Euclid's "Elements" and Foundations of Mathematics

TEXNH 2010

5 août 2010

Content :

Euclides Restitutus Denuo Limatus ab Omni Naevo Vindicatus

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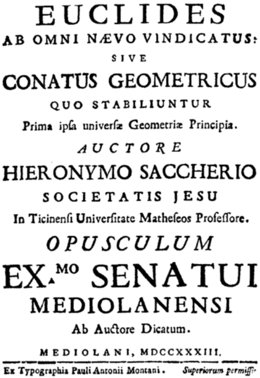
EUCLIDES
AB OMNI NÆVO VINDICATUS:
SIVE
CONATUS GEOMETRICUS
QUO STABILIUNTUR
Prima ipsa universæ Geometriæ Principia.
AUCTORE
HIERONYMO SACCHERIO
SOCIETATIS JESU
In Ticinensi Universitate Matheseos Professore.
OPUSCULUM
EX.MO SENATUI
MEDIOLANENSI
Ab Auctore Dicatum.
MEDIOLANI, MDCCXXXIII.
Ex Typographia Pauli Antonii Montani. Superiorum permissu.

Historical anecdote

Comparing once popular *Elements of Geometry* published by A. Tacquet in 1654 and the edition of Euclid's *Elements* (the first eight books thereof) published by M. Dechales 6 years later in 1660 it is difficult to say why the later work has Euclid's name in its title while the former doesn't. The difference between the two titles seems to be unrelated to the content of the two books although it might point to different intentions of their authors. When Tacquet's book was republished in 1725 (long after the authors death) it actually got Euclid's name on its cover !

Some more "Elements"

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- I. Barrow : 1733

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- J. Keill : 1754
- A. Arnauld :1667
- A.-M. Legendre : 1793 (AN II)

The Urtext

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Heiberg was Professor of Classical Philology at the University of Copenhagen from 1896 until 1924. Among his more than 200 publications were editions of the works of Archimedes (1880 and 1912), Euclid (with Heinrich Menge) (1883-1916), Apollonius of Perga (1891-93), Serenus of Antinouplis (1896), Ptolemy (1898), and Hero of Alexandria (1899). Many of his editions are still in use today.

Modern commented translations based on the Urtext

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First English translation :
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Vitrac, continued

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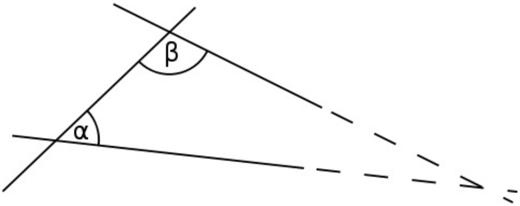
Today's Elements (also outdated but having no better replacement so far..)

D. Hilbert, *Grundlagen der Geometrie*, Leipzig 1899
N. Bourbaki, *Éléments de mathématique* (sic !), Paris 1939 - *circa* 2000

Invention of "non-Euclidean" geometries

People tried to prove or replace Fifth Postulate of Euclid's *Elements* because unlike other Principles of *Elements* this particular Postulate did not seem to be self-evident. (The popular view according to which the "usual" geometrical intuition is Euclidean doesn't stand against this historical evidence.)



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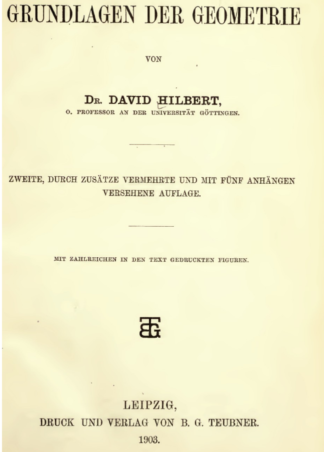
- Beltrami in 1868 discovered a link between the problem of parallels (Lobachevsky) and the geometry of curved surfaces (Gauss) and curve spaces (Riemann).

D. Hilbert : "Grundlagen der Geometrie"



GRUNDLAGEN DER GEOMETRIE

VON

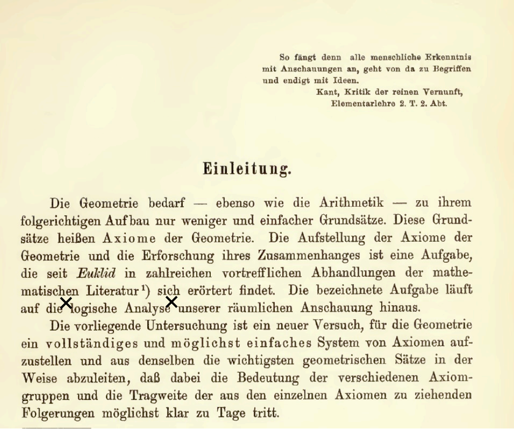
DR. DAVID HILBERT,
O. PROFESSOR AN DER UNIVERSITÄT GÖTTINGEN.

ZWEITE, DURCH ZUSÄTZE VERMEHRTE UND MIT FÜNF ANHÄNGEN VERSEHENE AUFLAGE.

MIT ZAHLREICHEN IN DEN TEXT GEDRUCKTEN FIGUREN.

LEIPZIG,
DRUCK UND VERLAG VON B. G. TEUBNER.
1903.

D. Hilbert : "Grundlagen der Geometrie"



So fängt denn alle menschliche Erkenntnis mit Anschauungen an, geht von da zu Begriffen und endigt mit Ideen.

Kant, Kritik der reinen Vernunft, Elementarlehre 2. T. 2. Abt.

Einleitung.

Die Geometrie bedarf — ebenso wie die Arithmetik — zu ihrem folgerichtigen Aufbau nur weniger und einfacher Grundsätze. Diese Grundsätze heißen Axiome der Geometrie. Die Aufstellung der Axiome der Geometrie und die Erforschung ihres Zusammenhanges ist eine Aufgabe, die seit *Euklid* in zahlreichen vortrefflichen Abhandlungen der mathematischen Literatur[1]) sich erörtert findet. Die bezeichnete Aufgabe läuft auf die logische Analyse unserer räumlichen Anschauung hinaus.

Die vorliegende Untersuchung ist ein neuer Versuch, für die Geometrie ein vollständiges und möglichst einfaches System von Axiomen aufzustellen und aus denselben die wichtigsten geometrischen Sätze in der Weise abzuleiten, daß dabei die Bedeutung der verschiedenen Axiomgruppen und die Tragweite der aus den einzelnen Axiomen zu ziehenden Folgerungen möglichst klar zu Tage tritt.

D. Hilbert : "Grundlagen der Geometrie"

Kapitel I.

Die fünf Axiomgruppen.

§ 1.

Die Elemente der Geometrie und die fünf Axiomgruppen.

Erklärung. Wir denken drei verschiedene Systeme von Dingen: die Dinge des ersten Systems nennen wir *Punkte* und bezeichnen sie mit $A, B, C, \ldots$; die Dinge des zweiten Systems nennen wir *Gerade* und bezeichnen sie mit $a, b, c, \ldots$; die Dinge des dritten Systems nennen wir *Ebenen* und bezeichnen sie mit $\alpha, \beta, \gamma, \ldots$; die Punkte heißen auch die *Elemente der linearen Geometrie*, die Punkte und Geraden heißen die *Elemente der ebenen Geometrie* und die Punkte, Geraden und Ebenen heißen die *Elemente der räumlichen Geometrie* oder *des Raumes*.

Wir denken die Punkte, Geraden, Ebenen in gewissen gegenseitigen Beziehungen und bezeichnen diese Beziehungen durch Worte wie „liegen", „zwischen", „parallel", „kongruent", „stetig"; die genaue und vollständige Beschreibung dieser Beziehungen erfolgt durch die *Axiome der Geometrie*.

Die Axiome der Geometrie gliedern sich in fünf Gruppen; jede einzelne dieser Gruppen drückt gewisse zusammengehörige Grundtatsachen unserer Anschauung aus. Wir benennen diese Gruppen von Axiomen in folgender Weise:

I 1—8. Axiome der *Verknüpfung*,
II 1—4. Axiome der *Anordnung*,
III 1—6. Axiome der *Kongruenz*,
IV. Axiom der *Parallelen*,
V 1—2. Axiome der *Stetigkeit*.

