Euclid’s “Elements” and Foundations of Mathematics

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Euclid's "Elements" and Foundations of Mathematics
Comparing once popular *Elements of Geometry* published by A. Tacquet in 1654 and the edition of Euclid’s *Elements* (the first eight books thereof) published by M. Dechales 6 years later in 1660 it is difficult to say why the later work has Euclid’s name in its title while the former doesn’t. The difference between the two titles seems to be unrelated to the content of the two books although it might point to different intentions of their authors. When Tacquet’s book was republished in 1725 (long after the authors death) it actually got Euclid’s name on its cover!
Some more “Elements”
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- I. Barrow : 1733
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- C.L. Dodgson (Lewis Carrol) : 1875
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- C.L. Dodgson (Lewis Carrol) : 1875
- J. Keill : 1754
- A. Arnauld : 1667
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- I. Barrow : 1733
- C.L. Dodgson (Lewis Carrol) : 1875
- J. Keill : 1754
- A. Arnauld : 1667
- A.-M. Legendre : 1793 (AN II)
The Urtext
I.L. HEIBERG and H. MENGE (an assistant) : Euclid’s complete works with new Latin translation : 1883-1916
The Urtext

I.L. HEIBERG and H. MENGE (an assistant) : Euclid’s complete works with new Latin translation : 1883-1916
Heiberg was Professor of Classical Philology at the University of Copenhagen from 1896 until 1924. Among his more than 200 publications were editions of the works of Archimedes (1880 and 1912), Euclid (with Heinrich Menge) (1883-1916), Apollonius of Perga (1891-93), Serenus of Antinouplis (1896), Ptolemy (1898), and Hero of Alexandria (1899). Many of his editions are still in use today.
Modern commented translations based on the Urtext
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First English translation:
Modern commented translations based on the Urtext

First English translation:  

First Russian translation:  
D.D. Morduhai-Boltovskoi, 1950
Modern commented translations based on the Urtext

First English translation:

First Russian translation:
D.D. Morduhai-Boltovskoi, 1950

First French translation:
Vitrac, continued
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A study of Euclid’s *Elements* in a **historical** (rather than purely mathematical) perspective begins with Heiberg’s publication of the Urtext. It seems me however very important to keep a **mathematical** (rather than purely historical) perspective on this document too.
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A study of Euclid’s *Elements* in a *historical* (rather than purely mathematical) perspective begins with Heiberg’s publication of the Urtext. It seems to me however very important to keep a mathematical (rather than purely historical) perspective on this document too. “Ancient mathematics” is mathematics at the first place! In fact the Euclidean tradition of producing mathematical “Elements” is still alive in pure mathematics!
Today’s Elements (also outdated but having no better replacement so far..)

D. Hilbert, *Grundlagen der Geometrie*, Leipzig 1899
Invention of “non-Euclidean” geometries

People tried to prove or replace Fifth Postulate of Euclid’s *Elements* because unlike other Principles of *Elements* this particular Postulate did not seem to be self-evident. (The popular view according to which the “usual” geometrical intuition is Euclidean doesn’t stand against this historical evidence.)
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- Beltrami in 1868 discovered a link between the problem of parallels (Lobachevsky) and the geometry of curved surfaces (Gauss) and curve spaces (Riemann).
D. Hilbert: “Grundlagen der Geometrie”
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Kapitel I.

Die fünf Axiomgruppen.

§ 1.

Die Elemente der Geometrie und die fünf Axiomgruppen.

Erklärung. Wir denken drei verschiedene Systeme von Dingen: die Dinge des ersten Systems nennen wir Punkte und bezeichnen sie mit \( A, B, C, \ldots \); die Dinge des zweiten Systems nennen wir Gerade und bezeichnen sie mit \( a, b, c, \ldots \); die Dinge des dritten Systems nennen wir Ebenen und bezeichnen sie mit \( a, b, c, \ldots \); die Punkte heißen auch die Elemente der linearen Geometrie, die Punkte und Geraden heißen die Elemente der ebenen Geometrie und die Punkte, Geraden und Ebenen heißen die Elemente der räumlichen Geometrie oder des Raumes.

Wir denken die Punkte, Geraden, Ebenen in gewissen gegenseitigen Beziehungen und bezeichnen diese Beziehungen durch Worte wie „liegen“ „zwischen“, „parallel“, „kongruent“, „stetig“; die genaue und vollständige Beschreibung dieser Beziehungen erfolgt durch die Axiome der Geometrie.

Die Axiome der Geometrie gliedern sich in fünf Gruppen; jede einzelne dieser Gruppen drückt gewisse zusammengehörige Grundtatsachen unserer Anschauung aus. Wir benennen diese Gruppen von Axiomen in folgender Weise:

I 1—8. Axiome der Verknüpfung,
II 1—4. Axiome der Anordnung,
III 1—6. Axiome der Kongruenz,
IV. Axiom der Parallelen,
V 1—2. Axiome der Stetigkeit.
D. Hilbert : letter to Frege

You say that my concepts, e.g. “point”, “between”, are not unequivocally fixed ... . But surely it is self-evident that every theory is merely a framework or schema of concepts together with their necessary relations to one another, and that basic elements can be construed as one pleases. If I think of my points as some system or other of things, e.g. the system of love, of law, or of chimney sweeps ... and then conceive of all my axioms as relations between these things, then my theorems, e.g. the Pythagorean one, will hold of these things as well. In other words, each and every theory can always be applied to infinitely many systems of basic elements. For one merely has to apply a univocal and reversible one-to-one transformation and stipulate that the axioms for the transformed things be correspondingly similar. Indeed this is frequently applied, for example in the principle of duality, etc.
The starting point of any strictly logical treatment of geometry (and indeed of any branch of mathematics) must then be a set of undefined elements and relations, and a set of unproved propositions (=axioms) involving them, and from these all other propositions (theorems) are to be derived by the methods of formal logic. Moreover, since we assumed the point of view of formal (i.e. symbolic) logic, the undefined elements are to be regarded as mere symbols devoid of content.
The notion of a *class* of objects is fundamental in logic and hence in any mathematical science. The object which make up the class are called the elements of the class. The notion of a class, moreover, and the relations of *belonging to a class* (being included in a class, being element of a class, etc.) are primitive notions of logic.
Euclid's "Elements" and Foundations of Mathematics

N. Bourbaki, Éléments de mathématique
Dans ce qui précède, nous laissons, à de rares exceptions, l'emploi du terme d'"élément", non dans le sens le plus vague, mais désignant seulement tout être susceptible de posséder les propriétés non contradictoires que nous lui prêtons. Les catégories d'éléments qui font ainsi l'objet d'une théorie mathématique constituent les ensembles fondamentaux de la théorie ; mais ces ensembles ne sont pas des agrégats arbitraires : ils présentent une certaine organisation ; nous entendons par ce dernier terme tout le complexe logique formé par les définitions des propriétés des éléments de ces ensembles, des relations qui les unissent, des constructions dont ils peuvent être les matériaux, et aussi par les propositions concernant ces propriétés, éléments, constructions, relations, qu'on regarde tout d'abord comme vraies. Cette organisation porte sous le nom de structure.

