

Doing and Showing (Axiomatic Method in Ancient and Modern Mathematics)

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Hilbert on Axiomatic method

When we assemble the facts of a definite field of knowledge, we soon notice that these facts are capable of being ordered. This ordering always comes about with the help of a certain framework of concepts ... The framework of concepts is nothing other than the theory of the field of knowledge.

Hilbert on Axiomatic method (continued)

If we consider a particular theory more closely, we always see that a few distinguished propositions of the field of knowledge underlie the construction of the framework of concepts, and these propositions then suffice by themselves for the construction, in accordance with logical principles, of the entire framework.

Hilbert on Axiomatic method (continued)

These fundamental propositions can be regarded ... as the axioms of the individual fields of knowledge : the progressive development of the individual field of knowledge then lies solely in the further logical construction of the already mentioned framework of concepts. This standpoint is especially predominant in pure mathematics.

Hilbert on Axiomatic method (continued)

[A]nything at all that can be the object of scientific thought becomes dependent on the axiomatic method, and thereby indirectly on mathematics.
Axiomatic Thought, 1918

Hilbert on Axiomatic method (continued)

The axiomatic method is and remains the indispensable tool, appropriate to our minds, for all exact research in any field whatsoever : it is logically incontestable and at the same time fruitful. ... To proceed axiomatically means in this sense nothing else than to think with consciousness.

The New Grounding of Mathematics, 1922

Claim :

The persisting gap between the formal and the informal mathematics is due to an inadequate notion of mathematical theory behind the current formalization techniques. I mean the (informal) notion of axiomatic theory according to which a mathematical theory consists of a set of axioms and further theorems deduced from these axioms according to certain rules of logical inference. Thus I claim that the usual notion of *axiomatic method* is inadequate and needs a replacement.

Claim (continued) :

In particular, I claim that *elementary theories* like ZFC are not adequate as foundations (albeit they may be useful for some other purposes).

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- ▶ Formalization and symbolization : a comparison with the early modern symbolic algebra ;
- ▶ Axiomatic mathematical theories do not apply in experimental sciences ; logical forms of mathematical reasoning are not fundamental ;
- ▶ Traditional mathematical constructivism does not meet this challenge ; constructive mathematical theories useful in natural sciences need a new method of theory-building.

Axiomatic Method

Problems and Theorems in Euclid

Modern Axiomatic Method

Euclid and Modern Mathematics

Formalization and Symbolization

Mathematical Constructivism

Theorem 1.5 of Euclid's ELEMENTS :

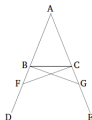
[*enunciation* :]

For isosceles triangles, the angles at the base are equal to one another, and if the equal straight lines are produced then the angles under the base will be equal to one another.

Theorem 1.5 (continued) :

[exposition] :

Let ABC be an isosceles triangle having the side AB equal to the side AC ; and let the straight lines BD and CE have been produced further in a straight line with AB and AC (respectively). [Post. 2].



Theorem 1.5 (continued) :

[*specification* :]

I say that the angle ABC is equal to ACB , and (angle) CBD to BCE .

Theorem 1.5 (continued) :

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I say that the angle ABC is equal to ACB , and (angle) CBD to BCE .

[construction :]

For let a point F be taken somewhere on BD , and let AG have been cut off from the greater AE , equal to the lesser AF [Prop. 1.3]. Also, let the straight lines FC , GB have been joined. [Post. 1]

Theorem 1.5 (continued) :

[proof :]

In fact, since AF is equal to AG , and AB to AC , the two (straight lines) FA , AC are equal to the two (straight lines) GA , AB , respectively. They also encompass a common angle FAG . Thus, the base FC is equal to the base GB , and the triangle AFC will be equal to the triangle AGB , and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) ACF to ABG , and AFC to AGB . And since the whole of AF is equal to the whole of AG , within which AB is equal to AC , the remainder BF is thus equal to the remainder CG [Ax.3]. But FC was also shown (to be) equal to GB . So the two (straight lines) BF , GC are equal to the two (straight lines) CG , GB .

Theorem 1.5 (continued) :

[*conclusion* :]

Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

Problem 1.1 of Euclid's ELEMENTS :

[*enunciation* :]

To construct an equilateral triangle on a given finite straight-line.