D. Hilbert : letter to Frege

You say that my concepts, e.g. "point", "between", are not unequivocally fixed ... . But surely it is self-evident that every theory is merely a framework or schema of concepts together with their necessary relations to one another, and that basic elements can be construed as one pleases. If I think of my points as some system or other of things, e.g. the system of love, of law, or of chimney sweeps ... and then conceive of all my axioms as relations between these things, then my theorems, e.g. the Pythagorean one, will hold of these things as well. In other words, each and every theory can always be applied to infinitely many systems of basic elements. For one merely has to apply a univocal and reversible one-to-one transformation and stipulate that the axioms for the transformed things be correspondingly similar. Indeed this is frequently applied, for example in the principle of duality, etc.

Veblen and Whitehead

The starting point of any strictly logical treatment of geometry (and indeed of any branch of mathematics) must then be a set of undefined elements and relations, and a set of unproved propositions (=axioms) involving them, and from these all other propositions (theorems) are to be derived by the methods of formal logic. Moreover, since we assumed the point of view of fromal (i.e. symbolic) logic, the undefined elements are to be regarded as mere symbols devoid of content..

Veblen and Whitehead

The notion of a *class* of objects is fundamental in logic and hence in any mathematical science. The object which make up the class are called the elements of the class. The notion of a class, moreover, and the relations of *belonging to a class* (being included in a class, being element of a class, etc.) are primitive notions of logic.

N. Bourbaki, Éléments de mathématique

Etat 2

ELEMENTS de la THEORIE des ENSEMBLES

Prolégomènes sur la notion de théorie mathématique

> La pensée qui contemple est le sujet de la réflexion,
> la pensée contemplée en est l'objet .
>
> (Victor Cousin)

Ce chapitre doit être considéré,par le lecteur, comme une préface nécessaire à toute théorie mathématique. Nous nous proposons d'y faire une description sommaire des êtres que l'on rencontre et des procédés que l'on utilise dans l'une quelconque de ces théories . Précisons d'abord ce que nous entendons par ce vocable: théorie mathématique. On appelle ainsi l'étude d'une ou plusieurs catégories d'éléments, de leurs propriétés, des relations qui les unissent, des constructions dont ils peuvent être les matériaux; cette étude ne se fait d'ailleurs qu'en admettant préalablement un certain nombre de propositions non contradictoires, concernant ces éléments, ces propriétés, ces relations, ces constructions . La tâche de la théorie est de déduire de ces prémisses d'autres propositions dont l'exactitude est seulement conditionnée par celle des propositions initialement admises, sans qu'il y ait besoin de faire de nouvelles hypothèses .

N. Bourbaki, Éléments de mathématique

Dans ce qui précède, nous laissons, à dessein, au au mot élément, son sens le plus vague. Il désignera seulement tout être susceptible de posséder les propriétés non contradictoires que nous lui prêtons. Les catégories d'éléments qui font ainsi l'objet d'une théorie mathématique constituent les ensembles fondamentaux de la théorie ; mais ces ensembles ne sont pas des agrégats amorphes; ils présentent une certaine organisation: nous entendons par ce dernier terme tout le complexe logique formé par les définitions des propriétés des éléments de ces ensembles, des relations qui les unissent, des constructions dont ils peuvent être les matériaux, et aussi par les propositions concernant ces propriétés, éléments, constructions, relations, qu'on regarde tout d'abord comme vraies . Cette organisation portera dans la suite le nom de structure.

Une théorie mathématique nous apparait donc comme résultant de la considération simultanée de deux entités bien distinctes : D'une part, les ensembles fondamentaux qui sont l'objet de la théorie, d'autre part, la structure qui forme le sujet de la théorie et qui en est la partie vivante et essentielle .

Le lecteur constatera par la suite qu'il est toujours très aisé de laisser indéterminée la nature des élé-

N. Bourbaki, Éléments de mathématique

ments constituant les ensembles fondamentaux , et qu'il y a le plus souvent intérêt à adopter cette position . De là à penser que seule la structure importe et que le véritable but de la théorie mathématique est l'étude d'une structure indépendamment des ensembles auxquels il est loisible de l'appliquer, il n'y a qu'un pas ; de fait, il est sans doute possible d'étudier les structures en elles-mêmes, en s'interdisant de considérer les ensembles fondamentaux ; mais pour des raisons de commodité de langage, et pour ne pas dérouter d'invincibles habitudes d'esprit, nous adopterons résolument le point de vue dit "ontologique", c'est-à-dire que nous envisagerons effectivement les ensembles fondamentaux de chaque théorie, nous les désignerons nommément ainsi que leurs éléments, par des symboles convenables, mais nous laisserons presque toujours leur nature tout-à-fait indéterminée.

Three versions of the (statement of the) Pythagorean theorem : Version 1 : Euclid

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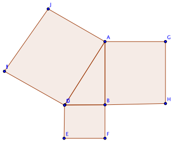
> *In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.*

(*Elements*, Proposition 1.47)



Three versions of the (statement of the) Pythagorean theorem : Version 2 : Arnauld (1667)

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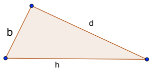
> *The square of hypothenuse is equal to (the sum of) squares of the two (other) sides (of the given rectangular triangle) : $bb + dd = hh$.*

(*New Elements of Geometry, Proposition 14.26.4*)

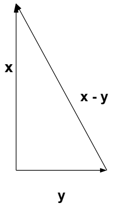


Three versions of the (statement of the) Pythagorean theorem : Version 3 : Doneddu (1965)

Three versions of the (statement of the) Pythagorean theorem : Version 3 : Doneddu (1965)

Two non-zero vectors x and y are orthogonal if and only if $(y - x)^2 = y^2 + x^2$

(Donnedu, *Euclidean plane geometry*)



Claim : Versions 1-3 of the Pythagorean theorem differ in their **foundations**, i.e., differ *radically*.

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Foundations change more rapidly than the rest of mathematics !

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Dialectical links between older and newer foundations are neither causal nor contingent. They represent an objective development of ideas.

A false counter-example : Euclid's *Elements*

It is often claimed that until recently Euclid's *Elements* used to be a Bible of mathematics. However, as we have seen, the literature published under the title of "Euclid's Elements" since the beginning of book printing is quite diverse. Revision of current versions of Euclid's book until very recently was a rule rather than an exception. The alleged stickiness to Euclid's letter NEVER existed in mathematics ! The history of revisions of Euclid's *Elements* still waits to be accounted for systematically !

Plato's philosophy of mathematics

WARNING : It has little if anything to do with "mathematical Platonism" that was first described by P. Bernays in 1935 and later became popular in the Analytic philosophy of mathematics.

Being and Becoming

[W]e must make a distinction and ask, What is that which always is and has no becoming ; and what is that which is always becoming and never is ? That which is apprehended by intelligence and reason is always in the same state ; but that which is conceived by opinion with the help of sensation and without reason, is always in a process of becoming and perishing and never really is. (Tim. 27d-28a)

Hypothetical knowledge

(Socrates talks to Gaucon)

"[Socrates :] - Next proceed to consider the manner in which the sphere of the intellectual is to be divided.

- In what manner ?

–There are two subdivisions, in the lower or which the soul uses the figures given by the former division as images ; the enquiry can only be hypothetical, and instead of going upwards to a principle descends to the other end ; in the higher of the two, the soul passes out of hypotheses, and goes up to a principle which is above hypotheses, making no use of images as in the former case, but proceeding only in and through the ideas themselves.

- I do not quite understand your meaning, he said.

Hypothetical knowledge

- Then I will try again ... . You are aware that students of geometry, arithmetic, and the kindred sciences assume the odd and the even and the figures and three kinds of angles and the like in their several branches of science ; these are their hypotheses, which they and everybody are supposed to know, and therefore they do not deign to give any account of them either to themselves or others ; but they begin with them, and go on until they arrive at last, and in a consistent manner, at their conclusion ?