Une théorie mathématique nous apparaît donc comme résultant de la considération simultanée de deux entités bien distinctes : d'une part, les ensembles fondamentaux qui sont l'objet de la théorie, d'autre part, la structure qui forme le sujet de la théorie et qui en est la partie vivante et essentielle.

Le lecteur constatera par la suite que l'il est toujours très cisé de laissez indéterminé le nature des éléments.
N. Bourbaki, Éléments de mathématique
Three versions of the (statement of the) Pythagorean theorem: Version 1: Euclid

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle. (Elements, Proposition 1.47)
Three versions of the (statement of the) Pythagorean theorem: Version 1: Euclid

*In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.*

(*Elements*, Proposition 1.47)
Three versions of the (statement of the) Pythagorean theorem: Version 2: Arnauld (1667)
Three versions of the (statement of the) Pythagorean theorem: Version 2: Arnauld (1667)

The square of hypothenuse is equal to (the sum of) squares of the two (other) sides (of the given rectangular triangle): \( bb + dd = hh \).

( New Elements of Geometry, Proposition 14.26.4)
Three versions of the (statement of the) Pythagorean theorem: Version 3: Doneddu (1965)
Three versions of the (statement of the) Pythagorean theorem: Version 3: Doneddu (1965)

Two non-zero vectors $x$ and $y$ are orthogonal if and only if

$$(y - x)^2 = y^2 + x^2$$

(Doneddu, *Euclidean plane geometry*)
Claim: Versions 1-3 of the Pythagorean theorem differ in their foundations, i.e., differ *radically*. 
Claim: Versions 1-3 of the Pythagorean theorem differ in their foundations, i.e., differ \textit{radically}.
Foundations change more rapidly than the rest of mathematics!
Question: What versions 1-3 of the Pythagorean theorem share in common?
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Claim: Versions 1-3 of the Pythagorean theorem share only a common history. Older versions translate into newer versions (but, generally, not the other way round!) They do not share an "essence" or a "structure".
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Dialectical links between older and newer foundations are neither causal nor contingent. They represent an objective development of ideas.
It is often claimed that until recently Euclid’s *Elements* used to be a Bible of mathematics. However, as we have seen, the literature published under the title of ”Euclid’s Elements” since the beginning of book printing is quite diverse. Revision of current versions of Euclid’s book until very recently was a rule rather than an exception. The alleged stickiness to Euclid’s letter NEVER existed in mathematics! The history of revisions of Euclid’s *Elements* still waits to be accounted for systematically!
Plato’s philosophy of mathematics

WARNING: It has little if anything to do with “mathematical Platonism” that was first described by P. Bernays in 1935 and later became popular in the Analytic philosophy of mathematics.
We must make a distinction and ask, What is that which always is and has no becoming; and what is that which is always becoming and never is? That which is apprehended by intelligence and reason is always in the same state; but that which is conceived by opinion with the help of sensation and without reason, is always in a process of becoming and perishing and never really is. (Tim. 27d-28a)
Hypothetical knowledge

(Socrates talks to Gaucon)
”[ Socrates :] - Next proceed to consider the manner in which the sphere of the intellectual is to be divided.
- In what manner?
–There are two subdivisions, in the lower or which the soul uses the figures given by the former division as images; the enquiry can only be hypothetical, and instead of going upwards to a principle descends to the other end; in the higher of the two, the soul passes out of hypotheses, and goes up to a principle which is above hypotheses, making no use of images as in the former case, but proceeding only in and through the ideas themselves.
- I do not quite understand your meaning, he said.
Hypothetical knowledge

- Then I will try again ... . You are aware that students of geometry, arithmetic, and the kindred sciences assume the odd and the even and the figures and three kinds of angles and the like in their several branches of science; these are their hypotheses, which they and everybody are supposed to know, and therefore they do not deign to give any account of them either to themselves or others; but they begin with them, and go on until they arrive at last, and in a consistent manner, at their conclusion?
Hypothetical knowledge

- Yes, he said, I know.
- And do you not know also that although they make use of the visible forms and reason about them, they are thinking not of these, but of the ideals which they resemble; not of the figures which they draw, but of the absolute square and the absolute diameter, and so on—the forms which they draw or make, and which have shadows and reflections in water of their own, are converted by them into images, but they are really seeking to behold the things themselves, which can only be seen with the eye of the mind?
Hypothetical knowledge

- That is true.
- And of this kind I spoke as the intelligible, although in the search after it the soul is compelled to use hypotheses; not ascending to a first principle, because she is unable to rise above the region of hypothesis, but employing the objects of which the shadows below are resemblances in their turn as images, they having in relation to the shadows and reflections of them a greater distinctness, and therefore a higher value.
- I understand, he said, that you are speaking of the province of geometry and the sister arts.
(Rep., 510b-511c)
Hypothetical knowledge IS mathematical knowledge!