Problem 1.1 (continued) :

[*exposition* :]

Let AB be the given finite straight-line.

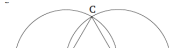
[*specification* :]

So it is required to construct an equilateral triangle on the straight-line AB .

Problem 1.1 (continued) :

[*construction* :]

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C , where the circles cut one another, to the points A and B [Post. 1].



Problem 1.1 (continued) :

[proof:]

And since the point A is the center of the circle CDB , AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE , BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB . Thus, CA and CB are each equal to AB . But things equal to the same thing are also equal to one another [Axiom 1]. Thus, CA is also equal to CB . Thus, the three (straight-lines) CA , AB , and BC are equal to one another.

Problem 1.1 (continued) :

[*conclusion* :]

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB . (Which is) the very thing it was required to do.

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- ▶ Axioms (Common Notions) :
play the role similar to that of logical rules restricted to mathematics : cf. the use of the term by Aristotle
- ▶ Postulates :
non-logical constructive rules

Common Notions

- A1. Things equal to the same thing are also equal to one another.
- A2. And if equal things are added to equal things then the wholes are equal.
- A3. And if equal things are subtracted from equal things then the remainders are equal.
- A4. And things coinciding with one another are equal to one another.
- A5. And the whole [is] greater than the part.

Axioms

Κοινὰ ἔννοιαι.

- α'. Τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἔστιν ἴσα.
β'. Καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἔστιν ἴσα.
γ'. Καὶ ἐὰν ἀπὸ ἴσων ἴσα ἀφαιρεθῇ, τὰ καταλειπόμενα
ἔστιν ἴσα.
δ'. Καὶ τὰ ἐφαρμόζοντα ἐπ' ἀλλήλα ἴσα ἀλλήλοις
ἔστιν.
ε'. Καὶ τὸ ὅλον τοῦ μέρους μείζον [ἔστιν].

Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.[†]
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

Common Notions (continued)

Euclid's Common Notions hold *both* for numbers and magnitudes (hence the title of “common”); they form the basis of a regional “mathematical logic” applicable throughout the mathematics. Aristotle transform them into laws of logic applicable throughout the *episteme*, which in Aristotle's view does not reduce to mathematics but also includes *physics*.

Common Notions (continued)

Euclid's Common Notions hold *both* for numbers and magnitudes (hence the title of “common”); they form the basis of a regional “mathematical logic” applicable throughout the mathematics. Aristotle transform them into laws of logic applicable throughout the *episteme*, which in Aristotle's view does not reduce to mathematics but also includes *physics*. Aristotle describes and criticizes a view according to which Common Notions constitute a basis for *Universal Mathematics*, which is a part of mathematics shared by all other mathematical disciplines. In 16-17th centuries the Universal Mathematics is often identified with Algebra and for this reason Euclid's Common Notions are viewed as *algebraic* principles.

Aristotle on Axioms (1) :

By first principles of proof [as distinguished from first principles in general] I mean the common opinions on which all men base their demonstrations, e.g. that one of two contradictories must be true, that it is impossible for the same thing both be and not to be, and all other propositions of this kind.” (Met. 996b27-32)

Here Aristotle refers to a logical principle as “common opinion”.

Aristotle on Axioms (1) :

ἀλλὰ μὴν καὶ περὶ τῶν ἀποδεικτικῶν ἀρχῶν, πότερον μιᾶς ἐστὶν ἐπιστήμης ἢ πλειόνων, ἀμφισβητήσιμόν ἐστιν (λέγω δὲ ἀποδεικτικὰς τὰς κοινὰς δόξας ἐξ ὧν ἅπαντες δεικνύουσιν) οἷον ὅτι πᾶν ἀναγκαῖον ἢ φάναι ἢ ἀποφάναι, καὶ ἀδύνατον ἅμα εἶναι καὶ μὴ εἶναι, καὶ ὅσαι ἄλλαι τοιαῦται προτάσεις, πότερον μία τούτων ἐπιστήμη καὶ τῆς οὐσίας ἢ ἕτερα, κἂν εἰ μὴ μία, ποτέραν χρὴ προσαγορεύειν τὴν ζητουμένην νῦν. μιᾶς μὲν οὖν οὐκ εὐλογον εἶναι: τί γὰρ μᾶλλον γεωμετρίας ἢ ὅποιασοῦν περὶ τούτων ἐστὶν ἴδιον τὸ ἐπαίειν;