Hypothetical knowledge

- Yes, he said, I know.
- And do you not know also that although they make use of the visible forms and reason about them, they are thinking not of these, but of the ideals which they resemble ; not of the figures which they draw, but of the absolute square and the absolute diameter, and so on –the forms which they draw or make, and which have shadows and reflections in water of their own, are converted by them into images, but they are really seeking to behold the things themselves, which can only be seen with the eye of the mind ?

Hypothetical knowledge

- That is true.
- And of this kind I spoke as the intelligible, although in the search after it the soul is compelled to use hypotheses ; not ascending to a first principle, because she is unable to rise above the region of hypothesis, but employing the objects of which the shadows below are resemblances in their turn as images, they having in relation to the shadows and reflections of them a greater distinctness, and therefore a higher value.
- I understand, he said, that you are speaking of the province of geometry and the sister arts.

(Rep., 510b-511c)

Hypothetical knowledge IS mathematical knowledge !

NO SPECIAL ROOM FOR NATURAL SCIENCE Physics is a "lower" section of mathematics (Quadrivium)

Critique of imagery

Plato criticize the geometrical reasoning (or more precisely the geometrical understanding) for using images. This critique follows from a more general notion, according to which opinion relies entirely on senses, reason operates with pure ideas without any help of sensual representations while mathematical understanding in general and geometrical understanding in particular do something in between. Geometry demonstrates the double nature of mathematics in the most explicit form. Plato's critique amounts to pushing mathematical understanding from the domain of opinion toward a dialectical pure reasoning.

Unwritten doctrine

Mathematical objects unlike their ideal prototypes exist in an indefinite number of copies (Met. 987b). There is an indefinite number of copies of mathematical number 2 (i.e. an indefinitely many of such numbers) all of which correspond to the same ideal number 2. The former unlike the latter cannot be a subject of arithmetical operations ; this in particular implies that ideal numbers unlike mathematical ones cannot be thought of as sums of units and so are indivisible (Met. 1081a-1082b). If one follows Plato's advise and "ascends" from mathematical objects to their ideal prototypes one certainly stops doing mathematics !

Aristotle's philosophy of mathematics : Nature of things and their Forms

Antiphon points out that if you planted a bed and the rotting wood acquired the power of sending up a shoot, it would not be a bed that would come up, but wood - which shows that the arrangement in accordance with the rules of the art is merely an incidental attribute, whereas the real nature is the other, which, further, persists continuously through the process of making. (Phys. 193a12-17)

Theory of Abstraction

[C]learly it is possible that there should also be both propositions and demonstrations about sensible magnitudes, not however qua sensible but qua possessed of certain definite qualities. For as there are many propositions about things merely considered as in motion, apart from what each such thing is and from their accidents, and as it is not therefore necessary that there should be either a mobile separate from sensibles, or a distinct mobile entity in the sensibles, so too in the case of mobiles there will be propositions and sciences, which treat them however not qua mobile but only qua bodies, or again only qua planes, or only qua lines, or qua divisibles, or qua indivisibles having position, or only qua indivisibles.

Theory of Abstraction

Thus since it is true to say without qualification that not only things which are separable but also things which are inseparable exist (for instance, that mobiles exist), it is true also to say without qualification that the objects of mathematics exist, and with the character ascribed to them by mathematicians. And as it is true to say of the other sciences too, without qualification, that they deal with such and such a subject - not with what is accidental to it (e.g. not with the pale, if the healthy thing is pale, and the science has the healthy as its subject), but with that which is the subject of each science - with the healthy if it treats its object qua healthy, with man if qua man : - so too is it with geometry ; if its subjects happen to be sensible, though it does not treat them qua sensible, the mathematical sciences will not for that reason be sciences of sensibles - nor, on the other hand, of other things separate from sensibles.

Theory of Abstraction

Many properties attach to things in virtue of their own nature as possessed of each such character ; e.g. there are attributes peculiar to the animal qua female or qua male (yet there is no 'female' nor 'male' separate from animals) ; so that there are also attributes which belong to things merely as lengths or as planes. And in proportion as we are dealing with things which are prior in reason and simpler, our knowledge has more accuracy, i.e. simplicity. Therefore a science which abstracts from spatial magnitude is more precise than one which takes it into account ; and a science is most precise if it abstracts from movement, but if it takes account of movement, it is most precise if it deals with the primary movement, for this is the simplest ; and of this again uniform movement is the simplest form. ...

Theory of Abstraction

Each question will be best investigated in this way - by setting up by an act of separation what is not separate, as the arithmetician and the geometer do. For a man qua man is one indivisible thing ; and the arithmetician supposed one indivisible thing, and then considered whether any attribute belongs to a man qua indivisible. But the geometer treats him neither qua man nor qua indivisible, but as a solid. For evidently the properties which would have belonged to him even if perchance he had not been indivisible, can belong to him even apart from these attributes. Thus, then, geometers speak correctly ; they talk about existing things, and their subjects do exist. (Met. 1077b16 - 1078a30)

Theory of Abstraction

δῆλον ὅτι ἐνδέχεται καὶ περὶ τῶν αἰσθητῶν μεγεθῶν εἶναι καὶ λόγους καὶ ἀποδείξεις, μὴ ᾗ δὲ αἰσθητὰ ἀλλ' ᾗ τοιαδί. ὥσπερ γὰρ καὶ ᾗ κινούμενα μόνον πολλοὶ λόγοι εἰσί, χωρὶς τοῦ τί ἕκαστόν ἐστι τῶν τοιούτων καὶ τῶν συμβεβηκότων αὐτοῖς, καὶ οὐκ ἀνάγκη διὰ ταῦτα ἢ κεχωρισμένον τι εἶναι κινούμενον τῶν αἰσθητῶν ἢ ἐν τούτοις τινὰ φύσιν εἶναι ἀφωρισμένην, οὕτω καὶ ἐπὶ τῶν κινουμένων ἔσονται λόγοι καὶ ἐπιστῆμαι, οὐχ ᾗ κινούμενα δὲ ἀλλ' ᾗ σώματα μόνον, καὶ πάλιν ᾗ ἐπίπεδα μόνον καὶ ᾗ μήκη μόνον, καὶ ᾗ διαιρετὰ καὶ ᾗ ἀδιαίρετα ἔχοντα δὲ θέσιν καὶ ᾗ ἀδιαίρετα μόνον, ὥστ' ἐπεὶ ἁπλῶς λέγειν ἀληθὲς μὴ μόνον τὰ χωριστὰ εἶναι ἀλλὰ καὶ τὰ μὴ χωριστά (οἷον κινούμενα εἶναι), καὶ τὰ μαθηματικὰ ὅτι ἔστιν ἁπλῶς ἀληθὲς εἰπεῖν, καὶ τοιαῦτά γε οἷα λέγουσιν.

Theory of Abstraction

καὶ ὥσπερ καὶ τὰς ἄλλας ἐπιστήμας ἁπλῶς ἀληθὲς εἰπεῖν τούτου εἶναι, οὐχὶ τοῦ συμβεβηκότος (οἷον ὅτι λευκοῦ, εἰ τὸ ὑγιεινὸν λευκόν, ἡ δ' ἔστιν ὑγιεινοῦ)

ἀλλ' ἐκείνου οὗ ἐστὶν ἑκάστη, εἰ <ᾗ> ὑγιεινὸν ὑγιεινοῦ, εἰ δ' ᾗ ἄνθρωπος ἀνθρώπου, οὕτω καὶ τὴν γεωμετρίαν: οὐκ εἰ συμβέβηκεν αἰσθητὰ εἶναι ὧν ἐστί, μὴ ἔστι δὲ ᾗ αἰσθητά, οὐ τῶν αἰσθητῶν ἔσονται αἱ μαθηματικαὶ ἐπιστῆμαι, οὐ μέντοι οὐσὲ παρὰ ταῦτα ἄλλων κεχωρισμένων. πολλὰ δὲ συμβέβηκε καθ' αὑτὰ τοῖς πράγμασιν ᾗ ἕκαστον ὑπάρχει τῶν τοιούτων, ἐπεὶ καὶ ᾗ θῆλυ τὸ ζῷον καὶ ᾗ ἄρρεν, ἴδια πάθη ἔστιν (καίτοι οὐκ ἔστι τι θῆλυ οὐδ' ἄρρεν κεχωρισμένον τῶν ζῴων) :