NO SPECIAL ROOM FOR NATURAL SCIENCE
Physics is a “lower” section of mathematics (Quadrivium)
Critique of imagery

Plato criticize the geometrical reasoning (or more precisely the geometrical understanding) for using images. This critique follows from a more general notion, according to which opinion relies entirely on senses, reason operates with pure ideas without any help of sensual representations while mathematical understanding in general and geometrical understanding in particular do something in between. Geometry demonstrates the double nature of mathematics in the most explicit form. Plato’s critique amounts to pushing mathematical understanding from the domain of opinion toward a dialectical pure reasoning.
Mathematical objects unlike their ideal prototypes exist in an indefinite number of copies (Met. 987b). There is an indefinite number of copies of mathematical number 2 (i.e. an indefinitely many of such numbers) all of which correspond to the same ideal number 2. The former unlike the latter cannot be a subject of arithmetical operations; this in particular implies that ideal numbers unlike mathematical ones cannot be thought of as sums of units and so are indivisible (Met. 1081a-1082b). If one follows Plato’s advise and “ascends” from mathematical objects to their ideal prototypes one certainly stops doing mathematics!
Aristotle’s philosophy of mathematics: Nature of things and their Forms

Antiphon points out that if you planted a bed and the rotting wood acquired the power of sending up a shoot, it would not be a bed that would come up, but wood - which shows that the arrangement in accordance with the rules of the art is merely an incidental attribute, whereas the real nature is the other, which, further, persists continuously through the process of making. (Phys. 193a12-17)
Theory of Abstraction

[C]learly it is possible that there should also be both propositions and demonstrations about sensible magnitudes, not however qua sensible but qua possessed of certain definite qualities. For as there are many propositions about things merely considered as in motion, apart from what each such thing is and from their accidents, and as it is not therefore necessary that there should be either a mobile separate from sensibles, or a distinct mobile entity in the sensibles, so too in the case of mobiles there will be propositions and sciences, which treat them however not qua mobile but only qua bodies, or again only qua planes, or only qua lines, or qua divisibles, or qua indivisibles having position, or only qua indivisibles.
Theory of Abstraction

Thus since it is true to say without qualification that not only things which are separable but also things which are inseparable exist (for instance, that mobiles exist), it is true also to say without qualification that the objects of mathematics exist, and with the character ascribed to them by mathematicians. And as it is true to say of the other sciences too, without qualification, that they deal with such and such a subject - not with what is accidental to it (e.g. not with the pale, if the healthy thing is pale, and the science has the healthy as its subject), but with that which is the subject of each science - with the healthy if it treats its object qua healthy, with man if qua man : - so too is it with geometry; if its subjects happen to be sensible, though it does not treat them qua sensible, the mathematical sciences will not for that reason be sciences of sensibles - nor, on the other hand, of other things separate from sensibles.
Theory of Abstraction

Many properties attach to things in virtue of their own nature as possessed of each such character; e.g. there are attributes peculiar to the animal qua female or qua male (yet there is no 'female' nor 'male' separate from animals); so that there are also attributes which belong to things merely as lengths or as planes. And in proportion as we are dealing with things which are prior in reason and simpler, our knowledge has more accuracy, i.e. simplicity. Therefore a science which abstracts from spatial magnitude is more precise than one which takes it into account; and a science is most precise if it abstracts from movement, but if it takes account of movement, it is most precise if it deals with the primary movement, for this is the simplest; and of this again uniform movement is the simplest form. ...
Theory of Abstraction

Each question will be best investigated in this way - by setting up by an act of separation what is not separate, as the arithmetician and the geometer do. For a man qua man is one indivisible thing; and the arithmetician supposed one indivisible thing, and then considered whether any attribute belongs to a man qua indivisible. But the geometer treats him neither qua man nor qua indivisible, but as a solid. For evidently the properties which would have belonged to him even if perchance he had not been indivisible, can belong to him even apart from these attributes. Thus, then, geometers speak correctly; they talk about existing things, and their subjects do exist. (Met. 1077b16 - 1078a30)
Theory of Abstraction

dήλον ότι ἔνδεχεται καὶ περὶ τῶν αἰσθητῶν μεγεθῶν εἶναι καὶ λόγους καὶ ἀποδείξεις, μὴ ἢ δὲ αἰσθητὰ ἄλλ᾽ ἢ τοιαδί. ὡσπερ γὰρ καὶ ἦ κινούμενα μόνον πολλοὶ λόγοι εἰσί, χωρὶς τοῦ τί ἐκαστὸν ἐστι τῶν τοιούτων καὶ τῶν συμβεβηκότων αὐτοῖς, καὶ οὐκ ἀνάγκη διὰ ταῦτα ἢ κεχωρισμένον τι εἶναι κινούμενον τῶν αἰσθητῶν ἢ ἐν τούτοις τινὰ φύσιν εἶναι ἀφωρισμένην, οὕτω καὶ ἐπὶ τῶν κινούμενων ἐσονται λόγοι καὶ ἐπιστήμαι, οὐχ ἦ κινούμενα δὲ ἄλλ᾽ ἢ σώματα μόνον, καὶ πάλιν ἦ ἐπίπεδα μόνον καὶ ἦ μήκη μόνον, καὶ ἦ διαιρετὰ καὶ ἦ ἀδιαιρετὰ ἑχοντα δὲ ϑέσιν καὶ ἦ ἀδιαιρετὰ μόνον, ὡςτ᾽ ἐπεὶ ἀπλῶς λέγειν ἁληθὲς μὴ μόνον τὰ χωριστά εἶναι ἀλλὰ καὶ τὰ μὴ χωριστά (οἰον κινούμενα εἶναι), καὶ τὰ μαθηματικὰ ότι ἔστιν ἀπλῶς ἁληθὲς εἰπεῖν, καὶ τοιαύτα γε οίᾳ λέγουσιν.
Theory of Abstraction

καὶ ὡσπερ καὶ τὰς άλλας ἐπιστήμας ἀπλῶς ἀληθὲς εἰπεῖν τούτο εἶναι, οὔχι τοῦ συμβεβηκότος (οἶν ὁτι λευκοῦ, εἰ τὸ ύγιεινὸν λευκὸν, ἥ δ᾽ ἔστιν ύγιεινοῦ) ἀλλ᾽ ἐκείνου οὐ ἔστιν ἐκάστη, εἰ <ἡ> ύγιεινὸν ύγιεινοῦ, εἰ δ᾽ ἥ ἀνθρώπος ἀνθρώπου, οὔτω καὶ τὴν γεωμετρίαν: οὔκ εἰ συμβέβηκεν αἰσθητὰ εἶναι ὕν ἐστί, μὴ ἔστι δὲ ἥ αἰσθητά, οὐ τῶν αἰσθητῶν ἔσονται αἱ μαθηματικαὶ ἐπιστήμαι, οὐ μέντοι οὐσὶ παρὰ ταῦτα ἄλλων κεχωρισμένων. πολλὰ δὲ συμβέβηκε καθ᾽ αὐτὰ τοῖς πράγμασιν ἥ ἐκαστὸν ὑπάρχει τῶν τοιούτων, ἐπεὶ καὶ ἥ θῆλυ τὸ ζώον καὶ ἥ ἄρρεν, ἴδια πάθη ἔστιν (καίτοι οὐκ ἔστι τι θῆλυ οὔδ᾽ ἄρρεν κεχωρισμένον τῶν ζῴων):
Theory of Abstraction