Aristotle on Axioms (2) :

Comparison of mathematical and logical axioms :

We have now to say whether it is up to the same science or to different sciences to inquire into what in mathematics is called axioms and into [the general issue of] essence. Clearly the inquiry into these things is up to the same science, namely, to the science of the philosopher. For axioms hold of everything that [there] is but not of some particular genus apart from others. Everyone makes use of them because they concern being qua being, and each genus is. But men use them just so far as is sufficient for their purpose, that is, within the limits of the genus relevant to their proofs.

Aristotle on Axioms (2), continued :

Since axioms clearly hold for all things qua being (for being is what all things share in common) one who studies being qua being also inquires into the axioms. This is why one who observes things partly [=who inquires into a special domain] like a geometer or an arithmetician never tries to say whether the axioms are true or false. But some of the physicists did so, quite naturally; for they alone professed to investigate nature as a whole, and Being. (Met. 1005a19-28)

Aristotle on Axioms (2) :

λεκτέον δὲ πότερον μιᾶς ἢ ἐτέρας ἐπιστήμης περί τε τῶν ἐν τοῖς μαθήμασι καλουμένων ἀξιομάτων καὶ περὶ τῆς οὐσίας. φανερόν δὴ ὅτι μιᾶς τε καὶ τῆς τοῦ φιλοσόφου καὶ ἡ περὶ τούτων ἐστὶ σκέψις: ἅπασι γὰρ ὑπάρχει τοῖς οὖσιν ἀλλ' οὐ γένει τινὶ χωρὶς ἰδίᾳ τῶν ἄλλων. καὶ χρῶνται μὲν πάντες, ὅτι τοῦ ὄντος ἐστὶν ἡ ὄν, ἕκαστον δὲ τὸ γένος ὄν: ἐπὶ τοσοῦτον δὲ χρῶνται ἐφ' ὅσον αὐτοῖς ἰκανόν, τοῦτο δ' ἔστιν ὅσον ἐπέχει τὸ γένος περὶ οὗ φέρουσι τὰς ἀποδείξεις: ὥστ' ἐπεὶ δῆλον ὅτι ἡ ὄντα ὑπάρχει πᾶσι (τοῦτο γὰρ αὐτοῖς τὸ κοινόν) , τοῦ περὶ τὸ ὄν ἡ ὄν γνωρίζοντος καὶ περὶ τούτων ἐστὶν ἡ θεωρία.

Aristotle on Axioms (2) :

διόπερ οὐθείς τῶν κατὰ μέρος ἐπισκοπούντων ὄν: ἐπὶ τοσοῦτον δὲ
χρῶνται ἐφ' ὅσον αὐτοῖς ἰκανόν, τοῦτο δ' ἔστιν ὅσον ἐπέχει τὸ γένος
περὶ οὗ φέρουσι τὰς ἀποδείξεις: ὥστ' ἐπεὶ δῆλον ὅτι ἡ ὄντα ὑπάρχει πᾶσι
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περὶ τούτων ἐστὶν ἡ θεωρία. διόπερ οὐθείς τῶν κατὰ μέρος
ἐπισκοπούντων ἐγχειρεῖ λέγειν τι περὶ αὐτῶν, εἰ ἀληθὴ ἢ μή, οὔτε
γεωμέτρης οὔτ' ἀριθμητικός , ἀλλὰ τῶν φυσικῶν ἔνιοι, εἰκότως τοῦτο
δρῶντες: μόνοι γὰρ ὥντο περὶ τε τῆς ὅλης φύσεως σκοπεῖν καὶ περὶ τοῦ
ὄντος

Aristotle on Axioms (3) :

Reference to Ax.3 :

Since the mathematician too uses common [axioms] only on the case-by-case basis, it must be the business of the first philosophy to investigate their fundamentals. For that, when equals are subtracted from equals, the remainders are equal is common to all quantities, but mathematics singles out and investigates some portion of its proper matter, as e.g. lines or angles or numbers, or some other sort of quantity, not however qua being, but as [...] continuous in one, two or three dimensions . (Met. 1061b)