Theory of Abstraction

ὥστε καὶ ᾗ μήκη μόνον καὶ ᾗ ἐπίπεδα. καὶ ὅσῳ δὴ ἂν περὶ προτέρων τῷ λόγῳ καὶ ἁπλουστέρων, τοσούτῳ μᾶλλον ἔχει τὸ ἀκριβές (τοῦτο δὲ τὸ ἁπλοῦν ἐστίν), ὥστε ἄνευ τε μεγέθους μᾶλλον ἢ μετὰ μεγέθους, καὶ μάλιστα ἄνευ κινήσεως, ἐὰν δὲ κίνησιν, μάλιστα τὴν πρώτην: ἁπλουστάτη γάρ, καὶ ταύτης ἡ ὁμαλή. ὁ δ' αὐτὸς λόγος καὶ περὶ ἁρμονικῆς καὶ ὀπτικῆς: οὐδετέρα γὰρ ᾗ ὄψις ἢ ᾗ φωνὴ θεωρεῖ, ἀλλ' ᾗ γραμμαὶ καὶ ἀριθμοί (οἰκεῖα μέντοι ταῦτα πάθη ἐκείνων), καὶ ἡ μηχανικὴ δὲ ὡσαύτως, ὥστ' εἴ τις θέμενος κεχωρισμένα τῶν συμβεβηκότων σκοπεῖ τι περὶ τούτων ᾗ τοιαῦτα, οὐθὲν διὰ τοῦτο ψεῦδος ψεύσεται, ὥσπερ οὐδ' ὅταν ἐν τῇ γῇ γράφῃ καὶ ποδιαίαν φῇ τὴν μὴ ποδιαίαν: οὐ γὰρ ἐν ταῖς προτάσεσι τὸ ψεῦδος.

Theory of Abstraction

ἄριστα δ' ἂν οὕτω θεωρηθείη ἕκαστον, εἴ τις τὸ μὴ κεχωρισμένον θείη χωρίσας, ὅπερ ὁ ἀριθμητικὸς ποιεῖ καὶ ὁ γεωμέτρης. ἓν μὲν γὰρ καὶ ἀδιαίρετον ὁ ἄνθρωπος ᾗ ἄνθρωπος: ὁ δ' ἔθετο ἓν ἀδιαίρετον, εἶτ' ἐθεώρησεν εἴ τι τῷ ἀνθρώπῳ συμβέβηκεν ᾗ ἀδιαίρετος. ὁ δὲ γεωμέτρης οὔθ' ᾗ ἄνθρωπος οὔθ' ᾗ ἀδιαίρετος ἀλλ' ᾗ στερεόν. ἃ γὰρ κἂν εἰ μή που ἦν ἀδιαίρετος ὑπῆρχεν αὐτῷ, δῆλον ὅτι καὶ ἄνευ τούτων ἐνδέχεται αὐτῷ ὑπάρχειν [τὸ δυνατόν], ὥστε διὰ τοῦτο ὀρθῶς οἱ γεωμέτραι λέγουσι, καὶ περὶ ὄντων διαλέγονται, καὶ ὄντα ἐστίν:

Trade between precision and abstraction

The more abstract is a given subject matter (i.e. the less is the number of features simultaneously taken into consideration) the more precise is the corresponding theory. This explains, in particular, why arithmetic is more precise than geometry. However on Aristotle's account the more abstract implies the less real. Thus unlike Plato Aristotle doesn't think of theoretical precision as a direct evidence of truth about what there is. He rather thinks of it as one specific epistemic criterion competing with other epistemic criteria, which are equally important.

Platonic Quadrivium upside down

Remind that in the Quadrivium the science of astronomy is given the lowest possible grade, which it shares with the science of harmonics. Aristotle, on the contrary, sees astronomy as a science, which achieves the best balance between mathematical precision and physical substantiality. This makes astronomy, by Aristotle's word "most akin to philosophy". As explains Aristotle "this science speculates about substance, which is perceptible but eternal, while the other mathematical sciences, i.e. arithmetic and geometry, treat of no substance". (Met. 1073b5-7)

Metabasis

One can reasonably constitute a research area like *female studies* provided that it will study only human females or only female individuals of some other particular species. But the notion of *general female studies*, which is a science about females of all biological species, is absurd. This is in spite of the fact that the general notion of female makes a perfect sense and applies across the species. Such generality doesn't allow one to abstract the property of being a female from the underlying species and make it into a subject matter of a special study.

Aristotelian doubt

DOES MATHEMATICS REALLY AVOIDS THE *METABASIS*?

Classical Model of Science (after Betti et al.)

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- Derived truths are obtained from fundamental truths through logical inference. Fundamental truths are taken as premises of inferences and derived truths are obtained as conclusions of inferences.
- Logical inferences are governed by laws of logic, which reflect general ontological principles and are universal for all sciences.

Ὅροι.

αʹ. Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.

βʹ. Γραμμὴ δὲ μῆκος ἀπλατές.

γʹ. Γραμμῆς δὲ πέρατα σημεῖα.

δʹ. Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται.

εʹ. Ἐπιφάνεια δέ ἐστιν, ὃ μῆκος καὶ πλάτος μόνον ἔχει.

ϛʹ. Ἐπιφανείας δὲ πέρατα γραμμαί.

ζʹ. Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς εὐθείαις κεῖται.

Definitions

1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points.
4. A straight-line is whatever lies evenly with points upon itself.
5. And a surface is that which has length and breadth alone.
6. And the extremities of a surface are lines.
7. A plane surface is whatever lies evenly with straight-lines upon itself.

η΄. Ἐπίπεδος δὲ γωνία ἐστὶν ἡ ἐν ἐπιπέδῳ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐπ᾽ εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις.

θ΄. Ὅταν δὲ αἱ περιέχουσαι τὴν γωνίαν γραμμαὶ εὐθεῖαι ὦσιν, εὐθύγραμμος καλεῖται ἡ γωνία.

ι΄. Ὅταν δὲ εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστι, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται, ἐφ᾽ ἣν ἐφέστηκεν.

ια΄. Ἀμβλεῖα γωνία ἐστὶν ἡ μείζων ὀρθῆς.

ιβ΄. Ὀξεῖα δὲ ἡ ἐλάσσων ὀρθῆς.

ιγ΄. Ὅρος ἐστίν, ὅ τινός ἐστι πέρας.

8. And a plane angle is the inclination of the lines, when two lines in a plane meet one another, and are not laid down straight-on with respect to one another.

9. And when the lines containing the angle are straight then the angle is called rectilinear.

10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called perpendicular to that upon which it stands.

11. An obtuse angle is greater than a right-angle.

12. And an acute angle is less than a right-angle.

ιδʹ. Σχῆμά ἐστι τὸ ὑπό τινος ἤ τινων ὅρων περιεχόμενον.

ιεʹ. Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον [ἣ καλεῖται περιφέρεια], πρὸς ἣν ἀφ' ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι [πρὸς τὴν τοῦ κύκλου περιφέρειαν] ἴσαι ἀλλήλαις εἰσίν.

ιϛʹ. Κέντρον δὲ τοῦ κύκλου τὸ σημεῖον καλεῖται.

ιζʹ. Διάμετρος δὲ τοῦ κύκλου ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ' ἑκάτερα τὰ μέρη ὑπὸ τῆς τοῦ κύκλου περιφερείας, ἥτις καὶ δίχα τέμνει τὸν κύκλον.

ιηʹ. Ἡμικύκλιον δέ ἐστι τὸ περιεχόμενον σχῆμα ὑπό τε τῆς διαμέτρου καὶ τῆς ἀπολαμβανομένης ὑπ' αὐτῆς περιφερείας. κέντρον δὲ τοῦ ἡμικυκλίου τὸ αὐτό, ὃ καὶ τοῦ κύκλου ἐστίν.