ὡστε καὶ ἦ μήκη μόνον καὶ ἦ ἐπίπεδα. καὶ ὅσφι δὴ ἀν περὶ προτέρων τῷ λόγῳ καὶ ἀπλουστέρων, τοσούτῳ μᾶλλον ἔχει τὸ ἀκριβές (τούτῳ δὲ τὸ ἄπλοῦν ἔστίν), ὡστε ἀνευ τε μεγέθους μᾶλλον ἦ μετὰ μεγέθους, καὶ μάλιστα ἀνευ κινήσεως, ἐὰν δὲ κίνησιν, μάλιστα τὴν πρώτην: ἀπλουστάτη γάρ, καὶ ταύτης ἢ ὁμαλή. ὃ δ᾽ αὐτὸς λόγος καὶ περὶ ἀρμονικῆς καὶ ὀπτικῆς: οὐδετέρα γάρ ἢ ὁψις ἢ φωνή θεωρεῖ, ἀλλ᾽ ἡ γραμμαί καὶ ἄριθμοί (οἰκεῖα μέντοι ταῦτα πάθη ἐκείνων), καὶ ἡ μηχανική δὲ ὡσαύτως, ὡστ᾽ εἰ τις θέμενος κεχωρισμένα τῶν συμβεβηκότων σκοπεῖ τι περὶ τούτων ἢ τοιαῦτα, οὐθὲν διὰ τὸ τοῦτο ψεύδος ψεύσεται, ὡσπερ οὐδ᾽ ὅταν ἐν τῇ γῇ γράφῃ καὶ ποδιαίαν φῇ τῇν μὴ ποδιαίαν: οὐ γάρ ἐν ταῖς προτάσεσι τὸ ψεύδος.
Theory of Abstraction

ἄριστα δὲ ἂν οὕτω θεωρηθείη ἕκαστον, εἰ τις τὸ μὴ κεχωρισμένον θείη χωρίσας, ὅπερ ὁ ἀριθμητικὸς ποιεῖ καὶ ὁ γεωμέτρης. ἐν μὲν γὰρ καὶ ἄδιαιρετον ὁ ἄνθρωπος ἢ ἄνθρωπος: ὁ δὲ ζητεῖ ἐν ἄδιαιρετον, εἰτ' ἐθεώρησεν εἰ τι τῷ ἄνθρωπῳ συμβεβηκεν ἢ ἄδιαιρετος. ὁ δὲ γεωμέτρης οὐθ' ἢ ἄνθρωπος οὐθ' ἢ ἄδιαιρετος ἀλλ' ἢ στερεόν. ἃ γὰρ κἂν εἰ μὴ ποὺ ἢν ἄδιαιρετος ὑπῆρχεν αὐτῷ, δῆλον ὅτι καὶ ἄνευ τούτων ἐνδέχεται αὐτῷ ὑπάρχειν [τὸ δυνατόν], ὥστε διὰ τούτῳ ὅρθως οἱ γεωμέτραι λέγουσι, καὶ περὶ ὄντων διαλέγονται, καὶ ὄντα ἐστίν:
Trade between precision and abstraction

The more abstract is a given subject matter (i.e. the less is the number of features simultaneously taken into consideration) the more precise is the corresponding theory. This explains, in particular, why arithmetic is more precise than geometry. However on Aristotle’s account the more abstract implies the less real. Thus unlike Plato Aristotle doesn’t think of theoretical precision as a direct evidence of truth about what there is. He rather thinks of it as one specific epistemic criterion competing with other epistemic criteria, which are equally important.
Platonic Quadrivium upside down

Remind that in the Quadrivium the science of astronomy is given the lowest possible grade, which it shares with the science of harmonics. Aristotle, on the contrary, sees astronomy as a science, which achieves the best balance between mathematical precision and physical substantiality. This makes astronomy, by Aristotle’s word “most akin to philosophy”. As explains Aristotle “this science speculates about substance, which is perceptible but eternal, while the other mathematical sciences, i.e. arithmetic and geometry, treat of no substance”. (Met. 1073b5-7)
Metabasis

One can reasonably constitute a research area like *female studies* provided that it will study only human females or only female individuals of some other particular species. But the notion of *general female studies*, which is a science about females of all biological species, is absurd. This is in spite of the fact that the general notion of female makes a perfect sense and applies across the species. Such generality doesn’t allow one to abstract the property of being a female from the underlying species and make it into a subject matter of a special study.
Aristotelian doubt

DOES MATHEMATICS REALLY AVOIDS THE *METABASIS*?
Classical Model of Science (after Betti et al.)
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- Every particular science accounts for a certain domain of being and claims certain truths (true propositions) about entities belonging to this domain of being.
Classical Model of Science (after Betti et al.)

- Every particular science accounts for a certain domain of being and claims certain truths (true propositions) about entities belonging to this domain of being.

- Scientific truths are divided into two classes: first (fundamental) truths taken for granted and secondary (derived) truths.

Logical inferences are governed by laws of logic, which reflect general ontological principles and are universal for all sciences.
Every particular science accounts for a certain domain of being and claims certain truths (true propositions) about entities belonging to this domain of being.

Scientific truths are divided into two classes: first (fundamental) truths taken for granted and secondary (derived) truths.

Derived truths are obtained from fundamental truths through logical inference. Fundamental truths are taken as premises of inferences and derived truths are obtained as conclusions of inferences.
Classical Model of Science (after Betti et al.)

- Every particular science accounts for a certain domain of being and claims certain truths (true propositions) about entities belonging to this domain of being.

- Scientific truths are divided into two classes: first (fundamental) truths taken for granted and secondary (derived) truths.

- Derived truths are obtained from fundamental truths through logical inference. Fundamental truths are taken as premises of inferences and derived truths are obtained as conclusions of inferences.

- Logical inferences are governed by laws of logic, which reflect general ontological principles and are universal for all sciences.
"Oroi.