Aristotle on Axioms (3) :

ἐπεὶ δὲ καὶ ὁ μαθηματικὸς χρήται τοῖς κοινοῖς ἰδίως, καὶ τὰς
τούτων ἀρχὰς ἂν εἴη θεωρῆσαι τῆς πρώτης φιλοσοφίας. ὅτι γὰρ
ἀπὸ τῶν ἴσων ἴσων ἀφαιρεθέντων ἴσα τὰ λειπόμενα, κοινὸν μὲν
ἐστὶν ἐπὶ πάντων τῶν ποσῶν, ἡ μαθηματικὴ δ' ἀπολαβοῦσα περὶ
τι μέρος τῆς οἰκείας ὕλης ποιεῖται τὴν θεωρίαν, οἷον περὶ
γραμμὰς ἢ γωνίας ἢ ἀριθμοὺς ἢ τῶν λοιπῶν τι ποσῶν, οὐχ ἢ δ'
ὄντα ἀλλ' ἢ συνεχὲς αὐτῶν ἕκαστον ἐφ' ἑν ἢ δύο ἢ τρία

Postulates 1-3 :

P1 : to draw a straight-line from any point to any point.

P2 : to produce a finite straight-line continuously in a straight-line.

P3 : to draw a circle with any center and radius.

Postulates

Αιτήματα.

α'. Ἡτιθήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

β'. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχές ἐπ' εὐθείας ἐκβαλεῖν.

γ'. Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράψασθαι.

δ'. Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

ε'. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένης τὰς δύο εὐθείας ἐπ' ἀπειρον συμπίπτειν, ἐφ' ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

Postulates

1. Let it have been postulated to draw a straight-line from any point to any point.

2. And to produce a finite straight-line continuously in a straight-line.

3. And to draw a circle with any center and radius.

4. And that all right-angles are equal to one another.

5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).[†]

[†] This postulate effectively specifies that we are dealing with the geometry of *flat*, rather than curved, space.

Postulates 1-3 (continued) :

Postulates 1-3 are NOT propositions ! They are not first truths.
They are basic (non-logical) *operations*.

Operational interpretation of Postulates

| Postulates | input | output |
|------------|-----------------------------------|------------------|
| P1 | two points | straight segment |
| P2 | straight segment | straight segment |
| P3 | straight segment and its endpoint | circle |

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Key points on Euclid :

- ▶ Problems et Theorems share a common 6-part structure (*enunciation, exposition, specification, construction, proof, conclusion*), which does NOT reduce to the binary structure *proposition - proof*.
- ▶ Postulates 1-3 and *enunciations* of Problems are NOT propositions but (non-logical) operations.
- ▶ Euclid's mathematics aims at *doing* AND *showing* but not only at showing and moreover not only to proving certain propositions

Doing and Shoing

ὅπερ ἔδει ποιῆσαι.

ὅπερ ἔδει δεῖξαι.

Aristotle's mathematical example

Let A be two right angles, B triangle, C isosceles. Then A is an attribute of C because of B, but it is not an attribute of B because of any other middle term; for a triangle has [its angles equal to] two right angles by itself, so that there will be no middle term between A and B, though AB is matter for demonstration.” (An. Pr. 48a33-37)

inadequacy of the syllogistic to geometrical proofs?

Aristotle's mathematical example

ἔστω τὸ Α δύο ὀρθαί, τὸ ἐφ' ᾧ Β τρίγωνον, ἐφ' ᾧ δὲ Γ ἰσοσκελές. τῷ μὲν οὖν Γ ὑπάρχει τὸ Α διὰ τὸ Β, τῷ δὲ Β οὐκέτι δι' ἄλλο (καθ' αὐτὸ γὰρ τὸ τρίγωνον ἔχει δύο ὀρθάς), ὥστ' οὐκ ἔσται μέσον τοῦ Α Β, ἀποδεικτοῦ ὄντος.

Conclusion on Euclid

The modern notion of axiomatic theory *prima facie* does not apply to the theory of Euclid's *Elements*

Modal and Existential Interpretation :

It is possible to translate the theory of Euclid's *Elements* into an (informal) axiomatic theory in the modern sense of the term through the Modal or the existential interpretation of Postulates and Problems. None of these two interpretations is *historically* grounded. None of them is innocent : both transform the content of Euclid's mathematics.