ιθʹ. Σχήματα εὐθύγραμμά ἐστι τὰ ὑπὸ εὐθειῶν περιεχόμενα, τρίπλευρα μὲν τὰ ὑπὸ τριῶν, τετράπλευρα δὲ τὰ ὑπὸ τεσσάρων, πολύπλευρα δὲ τὰ ὑπὸ πλειόνων ἢ τεσσάρων εὐθειῶν περιεχόμενα.

κʹ. Τῶν δὲ τριπλεύρων σχημάτων ἰσόπλευρον μὲν τρίγωνόν ἐστι τὸ τὰς τρεῖς ἴσας ἔχον πλευράς, ἰσοσκελὲς δὲ τὸ τὰς δύο μόνας ἴσας ἔχον πλευράς, σκαληνὸν δὲ τὸ τὰς τρεῖς ἀνίσους ἔχον πλευράς.

καʹ Ἔτι δὲ τῶν τριπλεύρων σχημάτων ὀρθογώνιον μὲν τρίγωνόν ἐστι τὸ ἔχον ὀρθὴν γωνίαν, ἀμβλυγώνιον

13. A boundary is that which is the extremity of something.

14. A figure is that which is contained by some boundary or boundaries.

15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from a single point lying inside the figure are equal to one another.

16. And the point is called the center of the circle.

17. And a diameter of the circle is any straight-line, being drawn through the center, which is brought to an end in each direction by the circumference of the circle. And any such (straight-line) cuts the circle in half.†

18. And a semi-circle is the figure contained by the diameter and the circumference it cuts off. And the center of the semi-circle is the same (point) as (the center of) the circle.

19. Rectilinear figures are those figures contained by straight-lines: trilateral figures being contained by three straight-lines, quadrilateral by four, and multilateral by more than four.

20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.

δὲ τὸ ἔχον ἀμβλεῖαν γωνίαν, ὀξυγώνιον δὲ τὸ τὰς τρεῖς ὀξείας ἔχον γωνίας.

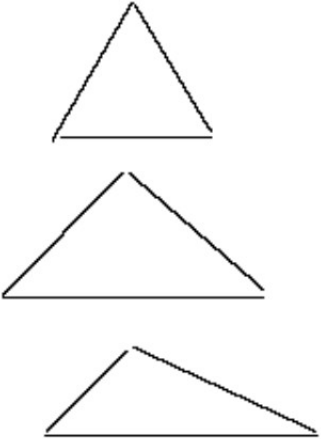
κβʹ. Τῶν δὲ τετραπλεύρων σχημάτων τετράγωνον μέν ἐστιν, ὃ ἰσόπλευρόν τέ ἐστι καὶ ὀρθογώνιον, ἑτερόμηκες δέ, ὃ ὀρθογώνιον μέν, οὐκ ἰσόπλευρον δέ, ῥόμβος δέ, ὃ ἰσόπλευρον μέν, οὐκ ὀρθογώνιον δέ, ῥομβοειδὲς δὲ τὸ τὰς ἀπεναντίον πλευράς τε καὶ γωνίας ἴσας ἀλλήλαις ἔχον, ὃ οὔτε ἰσόπλευρόν ἐστιν οὔτε ὀρθογώνιον· τὰ δὲ παρὰ ταῦτα τετράπλευρα τραπέζια καλείσθω.

κγʹ. Παράλληλοί εἰσιν εὐθεῖαι, αἵτινες ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ' ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις.

21. And further of the trilateral figures: a right-angled triangle is that having a right-angle, an obtuse-angled (triangle) that having an obtuse angle, and an acute-angled (triangle) that having three acute angles.

22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.

23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).



Definitions of the 7th Book

Ὅροι.

αʹ. Μονάς ἐστιν, καθ᾽ ἣν ἕκαστον τῶν ὄντων ἓν λέγεται.

βʹ. Ἀριθμὸς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος.

γʹ. Μέρος ἐστὶν ἀριθμὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος, ὅταν καταμετρῇ τὸν μείζονα.

δʹ. Μέρη δέ, ὅταν μὴ καταμετρῇ.

εʹ. Πολλαπλάσιος δὲ ὁ μείζων τοῦ ἐλάσσονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάσσονος.

ϛʹ. Ἄρτιος ἀριθμός ἐστιν ὁ δίχα διαιρούμενος.

ζʹ. Περισσὸς δὲ ὁ μὴ διαιρούμενος δίχα ἢ [ὁ] μονάδι διαφέρων ἀρτίου ἀριθμοῦ.

ηʹ. Ἀρτιάκις ἄρτιος ἀριθμός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν.

θʹ. Ἀρτιάκις δὲ περισσός ἐστιν ὁ ὑπὸ ἀρτίου

Definitions

1. A unit is (that) according to which each existing (thing) is said (to be) one.

2. And a number (is) a multitude composed of units.†

3. A number is part of a(nother) number, the lesser of the greater, when it measures the greater.‡

4. But (the lesser is) parts (of the greater) when it does not measure it.§

5. And the greater (number is) a multiple of the lesser when it is measured by the lesser.

6. An even number is one (which can be) divided in half.

7. And an odd number is one (which can)not (be) divided in half, or which differs from an even number by a unit.

WARNING : ONE IS NOT A NUMBER !

WARNING : ONE IS NOT A NUMBER !
number = finite set ?

Αἰτήματα.

αʹ. Ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

βʹ. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ' εὐθείας ἐκβαλεῖν.

γʹ. Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

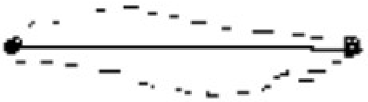
δʹ. Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

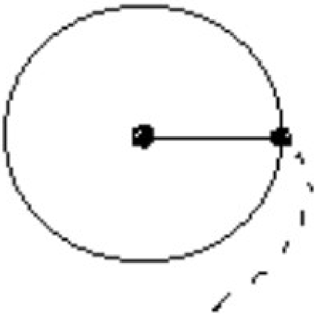
εʹ. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ' ἄπειρον συμπίπτειν, ἐφ' ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

Postulates

1. Let it have been postulated to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).†

† This postulate effectively specifies that we are dealing with the geometry of *flat*, rather than curved, space.





Postulates 1-3

The drawing of a line from any point to any point follows from the conception of the line as the flowing of a point and of the straight line as its uniform and undeviating flowing. For if we think of the point as moving uniformly over the shortest path, we shall come to the other point and so shall have got the first postulate without any complicated process of thought. And if we take a straight line as limited by a point and similarly imagine its extremity as moving uniformly over the shortest route, the second postulate will have been established by a simple and facile reflection. And if we think of a finite line as having one extremity stationary and the other extremity moving about this stationary point, we shall have produced the third postulate. (Proclus, Commentary 185.8-2)

Mathematical Becoming

Postulates 1-3 are <u>not</u> propositions but generic principles.
Postulates 4-5 are problematic.

Κοιναὶ ἔννοιαι.

α΄. Τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.

β΄. Καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἴσα.

γ΄. Καὶ ἐὰν ἀπὸ ἴσων ἴσα ἀφαιρεθῇ, τὰ καταλειπόμενά ἐστιν ἴσα.

δ΄. Καὶ τὰ ἐφαρμόζοντα ἐπ' ἀλλήλα ἴσα ἀλλήλοις ἐστίν.

ε΄. Καὶ τὸ ὅλον τοῦ μέρους μεῖζόν [ἐστιν].

Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.†
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

Common notions....

are common for all mathematical disciplines. (Also Ax. 5 !) In geometry "equality" means (roughly) "equicomposability".

Mathematical Being ...

... is defined "up to mathematical equality" rather than strict identity. Numbers and magnitudes exist in an indefinite number of "copies". How many 2s are there ?