α'. Σημεῖον ἐστιν, οὐ μέρος οὐδὲν.
β'. Γραμμή δὲ μήκος ἀπλῶς.
γ'. Γραμμῆς δὲ πέρατα σημεῖα.
δ'. Εὐθεία γραμμὴ ἐστιν, ἣτις ἐξ ἵς τοῖς ἐπὶ ἑκάτης

Σημείωσις καὶ ταῦτα.
ε'. Ἑπαράνθενά δὲ ἐστιν, ὁ μήκος καὶ πλάτος μόνον ἐχθεῖ.
ζ'. Ἐπιπέδου ἐπιφάνεια ἐστιν, ἣτις ἐξ ἵς τοῖς ἐπὶ ἑκάτης

εὐθείας εὐθείας καὶ ταῦτα.

Definitions

1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points.
4. A straight-line is whatever lies evenly with points upon itself.
5. And a surface is that which has length and breadth alone.
6. And the extremities of a surface are lines.
7. A plane surface is whatever lies evenly with straight-lines upon itself.
8. And a plane angle is the inclination of the lines, when two lines in a plane meet one another, and are not laid down straight-on with respect to one another.

9. And when the lines containing the angle are straight then the angle is called rectilinear.

10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called perpendicular to that upon which it stands.

11. An obtuse angle is greater than a right-angle.

12. And an acute angle is less than a right-angle.
Definitions of the 1st Book

Postulates
Axioms
Problems and Theorems

Euclid’s “Elements” and Foundations of Mathematics
 Definitions of the 1st Book
 Postulates
 Axioms
 Problems and Theorems

Euclid’s “Elements” and Foundations of Mathematics

21. And further of the trilateral figures: a right-angled triangle is that having a right-angle, an obtuse-angled (triangle) that having an obtuse angle, and an acute-angled (triangle) that having three acute angles.

22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.

23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).
Definitions of the 1st Book
Postulates
Axioms
Problems and Theorems

Euclid’s “Elements” and the problem of Mathesis Universalis
Definitions of the 7th Book

"Ὄροι.

α'. Μονάς ἐστιν, καθ' ἣν ἔκαστον τῶν ὄντων ἐν λέγεται.

β'. Ἄριθμος δὲ τὸ ἐκ μονάδων συγκείμενον πλήθος.

γ'. Μέρος ἐστὶν ἁριθμὸς ἀριθμοῦ ὁ ἐλάσσον τῷ μεῖζονος, ὅταν καταμετρήθη τὸν μεῖζονα.

δ'. Μέρη δὲ, ὅταν μὴ καταμετρήθη.

ε'. Πολλαπλασίας δὲ ὁ μεῖζον τοῦ ἐλάσσονος, ὅταν καταμετρήθηται ὑπὸ τοῦ ἐλάσσονος.

ζ'. Ἄρτιος ἁριθμὸς ἐστὶν ὁ δίχα διαιρούμενος.

η'. Περισσός δὲ ὁ μὴ διαιρούμενος δίχα ἡ [ὁ] μονάδι διαιρέων ἁρτίου ἁριθμοῦ.

ζ'. Ἀρτιάς ἁρτιός ἁριθμὸς ἐστὶν ὁ ὑπὸ ἁρτίου ἁριθμοῦ μετρούμενος κατὰ ἁρτίον ἁριθμὸν.

ζ`. Ἀρτιάς δὲ περισσός ἐστὶν ὁ ὑπὸ ἁρτίου

1. A unit is (that) according to which each existing thing is said (to be) one.

2. And a number (is) a multitude composed of units.

3. A number is part of another number, the lesser of the greater, when it measures the greater.

4. But (the lesser is) parts (of the greater) when it does not measure it.

5. And the greater (number is) a multiple of the lesser when it is measured by the lesser.

6. An even number is one (which can be) divided in half.

7. And an odd number is one (which cannot) be divided in half, or which differs from an even number by a unit.
WARNING: ONE IS NOT A NUMBER!
WARNING : ONE IS NOT A NUMBER!

number = finite set?
Definitions of the 1st Book
Postulates
Axioms
Problems and Theorems

Euclid’s “Elements” and Foundations of Mathematics

Postulates

1. Let it have been postulated to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).

† This postulate effectively specifies that we are dealing with the geometry of flat, rather than curved, space.
Foundations of mathematics from Euclid to Hilbert to Bourbaki
Plato's and Aristotle's philosophies of mathematics
Reading of Euclid's "Elements": Definitions, Postulates and Axioms
Euclid’s "Elements" and the problem of Mathesis Universalis

Definitions of the 1st Book
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Euclid's "Elements" and Foundations of Mathematics
Postulates 1-3

The drawing of a line from any point to any point follows from the conception of the line as the flowing of a point and of the straight line as its uniform and undeviating flowing. For if we think of the point as moving uniformly over the shortest path, we shall come to the other point and so shall have got the first postulate without any complicated process of thought. And if we take a straight line as limited by a point and similarly imagine its extremity as moving uniformly over the shortest route, the second postulate will have been established by a simple and facile reflection. And if we think of a finite line as having one extremity stationary and the other extremity moving about this stationary point, we shall have produced the third postulate. (Proclus, Commentary 185.8-2)
Postulates 1-3 are **not** propositions but generic principles.
Postulates 4-5 are problematic.
Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.\(^\dagger\)
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

\(\alpha\). Τὰ τῷ ἑαυτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.
\(\beta\). Καὶ ἕαν ἴσοις ἴσα προστέθη, τὰ ἕλθα ἐστὶν ἴσα.
\(\gamma\). Καὶ ἓκα ἀπὸ ἴσων ἴσα ἀφαίρεθη, τὰ καταλείπομενα ἐστὶν ἴσα.
\(\delta\). Καὶ τὰ ἐφαρμόζοντα ἐπὶ ἀλλήλα ἴσα ἀλλήλοις ἐστὶν.
\(\epsilon\). Καὶ τὸ ἕλον τοῦ μέρους μεῖζὸν [ἔστιν].
are common for all mathematical disciplines. (Also Ax. 5!) In geometry “equality” means (roughly) “equicomposability”.
... is defined “up to mathematical equality” rather than strict identity. Numbers and magnitudes exist in an indefinite number of “copies”. How many 2s are there?
Being and Becoming