Universal Instantiation and Generalization :

$$\forall x P(x) \implies P(a/x)$$

$$P(a) \implies \forall x P(x)$$

Kant on geometry

Give a philosopher the concept of a triangle, and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on this concept as long as he wants, yet he will never produce anything new. He can analyze and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts.

Kant on geometry (continued)

But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle, and obtains two adjacent angles that together are equal to two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that here there arises an external adjacent angle which is equal to an internal one, etc. In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question. (A7167/B7445)

Grundlagen der Mathematik, 1934

I. Axioms of Connection.

1. $(x)(y) Gr(x, x, y)$.
“ x, x, y always lie on one straight line.”^{5.4}
2. $(x)(y)(z) (Gr(x, y, z) \rightarrow Gr(y, x, z) \ \& \ Gr(x, z, y))$.
“Whenever x, y, z lie on a straight line, then so do y, x, z as well as x, z, y .”
3. $(x)(y)(z)(u) (Gr(x, y, z) \ \& \ Gr(x, y, u) \ \& \ x \neq y \rightarrow Gr(x, z, u))$.
“For any x, y, z, u , if x, y are distinct points and x, y, z as well as x, y, u lie on straight lines, then x, z, u lie on a straight line as well.”

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“For any x, y, z, u , if x, y are distinct points and x, y, z as well as x, y, u lie on straight lines, then x, z, u lie on a straight line as well.”
4. $(Ex)(Ey)(Ez) \overline{Gr(x, y, z)}$.
“There are points x, y, z that do not lie on a straight line.”

Grundlagen der Mathematik, 1934

III. Axiom of Parallels.

Since we omit the congruence axioms, we have to take the Axiom of Parallels in its sharper form:^{6.4} “For every straight line, through every point outside of it, there is one and only one non-intersecting straight line.”¹

To simplify the symbolic formulation, the symbol

$$Par(x, y; u, v)$$

will be used as an abbreviation for:

$$\overline{(Ew)} (Gr(x, y, w) \ \& \ Gr(u, v, w))$$

“There is no point w that is on a straight line with x and y as well as with u and v .”

The axiom then reads:

$$(x)(y)(z) \left(\overline{Gr(x, y, z)} \rightarrow (Eu) \left\{ \& \begin{array}{l} Par(x, y; z, u) \\ (v) (Par(x, y; z, v) \rightarrow Gr(z, u, v)) \end{array} \right\} \right)$$

Grundlagen der Mathematik, 1934

If we consider all the axioms listed here to be consecutively conjoined with $\&$, we obtain a single formula representing a proposition about the predicates Gr , Zw . This formula will be designated by

$$\mathfrak{A}(Gr, Zw).$$

Similarly, we can represent any theorem of plane geometry that is only about the relations of position and order^{6.5} as a formula

$$\mathfrak{S}(Gr, Zw).$$

Grundlagen der Mathematik, 1934

The part of axiomatic geometry we considered so far contains two variable ternary predicates

$$R(x, y, z), \quad S(x, y, z).$$

The system of axioms consists of a requirement on two such predicates, expressed in the logical formula $\mathfrak{A}(R, S)$, obtained from $\mathfrak{A}(Gr, Zw)$ by replacing $Gr(x, y, z)$ with $R(x, y, z)$, and $Zw(x, y, z)$ with $S(x, y, z)$. In addition to the variable predicates, this formula contains the identity relation $x = y$, which is to be interpreted contentually.^{7.4}

Grundlagen der Mathematik, 1934

According to this view, a sentence of the form $\mathfrak{S}(Gr, Zw)$ corresponds to a statement of the following logical content: For *arbitrary* predicates $R(x, y, z)$, $S(x, y, z)$ satisfying the condition $\mathfrak{A}(R, S)$, the relation $\mathfrak{S}(R, S)$ holds as well, i.e., for any two predicates $R(x, y, z)$, $S(x, y, z)$, the formula

$$\mathfrak{A}(R, S) \rightarrow \mathfrak{S}(R, S)$$

is a true proposition. In this way, a geometrical sentence is transformed into a sentence of pure predicate logic.