Being and Becoming

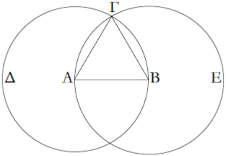
Science as a whole has two parts : in one it occupies itself with immediate enunciations, while in the other it treats systematically the things that can be demonstrated or constructed from these first principles, or in general are consequences of them. In the geometrical reasoning this second part is again divided into solving problems and finding theorems. The name "problem" is appropriate where what in a sense doesn't exist is produced, set, brought into view and arranged, while the name "theorem" is appropriate where something that is attributed or not attributed is seen, known and proved. The former [has to do with] generation, setting, application, ascription, inscription, insertion, touching and the like ; the latter [has to do with] properties and essential attributes of geometrical objects, which are grasped and firmly bound by demonstration. (Commentary, 200.20-201.14)

Structure

Every Problem and every Theorem that is furnished with all its parts should contain the following elements : [i] an enunciation, [ii] an exposition, [iii] a specification, [iv] a construction, [v] a proof, and [vi] a conclusion. Of these enunciation states what is given and what is being sought from it, a perfect enunciation consists of both these parts. The exposition takes separately what is given and prepares it in advance for use in the investigation. The specification takes separately the thing that is sought and makes clear precisely what it is. The construction adds what is lacking in the given for finding what is sought. The proof draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The conclusion reverts to the enunciation, confirming what has been proved." (Commentary, 203.1-15)

α΄.

Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι.



Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ.

Δεῖ δὴ ἐπὶ τῆς ΑΒ εὐθείας τρίγωνον ἰσόπλευρον συστήσασθαι.

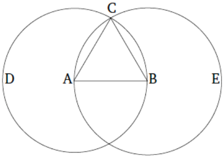
Κέντρῳ μὲν τῷ Α διαστήματι δὲ τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΓΔ, καὶ πάλιν κέντρῳ μὲν τῷ Β διαστήματι δὲ τῷ ΒΑ κύκλος γεγράφθω ὁ ΑΓΕ, καὶ ἀπὸ τοῦ Γ σημείου, καθ' ὃ τέμνουσιν ἀλλήλους οἱ κύκλοι, ἐπί τὰ Α, Β σημεῖα ἐπεζεύχθωσαν εὐθεῖαι αἱ ΓΑ, ΓΒ.

Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΓΔΒ κύκλου, ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ· πάλιν, ἐπεὶ τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΑΕ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΑ. ἐδείχθη δὲ καὶ ἡ ΓΑ τῇ ΑΒ ἴση· ἑκατέρα ἄρα τῶν ΓΑ, ΓΒ τῇ ΑΒ ἐστιν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΓΑ ἄρα τῇ ΓΒ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΓΑ, ΑΒ, ΒΓ ἴσαι ἀλλήλαις εἰσίν.

Ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον. καὶ συνέσταται ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τῆς ΑΒ· ὅπερ ἔδει ποιῆσαι.

Proposition 1

To construct an equilateral triangle on a given finite straight-line.



Let AB be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line AB.

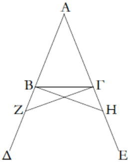
Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another,† to the points A and B (respectively) [Post. 1].

And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which

ε΄.

Τῶν ἰσοσκελῶν τριγώνων αἱ πρὸς τῇ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ προσεκβληθεισῶν τῶν ἴσων εὐθειῶν αἱ ὑπὸ τὴν βάσιν γωνίαι ἴσαι ἀλλήλαις ἔσονται.



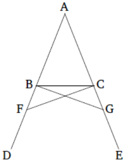
Ἔστω τρίγωνον ἰσοσκελὲς τὸ ΑΒΓ ἴσην ἔχον τὴν ΑΒ πλευρὰν τῇ ΑΓ πλευρᾷ, καὶ προσεκβεβλήσθωσαν ἐπ᾽ εὐθείας ταῖς ΑΒ, ΑΓ εὐθεῖαι αἱ ΒΔ, ΓΕ· λέγω, ὅτι ἡ μὲν ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΑΓΒ ἴση ἐστίν, ἡ δὲ ὑπὸ ΓΒΔ τῇ ὑπὸ ΒΓΕ.

Εἰλήφθω γὰρ ἐπὶ τῆς ΒΔ τυχὸν σημεῖον τὸ Ζ, καὶ ἀφῃρήσθω ἀπὸ τῆς μείζονος τῆς ΑΕ τῇ ἐλάσσονι τῇ ΑΖ ἴση ἡ ΑΗ, καὶ ἐπεζεύχθωσαν αἱ ΖΓ, ΗΒ εὐθεῖαι.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΑΖ τῇ ΑΗ ἡ δὲ ΑΒ τῇ

Proposition 5

For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another.



Let ABC be an isosceles triangle having the side AB equal to the side AC, and let the straight-lines BD and CE have been produced in a straight-line with AB and AC (respectively) [Post. 2]. I say that the angle ABC is equal to ACB, and (angle) CBD to BCE.

For let the point F have been taken somewhere on BD, and let AG have been cut off from the greater AE, equal to the lesser AF [Prop. 1.3]. Also, let the straight-lines FC and GB have been joined [Post. 1].

ΑΓ, δύο δὴ αἱ ΖΑ, ΑΓ δυσὶ ταῖς ΗΑ, ΑΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνίαν κοινὴν περιέχουσι τὴν ὑπὸ ΖΑΗ· βάσις ἄρα ἡ ΖΓ βάσει τῇ ΗΒ ἴση ἐστίν, καὶ τὸ ΑΖΓ τρίγωνον τῷ ΑΗΒ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾿ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν, ἡ μὲν ὑπὸ ΑΓΖ τῇ ὑπὸ ΑΒΗ, ἡ δὲ ὑπὸ ΑΖΓ τῇ ὑπὸ ΑΗΒ. καὶ ἐπεὶ ὅλη ἡ ΑΖ ὅλῃ τῇ ΑΗ ἐστιν ἴση, ὧν ἡ ΑΒ τῇ ΑΓ ἐστιν ἴση, λοιπὴ ἄρα ἡ ΒΖ λοιπῇ τῇ ΓΗ ἐστιν ἴση. ἐδείχθη δὲ καὶ ἡ ΖΓ τῇ ΗΒ ἴση· δύο δὴ αἱ ΒΖ, ΖΓ δυσὶ ταῖς ΓΗ, ΗΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΒΖΓ γωνίᾳ τῃ ὑπὸ ΓΗΒ ἴση, καὶ βάσις αὐτῶν κοινὴ ἡ ΒΓ· καὶ τὸ ΒΖΓ ἄρα τρίγωνον τῷ ΓΗΒ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾿ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ΖΒΓ τῇ ὑπὸ ΗΓΒ ἡ δὲ ὑπὸ ΒΓΖ τῇ ὑπὸ ΓΒΗ. ἐπεὶ οὖν ὅλη ἡ ὑπὸ ΑΒΗ γωνία ὅλῃ τῇ ὑπὸ ΑΓΖ γωνίᾳ ἐδείχθη ἴση, ὧν ἡ ὑπὸ ΓΒΗ τῇ ὑπὸ ΒΓΖ ἴση, λοιπὴ ἄρα ἡ ὑπὸ ΑΒΓ λοιπῇ τῇ ὑπὸ ΑΓΒ ἐστιν ἴση· καί εἰσι πρὸς τῇ βάσει τοῦ ΑΒΓ τριγώνου. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΖΒΓ τῇ ὑπὸ ΗΓΒ ἴση· καί εἰσιν ὑπὸ τὴν βάσιν.

Τῶν ἄρα ἰσοσκελῶν τριγώνων αἱ πρὸς τῇ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ προσεκβληθεισῶν τῶν ἴσων εὐθειῶν αἱ ὑπὸ τὴν βάσιν γωνίαι ἴσαι ἀλλήλαις ἔσονται· ὅπερ ἔδει δεῖξαι.