Science as a whole has two parts: in one it occupies itself with immediate enunciations, while in the other it treats systematically the things that can be demonstrated or constructed from these first principles, or in general are consequences of them. In the geometrical reasoning this second part is again divided into solving problems and finding theorems. The name “problem” is appropriate where what in a sense doesn’t exist is produced, set, brought into view and arranged, while the name “theorem” is appropriate where something that is attributed or not attributed is seen, known and proved. The former [has to do with] generation, setting, application, ascription, inscription, insertion, touching and the like; the latter [has to do with] properties and essential attributes of geometrical objects, which are grasped and firmly bound by demonstration. (Commentary, 200.20-201.14)
Structure

Every Problem and every Theorem that is furnished with all its parts should contain the following elements: [i] an enunciation, [ii] an exposition, [iii] a specification, [iv] a construction, [v] a proof, and [vi] a conclusion. Of these enunciation states what is given and what is being sought from it, a perfect enunciation consists of both these parts. The exposition takes separately what is given and prepares it in advance for use in the investigation. The specification takes separately the thing that is sought and makes clear precisely what it is. The construction adds what is lacking in the given for finding what is sought. The proof draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The conclusion reverts to the enunciation, confirming what has been proved.” (Commentary, 203.1-15)
ΣΤΟΙΧΕΙΩΝ α΄.

α΄.

Επί τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ισόπλευρον συστήσαται.

Proposition 1

To construct an equilateral triangle on a given finite straight-line.

Let \( AB \) be the given finite straight-line. So it is required to construct an equilateral triangle on the straight-line \( AB \).

Let the circle \( BCD \) with center \( A \) and radius \( AB \) have been drawn [Post. 3], and again let the circle \( ACE \) with center \( B \) and radius \( BA \) have been drawn [Post. 3]. And let the straight-lines \( CA \) and \( CB \) have been joined from the point \( C \), where the circles cut one another, \(^1\) to the points \( A \) and \( B \) (respectively) [Post. 1].

And since the point \( A \) is the center of the circle \( CDB \), \( AC \) is equal to \( AB \) [Def. 1.15]. Again, since the point \( B \) is the center of the circle \( CAE \), \( BC \) is equal to \( BA \) [Def. 1.15]. But \( CA \) was also shown (to be) equal to \( AB \). Thus, \( CA \) and \( CB \) are each equal to \( AB \). But things equal to the same thing are also equal to one another [CN. 1]. Thus, \( CA \) is also equal to \( CB \). Thus, the three (straight-lines) \( CA \), \( AB \), and \( BC \) are equal to one another.

Thus, the triangle \( ABC \) is equilateral, and has been constructed on the given finite straight-line \( AB \). (Which
Proposition 5

For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another.

Let $ABC$ be an isosceles triangle having the side $AB$ equal to the side $AC$, and let the straight-lines $BD$ and $CE$ have been produced in a straight-line with $AB$ and $AC$ (respectively) [Post. 2]. I say that the angle $ABC$ is equal to $ACB$, and (angle) $CBD$ to $BCE$.

For let the point $F$ have been taken somewhere on $BD$, and let $AG$ have been cut off from the greater $AE$, equal to the lesser $AF$ [Prop. 1.3]. Also, let the straight-lines $FC$ and $GB$ have been joined [Post. 1].
In fact, since $AF$ is equal to $AG$, and $AB$ to $AC$, the two (straight-lines) $FA$, $AC$ are equal to the two (straight-lines) $GA$, $AB$, respectively. They also encompass a common angle $FAG$. Thus, the base $FC$ is equal to the base $GB$, and the triangle $AFC$ will be equal to the triangle $AGB$, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) $ACF$ to $ABG$, and $AFC$ to $AGB$. And since the whole of $AF$ is equal to the whole of $AG$, within which $AB$ is equal to $AC$, the remainder $BF$ is thus equal to the remainder $CG$ [C.N. 3]. But $FC$ was also shown (to be) equal to $GB$. So the two (straight-lines) $BF$, $FC$ are equal to the two (straight-lines) $CG$, $GB$, respectively, and the angle $BFC$ (is) equal to the angle $CGB$, and the base $BC$ is common to them. Thus, the triangle $BFC$ will be equal to the triangle $CGB$, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. Thus, $FBC$ is equal to $GCB$, and $BCF$ to $CGB$. Therefore, since the whole angle $ABC$ was shown (to be) equal to the whole angle $ACF$, within which $CGB$ is equal to $BCF$, the remainder $ABC$ is thus equal to the remainder $ACB$ [C.N. 3]. And they are at the base of triangle $ABC$. And $FBC$ was also shown (to be) equal to $GCB$. And they are under the base.

Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.
Double-thinking

Mathematicians are accustomed to draw what is in a way a double conclusion. For when they have shown something to be true of the given figure, they infer that it is true in general, going from the particular to the universal conclusion. Because they do not make use of the particular qualities of the subjects but draw the angle or the straight line in order to place what is given before our eyes, they consider that what they infer about the given angle or straight line can be identically asserted for every similar case. They pass therefore to the universal conclusion in order that we may not suppose that the result is confined to the particular instance. This procedure is justified, since for the demonstration they use the objects set out in the diagram not as these particular figures, but as figures resembling others of the same sort.
Double-thinking

It is not as having such-and-such size that the angle before me is bisected, but as being rectilinear and nothing more. Its particular size is a character of the given angle, but its having rectilinear sides is a common feature of all rectilinear angles. Suppose the given angle is a right angle. If I used its rightness for my demonstration, I should not be able to infer anything about the whole class of rectilinear angles; but if I make no use of its rightness and consider only its rectilinear character, the proposition will apply equally to all angles with rectilinear sides. (Commentary 207.4-25)
Structure
Structure

- Objective and subjective individuation
Structure

- Objective and subjective individuation
- Naming
Structure

- Objective and subjective individuation
- Naming
- The role of proof
Structure

- Objective and subjective individuation
- Naming
- The role of proof
- Return
Proportion of numbers

20. Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third (is) of the fourth.
Proportion of magnitudes

"Ὁροι."

α’. Μέρος ἐστὶ μέγεθος μεγέθους τὸ ἐλασσὸν τοῦ μείζονος, ὅταν καταμετρῇ τὸ μεῖζον.

β’. Πολλαπλάσιον δὲ τὸ μεῖζον τοῦ ἐλάσσονος, ὅταν καταμετρῇ τὸ ἐλάσσονος.