Kolmogorov and Fomin, 1976

[*enunciation* :]

Any closed subset of a compact space is compact

[*exposition* :]

Let F be a closed subset of compact space T

[*specification* : I say that F is a compact space]

Kolmogorov and Fomin, 1976 (continued) :

[*construction :*]

[Let] $\{F_\alpha\}$ [be] an arbitrary centered system of closed subsets of subspace $F \subset T$.

[*proof :*]

[E]very F_α is also closed in T , and hence $\{F_\alpha\}$ is a centered system of closed sets in T . Therefore $\cap F_\alpha \neq \emptyset$. By Theorem 1 it follows that F is compact.

[*conclusion :* Thus any closed subset of a compact space is compact. (Which is) the very thing it was required to show.]

ELEMENTS DE LA THEORIE DES ENSEMBLES, Rédaction 50

ELEMENTS DE LA THEORIE DES
ENSEMBLES.

Rédaction n° 050

Philosophical Statement

First of all let us clarify what we understand under the name of mathematical theory. A mathematical theory is a study of one or more categories of elements, of their properties and relations that unify them, and of constructions made out of them. Such a study cannot proceed without assuming a number of mutually consistent propositions concerning these elements, these properties, these relations and these constructions. The purpose of the theory is to deduce from these premises some other propositions, so that their exactness depends only on the exactness of the premises but does not require any further hypothesis.

Axiomatic theory ?

Basic Constructions

In any mathematical theory one begins with a number of fundamental sets, each of which consists of elements of a certain type that needs to be considered. Then on the basis of types that are already known one introduces new types of elements (for example, the subsets of a set of elements, pairs of elements) and for each of those new types of elements one introduces sets of elements of those types.

Basic Constructions (continued)

So one forms a family of sets constructed from the fundamental sets. Those constructions are the following :

- 1) given set A , which is already constructed, take the set $P(A)$ of the subsets of A ;*
- 2) given sets A, B , which are already constructed, take the cartesian product $A \times B$ of these sets.*

The sets of objects, which are constructed in this way, are introduced into a theory step by step when it is needed. Each proof involves only a finite number of sets. We call such sets types of the given theory ; their infinite hierarchy constitutes a scale of types.

Structure

On this basis Bourbaki describes the concept of *structure* as follows :

We begin with a number of fundamental sets : A, B, C, \dots, L that we call base sets. To be given a structure on this base amounts to this :

1) be given properties of elements of these sets ; 2) be given relations between elements of these sets ; 3) be given a number of types making part of the scale of types constructed on this base ; 4) be given relations between elements of certain types constructed on this base ; 5) assume as true a number of mutually consistent propositions about these properties and these relations.

(Informal) Bourbaki and Euclid

Principles of building mathematical theory described in the Bourbaki's draft are not so different from Euclid's (in spite of the above statement). These principles adequately describe what is done in a large part of research mathematics of the 20th century. Like Euclid Bourbaki begins his exposition with principles of building mathematical objects but not with certain propositions about some abstract entities assumed as axioms. Propositions appear only in the very end (the 5th item of the above quote), and even it is usual to call them “axioms” (like in the case of axioms of group theory') it is clear that they are rather analogous to Euclid's *definitions*.

(Informal) Bourbaki and Euclid (continued)

While for Euclid the basic data is a finite family of *points* (everything else is constructed through Postulates) for Bourbaki the basic data is a finite family of *sets* and everything else is constructed as just described. While for Euclid the basic type of geometrical object is a *figure* for Bourbaki the basic type of mathematical object is a *structure*. In both cases the constructed objects come with certain propositions that can be asserted about these objects without proofs because they immediately follow from corresponding definitions. In both cases the construction of objects is a subject of certain *rules* but not the matter of a mere stipulation.

HOWEVER ALL OF THAT REMAINS “UNOFFICIAL” !

The Published “Official” Version

The first chapter of the treatise, which has the title *Description of Formal Mathematics*, begins with an account of *signs* and *assemblies* (strings) of signs provided with a definition of mathematical theory according to which such a theory

... contains rules which allow us to assert that certain assemblies of signs are terms or relations of the theory, and other rules which allow us to assert that certain assemblies are theorems of the theory.