In fact, since AF is equal to AG, and AB to AC, the two (straight-lines) FA, AC are equal to the two (straight-lines) GA, AB, respectively. They also encompass a common angle FAG. Thus, the base FC is equal to the base GB, and the triangle AFC will be equal to the triangle AGB, and the remaining angles subtendend by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) ACF to ABG, and AFC to AGB. And since the whole of AF is equal to the whole of AG, within which AB is equal to AC, the remainder BF is thus equal to the remainder CG [C.N. 3]. But FC was also shown (to be) equal to GB. So the two (straight-lines) BF, FC are equal to the two (straight-lines) CG, GB, respectively, and the angle BFC (is) equal to the angle CGB, and the base BC is common to them. Thus, the triangle BFC will be equal to the triangle CGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. Thus, FBC is equal to GCB, and BCF to CBG. Therefore, since the whole angle ABG was shown (to be) equal to the whole angle ACF, within which CBG is equal to BCF, the remainder ABC is thus equal to the remainder ACB [C.N. 3]. And they are at the base of triangle ABC. And FBC was also shown (to be) equal to GCB. And they are under the base.

Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

Double-thinking

Mathematicians are accustomed to draw what is in a way a double conclusion. For when they have shown something to be true of the given figure, they infer that it is true in general, going from the particular to the universal conclusion. Because they do not make use of the particular qualities of the subjects but draw the angle or the straight line in order to place what is given before our eyes, they consider that what they infer about the given angle or straight line can be identically asserted for every similar case. They pass therefore to the universal conclusion in order that we may not suppose that the result is confined to the particular instance. This procedure is justified, since for the demonstration they use the objects set out in the diagram not as these particular figures, but as figures resembling others of the same sort.

Double-thinking

It is not as having such-and-such size that the angle before me is bisected, but as being rectilinear and nothing more. Its particular size is a character of the given angle, but its having rectilinear sides is a common feature of all rectilinear angles. Suppose the given angle is a right angle. If I used its rightness for my demonstration, I should not be able to infer anything about the whole class of rectilinear angles ; but if I make no use of its rightness and consider only its rectilinear character, the proposition will apply equally to all angles with rectilinear sides. (Commentary 207.4-25)

Structure

Structure

- Objective and subjective individuation

Structure

- Objective and subjective individuation
- Naming

Structure

- Objective and subjective individuation
- Naming
- The role of proof

Structure

- Objective and subjective individuation
- Naming
- The role of proof
- Return

Proportion of numbers

20. Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third (is) of the fourth.

Proportion of magnitudes

Ὅροι.

αʹ. Μέρος ἐστὶ μέγεθος μεγέθους τὸ ἔλασσον τοῦ μείζονος, ὅταν καταμετρῇ τὸ μεῖζον.

βʹ. Πολλαπλάσιον δὲ τὸ μεῖζον τοῦ ἐλάττονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάττονος.

γʹ. Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητά ποια σχέσις.

δʹ. Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.

εʹ. Ἐν τῷ αὐτῷ λόγῳ μεγέθη λέγεται εἶναι πρῶτον πρὸς δεύτερον καὶ τρίτον πρὸς τέταρτον, ὅταν τὰ τοῦ πρώτου καί τρίτου ἰσάκις πολλαπλάσια τῶν τοῦ δευτέρου καὶ τετάρτου ἰσάκις πολλαπλασίων καθ᾽ ὁποιονοῦν πολλαπλασιασμὸν ἑκάτερον ἑκατέρου ἢ ἅμα ὑπερέχῃ ἢ ἅμα ἴσα ᾖ ἢ ἅμα ἐλλείπῃ ληφθέντα κατάλληλα.

ϛʹ. Τὰ δὲ τὸν αὐτὸν ἔχοντα λόγον μεγέθη ἀνάλογον καλείσθω.

ζʹ. Ὅταν δὲ τῶν ἰσάκις πολλαπλασίων τὸ μὲν τοῦ πρώτου πολλαπλάσιον ὑπερέχῃ τοῦ τοῦ δευτέρου πολ-

Definitions

1. A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater.†

2. And the greater (magnitude is) a multiple of the lesser when it is measured by the lesser.

3. A ratio is a certain type of condition with respect to size of two magnitudes of the same kind.‡

4. (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another.§

5. Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and the third either both exceed, are both equal to, or are both less than, equal multiples of the second and the fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.¶

6. And let magnitudes having the same ratio be called proportional.*

METABASIS ?

ε´.

Τὰ σύμμετρα μεγέθη πρὸς ἄλληλα λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.

Proposition 5

Commensurable magnitudes have to one another the ratio which (some) number (has) to (some) number.

Proclus on Universal Mathematics

Let us enumerate the simple theorems .... generated by the single science that embraces alike all forms of mathematical knowledge ; and let us see how they fit into all these sciences and can be observed alike in numbers, magnitudes, and motions. Such are the theorems governing proportion, namely, the rules of compounding, dividing, converting, and alternating ; likewise the theorems concerning ratios of all kinds, multiple, superparticular, superpartient, and their counterparts ; and the theorems about equality and inequality in their most general and universal aspects, not equality or inequality of figures, numbers, or motions, but each of the two by itself as having a nature common to all its forms and capable of more simple apprehension.

Proclus on Universal Mathematics

.... We must not regard these common theorems as subsisting their origin from them (= the particular sciences) , but as prior to their instances and superior in simplicity and exactness. For this reason, knowledge of them takes precedence over the particular sciences and furnishes to them their principles ; that is, these several sciences are based upon this prior science and refer back to it.

Aristotle on Universal Mathematics

The question may be asked whether first philosophy is universal or deals with some particular genus or some one class of things. For not even in mathematical sciences is the method one and the same ; geometry and astronomy, for instance, deal with a certain class of thing, but the universal science of mathematics is common to all branches. (Met. 1026a3-7)

Aristotle on Universal Mathematics

For each of the mathematical sciences is concerned with some distinct genus, but universal science of mathematics is common to all (Met.1064b8-9)

Aristotle on Universal Mathematics

Further some propositions are proved universally by mathematicians, which extend beyond these substances [belonging to special mathematical sciences] (Met.1077a9-10)

Aristotle on Universal Mathematics

Just as the universal part of mathematics deals not with objects which exist separately, apart from extended magnitudes and numbers, but with magnitudes and numbers, not however qua such as to have magnitude or to be divisible, clearly it is possible that there should also be both propositions and demonstrations about sensible magnitudes, not however qua sensible but qua possessed of certain definite qualities." (Met. 1077b17-22)

Ontological worry

What is the subject-matter (genus) of Universal Mathematics ?

Ontological worry

What is the subject-matter (genus) of Universal Mathematics ? The answer given in 17th century : the general notion of magnitude that includes that of *number* as a special case of *discrete* magnitude along with *continuous* geometrical magnitude (Arnauld)

Epistemological worry

If the notion of Universal Mathematics is sound then the Universal Mathematics must be treated first.

Epistemological worry

If the notion of Universal Mathematics is sound then the Universal Mathematics must be treated first.
Solution of 17th century : Algebra is Universal Mathematics, i.e., a general theory of magnitude.

To hold universally

Something holds universally when it is proved of an arbitrary and primitive case. E.g. having [the sum of internal angles equal to] two right angles doesn't hold universally of figures - you may indeed prove of a figure that it has two right angles, but not of an arbitrary figure, nor can you use an arbitrary figure in proving it ; for quadrangles are figures but do not have angles equal to two right angles. An arbitrary isosceles [triangle] does have angles equal to two right angles - but it is not primitive : triangles are prior. Thus if an arbitrary primitive case is proved to have two right angles (or whatever else), then it holds universally of this primitive item, and the demonstration applies to it universally [...] [I]t does not apply to the isosceles [triangles] universally, but extends further. (An.Pr. 73b33-74a4)

To hold universally

τὸ καθόλου δὲ ὑπάρχει τότε, ὅταν ἐπὶ τοῦ τυχόντος καὶ πρώτου δεικνύηται. οἷον τὸ δύο ὀρθὰς ἔχειν οὔτε τῷ σχήματί ἐστι καθόλου (καίτοι ἔστι δεῖξαι κατὰ σχήματος ὅτι δύο ὀρθὰς ἔχει, ἀλλ' οὐ τοῦ τυχόντος σχήματος, οὐδὲ χρῆται τῷ τυχόντι σχήματι δεικνύς· τὸ γὰρ τετράγωνον σχῆμα μέν, οὐκ ἔχει δὲ δύο ὀρθαῖς ἴσας) τὸ δ' ἰσοσκελὲς ἔχει μὲν τὸ τυχὸν δύο ὀρθαῖς ἴσας, ἀλλ'οὐ πρῶτον, ἀλλὰ τὸ τρίγωνον πρότερον. ὃ τοίνυν τὸ τυχὸν πρῶτον δείκνυται δύο ὀρθὰς ἔχον ἢ ὁτιοῦν ἄλλο, τούτῳ πρώτῳ ὑπάρχει καθόλου, καὶ ἡ ἀπόδειξις καθ' αὑτὸ τούτου καθόλου ἐστί, τῶν δ' ἄλλων τρόπον τινὰ οὐ καθ' αὑτό, οὐδὲ τοῦ ἰσοσκελοῦς οὐκ ἔστι καθόλου ἀλλ' ἐπὶ πλέον.