γ’. Λόγος ἐστὶ δύο μεγεθῶν ὑμοιογενῶν ἢ κατὰ πηλικότητα ποια σχέσεις.

δ’. Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἤ διόντα πολλαπλασιάζοντα ἄλληλων ὑπερέχειν.

ε’. Ἐν τῷ αὐτῷ λόγῳ μεγέθη λέγεται εἶναι πρῶτον πρὸς διότερον καὶ τρίτον πρὸς τέταρτον, ὅταν τὰ τοῦ πρώτου καὶ τρίτου ἰσοί μοι πολλαπλάσια τῶν τοῦ δευτέρου καὶ τέταρτου ἰσοί μοι πολλαπλασιασμὸν ἕκατον ἢ ἄρα ὑπερέχῃ ἢ ἄρα ἰσα ἢ ἂν ἄλλη περισσῆ ἡ ἁμαλίκη αὐτός ἢ ἀκομὴ ἐλλιστῇ ληθάνετα κατάλληλα.

ξ’. Τὰ δὲ τῶν αὐτῶν ἔχουσα λόγων μεγεθή τάξειν ὑπάρχουσαν καθέσθαι.

ζ’. Ἡ ὅταν δὲ τῶν ἰσοί μοι πολλαπλασιασμὸν τὸ μὲν τοῦ πρώτου πολλαπλασιασμὸν ὑπερέχῃ τοῦ τοῦ δευτέρου πολ-

**Definitions**

1. A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater.†

2. And the greater (magnitude) is a multiple of the lesser when it is measured by the lesser.

3. A ratio is a certain type of condition with respect to size of two magnitudes of the same kind.‡

4. (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another..§

5. Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and the third either both exceed, are both equal to, or are both less than, equal multiples of the second and the fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.¶

6. And let magnitudes having the same ratio be called proportional.*
Euclid's "Elements" and Foundations of Mathematics

Proposition 5

Commensurable magnitudes have to one another the ratio which (some) number (has) to (some) number.

Γὰρ Ἀριθμὸς πρὸς Ἀριθμόν.
Proclus on Universal Mathematics

Let us enumerate the simple theorems .... generated by the single science that embraces alike all forms of mathematical knowledge; and let us see how they fit into all these sciences and can be observed alike in numbers, magnitudes, and motions. Such are the theorems governing proportion, namely, the rules of compounding, dividing, converting, and alternating; likewise the theorems concerning ratios of all kinds, multiple, superparticular, superpartient, and their counterparts; and the theorems about equality and inequality in their most general and universal aspects, not equality or inequality of figures, numbers, or motions, but each of the two by itself as having a nature common to all its forms and capable of more simple apprehension.
Proclus on Universal Mathematics

.... We must not regard these common theorems as subsisting their origin from them (= the particular sciences), but as prior to their instances and superior in simplicity and exactness. For this reason, knowledge of them takes precedence over the particular sciences and furnishes to them their principles; that is, these several sciences are based upon this prior science and refer back to it.
Aristotle on Universal Mathematics

The question may be asked whether first philosophy is universal or deals with some particular genus or some one class of things. For not even in mathematical sciences is the method one and the same; geometry and astronomy, for instance, deal with a certain class of thing, but the universal science of mathematics is common to all branches. (Met. 1026a3-7)
Aristotle on Universal Mathematics

For each of the mathematical sciences is concerned with some distinct genus, but universal science of mathematics is common to all (Met.1064b8-9)
Aristotle on Universal Mathematics

Further some propositions are proved universally by mathematicians, which extend beyond these substances [belonging to special mathematical sciences] (Met.1077a9-10)
Aristotle on Universal Mathematics

Just as the universal part of mathematics deals not with objects which exist separately, apart from extended magnitudes and numbers, but with magnitudes and numbers, not however qua such as to have magnitude or to be divisible, clearly it is possible that there should also be both propositions and demonstrations about sensible magnitudes, not however qua sensible but qua possessed of certain definite qualities.” (Met. 1077b17-22)
Ontological worry

What is the subject-matter (genus) of Universal Mathematics?
What is the subject-matter (genus) of Universal Mathematics? The answer given in 17th century: the general notion of magnitude that includes that of *number* as a special case of *discrete* magnitude along with *continuous* geometrical magnitude (Arnauld)
If the notion of Universal Mathematics is sound then the Universal Mathematics must be treated first.
Epistemological worry

If the notion of Universal Mathematics is sound then the Universal Mathematics must be treated first. Solution of 17th century: Algebra is Universal Mathematics, i.e., a general theory of magnitude.
To hold universally

Something holds universally when it is proved of an arbitrary and primitive case. E.g. having [the sum of internal angles equal to] two right angles doesn’t hold universally of figures - you may indeed prove of a figure that it has two right angles, but not of an arbitrary figure, nor can you use an arbitrary figure in proving it; for quadrangles are figures but do not have angles equal to two right angles. An arbitrary isosceles [triangle] does have angles equal to two right angles - but it is not primitive: triangles are prior. Thus if an arbitrary primitive case is proved to have two right angles (or whatever else), then it holds universally of this primitive item, and the demonstration applies to it universally […] [I]t does not apply to the isosceles [triangles] universally, but extends further. (An.Pr. 73b33-74a4)
Euclid’s “Elements” and the problem of Mathesis Universalis

To hold universally

τὸ καθόλου δὲ ύπάρχει τότε, ὅταν ἐπὶ τοῦ τυχόντος καὶ πρώτου δεικνύηται. οἶον τὸ δύο ὀρθὰς ἔχειν οὔτε τῷ σχήματι ἔστι καθόλου (καίτοι ἐστὶ δεῖξαι κατὰ σχήματος ὅτι δύο ὀρθὰς ἔχει, ἀλλ’ οὐ τοῦ τυχόντος σχήματος, οὐδὲ χρήται τῷ τυχόντι σχήματι δεικνύς· τὸ γὰρ τετράγωνον σχῆμα μέν, οὐκ ἔχει δὲ δύο ὀρθαῖς ἴσας) τὸ δ’ ἰσοσκελὲς ἔχει μὲν τὸ τυχόν δύο ὀρθαῖς ἴσας, ἀλλ’ οὔ πρῶτον, ἀλλὰ τὸ τρίγωνον πρότερον. ὃ τοίνυν τὸ τυχόν πρῶτον δεικνυται δύο ὀρθὰς ἔχον ἢ ὅτιοῦν ἄλλο, τούτω πρώτω ύπάρχει καθόλου, καὶ ἢ ἀπόδειξις καθ’ αὐτὸ τούτου καθόλου ἔστι, τῶν δ’ ἄλλων τρόπων τινὰ οὐ καθ’ αὐτό, οὐδὲ τοῦ ἰσοσκελοῦς οὐκ ἔστι καθόλου ἀλλ’ ἐπὶ πλέον.
Specific and Universal Principles after Aristotle

