gets a <u>disk</u> $d : C \rightarrow ESPACE$ living
in the 3D space. This combination can be shown with a diagram:

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Categorification

In order to make our "manifold of intuition" into a *category* in the usual mathematical sense of this term there remains:

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Categorification

In order to make our "manifold of intuition" into a *category* in the usual mathematical sense of this term there remains:

- ▶ to postulate that each *type* instantiates itself in itself (as in a *space*) and that among such instantiations there is one, which is neutral with respect to combination with any other object (the identity morphism)

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- ▶ to grant the associativity of combination of objects (of morphisms).

I leave the transcendental deduction of these final steps of categorification for a future work.

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Categorification

An appropriate specification of (categorical) properties of a
category of spaces fully determines "what those spaces are".
Compare with Hilbert's approach.

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A comparison with Kant and Hilbert

We have seen that the mathematical notion of category provides a
novel way of unification of a given intuitive manifold. It brings
about a novel way of thinking about objecthood and objectivity.
The key features of this approach are these:

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- the categorical objecthood remains relevant to sensual
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- the categorical objectivity and objecthood allow for a
  multiplicity of representation spaces, which is not the case in
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  organization of intuitive representations like in Kant
- ▶ the categorical objecthood remains relevant to sensual
  perception like in Kant
- ▶ the categorical objectivity and objecthood allow for a
  multiplicity of representation spaces, which is not the case in
  Kant
- ▶ the categorical notion of object is *relational* in the sense that
  it relates *what* is represented to *where* it is represented (which
  is not the case either in Kant or Hilbert et al.).

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An open problem

Is it possible to make alternative choices of the two spaces for the same object? Here is a hint. By presenting a given object O as a bare set of *points* living in some given space S and assuming that the intrinsic geometry of a point is null one gets a "purely extrinsic" presentation of this object. Alternatively one may think of a given object "purely intrinsically" as a *space* inhabited by some other objects and independent of any "external" space. Such "extreme" presentations in different ways "kill" the given object. A bare set of points in a space is no longer a single object; in this case one reduces the *type* of the given object. A space is not an object either. Considering the given object "purely intrinsically" as a space one forgets about the "ambient" space, in which this given object is represented. It is interesting to study the spectrum of all intermediate cases.