Specific and Universal Principles after Aristotle

Instances of first principles peculiar to a science are the assumptions that a line is of such-and- such a character, and similarly for the straight line ; whereas it is a common principle, for instance, that if equals be subtracted from equals, the remainders are equal." (An. Post. 76a38- 43)

From Mathematics to Logic

By first principles of proof I mean the common opinions on which all men base their demonstrations, e.g. that one of two contradictories must be true, that it is impossible for the same thing both be and not to be, and all other propositions of this kind." (Met. 996b27-32)

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- A1 : If $A = B$ and $C = B$ then $A = C$

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- A1 : If $A = B$ and $C = B$ then $A = C$
- PS : If all A are B and all B are C then all A are$C$

Changing meaning of the term "axiom"

We have now to say whether it is up to the same science or to different sciences to inquire into what in mathematics is called axioms and into the essence. Clearly the inquiry into these things is up to the same science, namely, to the science of the philosopher. For axioms hold of everything that [there] is but not of some particular genus apart from others. Everyone makes use of them because they concern being qua being, and each genus is. But men use them just so far as is sufficient for their purpose, that is, within the limits of the genus relevant to their proofs. Since axioms clearly hold for all things qua being (for being is what all things share in common) one who studies being qua being also inquires into the axioms. This is why one who observes things partly [=who inquires into a special domain] like a geometer or a arithmetician never tries to say whether the axioms are true or false. (Met. 1005a19-28)

Changing meaning of the term "axiom"

Since the mathematician too uses common [axioms] only on the case-by-case basis, it must be the business of the first philosophy to investigate their fundamentals. For that, when equals are subtracted from equals, the remainders are equal is common to all quantities, but mathematics singles out and investigates some portion of its proper matter, as e.g. lines or angles or numbers, or some other sort of quantity, not however qua being, but as [...] continuous. (Met. 1061b)

CLAIM : Aristotle's Logic/Metaphysics is a generalization of Universal Mathematics ; Logic/Metaphysics is supposed to provide a foundation for all sciences <u>including physics</u>

CLAIM : Aristotle's Logic/Metaphysics is a generalization of Universal Mathematics ; Logic/Metaphysics is supposed to provide a foundation for all sciences <u>including physics</u>
QUESTION : Does Aristotle's logic apply to Euclid's mathematics ?

Aristotle's controversial mathematical example

Let A be two right angles, B triangle, C isosceles. Then A is an attribute of C because of B, but it is not an attribute of B because of any other middle term ; for a triangle has [its angles equal to] two right angles by itself, so that there will be no middle term between A and B, AB is matter for demonstration. (An. Pr. 48a33-37)

Aristotle's controversial mathematical example

Aristotle's controversial mathematical example

- All triangles have two right angles (Premise AB)

Aristotle's controversial mathematical example

- All triangles have two right angles (Premise AB)
- All isosceles triangles are triangles (Premise AC)

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- All triangles have two right angles (Premise AB)
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Aristotle's controversial mathematical example

- All triangles have two right angles (Premise AB)
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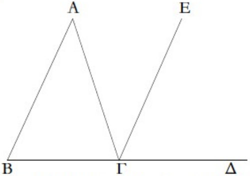
- All isosceles triangles have two right angles (Conclusion BC)

AB is immediate BUT still is a matter of demonstration ? ? ! !

λβʹ.

Παντὸς τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ἴση ἐστίν, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.



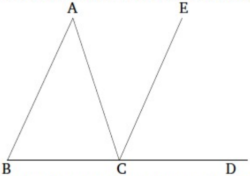
Ἔστω τρίγωνον τὸ ΑΒΓ, καὶ προσεκβεβλήσθω αὐτοῦ μία πλευρὰ ἡ ΒΓ ἐπὶ τὸ Δ· λέγω, ὅτι ἡ ἐκτὸς γωνία ἡ ὑπὸ ΑΓΔ ἴση ἐστὶ δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ταῖς ὑπὸ ΓΑΒ, ΑΒΓ, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΓΑ, ΓΑΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Ἤχθω γὰρ διὰ τοῦ Γ σημείου τῇ ΑΒ εὐθείᾳ παράλληλος ἡ ΓΕ.

Καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΕ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΑΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΒΑΓ, ΑΓΕ ἴσαι ἀλλήλαις εἰσίν. πάλιν, ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΕ, καὶ εἰς αὐτὰς ἐμπέπτωκεν εὐθεῖα ἡ ΒΔ, ἡ ἐκτὸς γωνία ἡ ὑπὸ ΕΓΔ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΑΒΓ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΑΓΕ τῇ ὑπὸ ΒΑΓ ἴση· ὅλη ἄρα ἡ ὑπὸ ΑΓΔ γωνία ἴση ἐστὶ δυσὶ ταῖς ἐντὸς

Proposition 32

For any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.



Let ABC be a triangle, and let one of its sides BC have been produced to D. I say that the external angle ACD is equal to the (sum of the) two internal and opposite angles CAB and ABC, and the (sum of the) three internal angles of the triangle—ABC, BCA, and CAB—is equal to two right-angles.

For let CE have been drawn through point C parallel to the straight-line AB [Prop. 1.31].

And since AB is parallel to CE, and AC has fallen across them, the alternate angles BAC and ACE are equal to one another [Prop. 1.29]. Again, since AB is parallel to CE, and the straight-line BD has fallen across them, the external angle ECD is equal to the internal and opposite (angle) ABC [Prop. 1.29]. But ACE was also shown (to be) equal to BAC. Thus, the whole an-καὶ ἀπεναντίον ταῖς ὑπὸ ΒΑΓ, ΑΒΓ.

Κοινὴ προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΔ, ΑΓΒ τρισὶ ταῖς ὑπὸ ΑΒΓ, ΒΓΑ, ΓΑΒ ἴσαι εἰσίν. ἀλλ᾽ αἱ ὑπὸ ΑΓΔ, ΑΓΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΑΓΒ, ΓΒΑ, ΓΑΒ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Παντὸς ἄρα τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ἴση ἐστίν, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν· ὅπερ ἔδει δεῖξαι.

gle ACD is equal to the (sum of the) two internal and opposite (angles) BAC and ABC.

Let ACB have been added to both. Thus, (the sum of) ACD and ACB is equal to the (sum of the) three (angles) ABC, BCA, and CAB. But, (the sum of) ACD and ACB is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) ACB, CBA, and CAB is also equal to two right-angles.

Thus, for any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles. (Which is) the very thing it was required to show.

Diagrammatic thinking

Diagrams are devised by an activity, namely by dividing-up. If they had already been divided, they would have been manifest to begin with ; but as it is this [clarity] presents itself [only] potentially. Why does the triangle has [the sum of its internal angles equal to] two right angles ? Because the angles about one point are equal to two right angles. If the parallel to the side had been risen [in advance], this would be seen straightforwardly (Met. 1051a21- 26)

Classical Model of Science

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Paradoxically, the genuine Platonic philosophy stresses the significance of the *constructive* aspect of mathematics !

THE END