The Published “Official” Version (continued)

In the published formalized version the set-theoretic constructions are replaced by syntactic constructions that formally *prove the existence* of certain sets. Basic objects of the formalized set theory are no longer sets and structures but symbolic expressions that can be interpreted as propositions *about* sets and structures.

Conclusion on Bourbaki

What makes the major difference between the formal and the informal versions of Bourbaki's set theory is the character of its *objects*; otherwise the two theories proceed similarly. Both follow Euclid's pattern. What remains unclear is this : In which sense if any the formal set theory tells us something about *sets* and further set-theoretic constructions. Prima facie it only tells us something about ways of *talking* about sets and set-theoretic constructions. It may only work if the St. John's Dogma is true. I shall now show that as far as mathematics is concerned this Dogma is false.

A comparison with the Symbolic Algebra

MacLaurin, *A Treatise of fluxions* :

The improvement that have been made by it [the doctrine of fluxions] ... are in a great measure owing to a facility, conciseness, and great extend of the method of computation, or algebraic part. It is for the sake of these advantages that so many symbols are employed in algebra. ... It [algebra] may have been employed to cover, under a complication of symbols, obstruse doctrines, that could not bear the light so well in a plain geometrical form ; but, without doubt, obscurity may be avoided in this art as well as in geometry, by defining clearly the import and use of the symbols, and proceeding with care afterwards.

Formalization (in the modern sense of the term) is supposed to play the same epistemic role as the symbolic algebraization as MacLaurin describes it : to make clear some “obstruse doctrines” through determining the “use of the symbols”. There is however an essential difference between the two approaches. While *algebraic* symbolic constructions mimic the constructions of mathematical *objects* referred to by the corresponding symbols the symbolic constructions of modern formal theories mimic informal *descriptions* of certain objects but not the construction of these objects themselves ! While the symbolic algebra represents forms of human constructive activities, and for this reason may guide human actions in the real material world, the formal mathematics reflects only *logical* forms of the pure speculative thought.

Speculative Thought and Modern (Galilean) Science

But the modern Galilean science requires an active intervention of humans into the nature rather than a passive observation of and speculation on natural phenomena. This is why the modern science turns to be so helpful for technology. This is why the formal mathematics is useless in the modern experimental science and technology (with the possible exception of the software engineering that does not deal with the hardware development).

St. John's Dogma (Lawvere)

The problem of the formal mathematics is that it takes the *logical* form to be fundamental. This dogma is incompatible with the Galilean science, which requires *doing* (experiments) rather than just *proving* (certain propositions on the basis on some plausible hypotheses).

Constructivism

The constructivist thinking in mathematics from the very beginning of the 20th century took a conservative bend and began a fight against then-new ways of mathematical thinking including the set-theoretic thinking. This tendency can be traced back to Kronecker who required every well-formed mathematical object to be constructible from natural numbers. More recently Bishop was inspired by similar ideas (and in particular by Kronecker's works).

Constructivism (continued)

Brouwer's *intuitionism* (which qualifies as a form of constructivism) also put rather severe restrictions on his contemporary mathematics as well as on essential parts of earlier established mathematical results. Even those constructivists who like Markov tried to develop constructive mathematics as a special part of mathematics rather than reform mathematics as a whole understood the notion of mathematical construction very restrictively and almost exclusively in computational terms.

Axiomatic Method

I claim that the general issue of the constructive thinking in mathematics concerns the very method of theory-building (the axiomatic method) rather than some particular principles like the principle of the excluded middle, etc. The modern axiomatic method is non-constructive by its very design because it doesn't require to provide rules of construction of mathematical objects (except formulae). What makes a system of postulates coherent?

Axiomatic Method (continued)

A genuinely constructive method of theory-building needs non-propositional principles similar to Euclid's Postulates and algebraic rules, which allow for building of and operating with objects and not only with formal expressions telling us something *about* these objects. Such a general method must not specify the postulates just like the standard axiomatic method does not specify axioms. Following Hilbert one should rather focus on questions concerning the mutual compatibility of postulates and the like.

Open Problems

What makes a system of postulates coherent? What is an appropriate analogue of the logical consistency for postulates? What is an appropriate notion of dependency for postulates? (Cf. the constructive type theory with dependent types.)

THE END