Instances of first principles peculiar to a science are the assumptions that a line is of such-and-such a character, and similarly for the straight line; whereas it is a common principle, for instance, that if equals be subtracted from equals, the remainders are equal.” (An. Post. 76a38-43)
By first principles of proof I mean the common opinions on which all men base their demonstrations, e.g. that one of two contradictories must be true, that it is impossible for the same thing both be and not to be, and all other propositions of this kind.” (Met. 996b27-32)
By first principles of proof I mean the common opinions on which all men base their demonstrations, e.g. that one of two contradictories must be true, that it is impossible for the same thing both be and not to be, and all other propositions of this kind.” (Met. 996b27-32)

- A1 : If $A = B$ and $C = B$ then $A = C$
From Mathematics to Logic

By first principles of proof I mean the common opinions on which all men base their demonstrations, e.g. that one of two contradictories must be true, that it is impossible for the same thing both be and not to be, and all other propositions of this kind.” (Met. 996b27-32)

- A1 : If $A = B$ and $C = B$ then $A = C$
- PS : If all $A$ are $B$ and all $B$ are $C$ then all $A$ are $C$
Changing meaning of the term “axiom”

We have now to say whether it is up to the same science or to different sciences to inquire into what in mathematics is called axioms and into the essence. Clearly the inquiry into these things is up to the same science, namely, to the science of the philosopher. For axioms hold of everything that [there] is but not of some particular genus apart from others. Everyone makes use of them because they concern being qua being, and each genus is. But men use them just so far as is sufficient for their purpose, that is, within the limits of the genus relevant to their proofs. Since axioms clearly hold for all things qua being (for being is what all things share in common) one who studies being qua being also inquires into the axioms. This is why one who observes things partly [=who inquires into a special domain] like a geometer or a arithmetician never tries to say whether the axioms are true or false. (Met. 1005a19-28)
Since the mathematician too uses common [axioms] only on the case-by-case basis, it must be the business of the first philosophy to investigate their fundamentals. For that, when equals are subtracted from equals, the remainders are equal is common to all quantities, but mathematics singles out and investigates some portion of its proper matter, as e.g. lines or angles or numbers, or some other sort of quantity, not however qua being, but as [...] continuous. (Met. 1061b)
CLAIM: Aristotle’s Logic/Metaphysics is a generalization of Universal Mathematics; Logic/Metaphysics is supposed to provide a foundation for all sciences including physics.
CLAIM: Aristotle’s Logic/Metaphysics is a generalization of Universal Mathematics; Logic/Metaphysics is supposed to provide a foundation for all sciences including physics.

QUESTION: Does Aristotle’s logic apply to Euclid’s mathematics?
Aristotle’s controversial mathematical example

Let $A$ be two right angles, $B$ triangle, $C$ isosceles. Then $A$ is an attribute of $C$ because of $B$, but it is not an attribute of $B$ because of any other middle term; for a triangle has [its angles equal to] two right angles by itself, so that there will be no middle term between $A$ and $B$, $AB$ is matter for demonstration. (An. Pr. 48a33-37)
Aristotle’s controversial mathematical example
Aristotle’s controversial mathematical example

- All triangles have two right angles (Premise $AB$)
Aristotle’s controversial mathematical example

- All triangles have two right angles (Premise $AB$)
- All isosceles triangles are triangles (Premise $AC$)
Aristotle’s controversial mathematical example

- All triangles have two right angles (Premise $AB$)
- All isosceles triangles are triangles (Premise $AC$)
- All isosceles triangles have two right angles (Conclusion $BC$)
Aristotle’s controversial mathematical example

- All triangles have two right angles (Premise $AB$)
- All isosceles triangles are triangles (Premise $AC$)
- All isosceles triangles have two right angles (Conclusion $BC$)

$AB$ is immediate BUT still is a matter of demonstration??!!
Euclid’s reasoning and Aristotelian Logic

Proposition 32

For any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.

Let $ABC$ be a triangle, and let one of its sides $BC$ have been produced to $D$. I say that the external angle $ACD$ is equal to the (sum of the) two internal and opposite angles $CAB$ and $ABC$, and the (sum of the) three internal angles of the triangle—$ABC$, $BCA$, and $CAB$—is equal to two right-angles.

For let $CE$ have been drawn through point $C$ parallel to the straight-line $AB$ [Prop. 1.31].

And since $AB$ is parallel to $CE$, and $AC$ has fallen across them, the alternate angles $BAC$ and $ACE$ are equal to one another [Prop. 1.29]. Again, since $AB$ is parallel to $CE$, and the straight-line $BD$ has fallen across them, the external angle $ECD$ is equal to the internal and opposite (angle) $ABC$ [Prop. 1.29]. But $ACE$ was also shown (to be) equal to $BAC$. Thus, the whole an-
Euclid’s “Elements” and the problem of Mathesis Universalis

Reading of Euclid’s “Elements” : Definitions, Postulates and Axioms.

Euclid’s reasoning and Aristotelian Logic

Universal Mathematics
Diagrammatic thinking

Diagrams are devised by an activity, namely by dividing-up. If they had already been divided, they would have been manifest to begin with; but as it is this [clarity] presents itself [only] potentially. Why does the triangle have [the sum of its internal angles equal to] two right angles? Because the angles about one point are equal to two right angles. If the parallel to the side had been risen [in advance], this would be seen straightforwardly (Met. 1051a21-26)
Classical Model of Science

Hilbert and Bourbaki realize the Classical Model of Science in Mathematics. Is this indeed a good idea?
Hilbert and Bourbaki realize the Classical Model of Science in Mathematics. Is this indeed a good idea?
Paradoxically, the genuine Platonic philosophy stresses the significance of the *constructive* aspect of mathematics!
THE END