OBJECTS ARE MAPS

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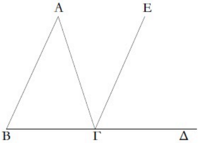
Content:

Euclidean Scheme

Every Problem and every Theorem that is furnished with all its parts should contain the following elements: [i] an proposition (προτασις), [ii] an exposition (εκθεσις), [iii] a specification (διαρισμος), [iv] a construction (κατασκευη), [v] a proof (αποδειξις), and [vi] a conclusion (συμπερασμα). (Proclus, *Commentary on Euclid*) λβʹ.

Παντὸς τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ἴση ἐστίν, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.



Ἔστω τρίγωνον τὸ ΑΒΓ, καὶ προσεκβεβλήσθω αὐτοῦ μία πλευρὰ ἡ ΒΓ ἐπὶ τὸ Δ· λέγω, ὅτι ἡ ἐκτὸς γωνία ἡ ὑπὸ ΑΓΔ ἴση ἐστὶ δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ταῖς ὑπὸ ΓΑΒ, ΑΒΓ, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΓΑ, ΓΑΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Ἤχθω γὰρ διὰ τοῦ Γ σημείου τῇ ΑΒ εὐθείᾳ παράλληλος ἡ ΓΕ.

Καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΕ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΑΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΒΑΓ, ΑΓΕ ἴσαι ἀλλήλαις εἰσίν. πάλιν, ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΕ, καὶ εἰς αὐτὰς ἐμπέπτωκεν εὐθεῖα ἡ ΒΔ, ἡ ἐκτὸς γωνία ἡ ὑπὸ ΕΓΔ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΑΒΓ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΑΓΕ τῇ ὑπὸ ΒΑΓ ἴση· ὅλη ἄρα ἡ ὑπὸ ΑΓΔ γωνία ἴση ἐστὶ δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ταῖς ὑπὸ ΒΑΓ, ΑΒΓ.

Κοινὴ προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΔ, ΑΓΒ τρισὶ ταῖς ὑπὸ ΑΒΓ, ΒΓΑ, ΓΑΒ ἴσαι εἰσίν. ἀλλ᾽ αἱ ὑπὸ ΑΓΔ, ΑΓΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΑΓΒ, ΓΒΑ, ΓΑΒ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν.

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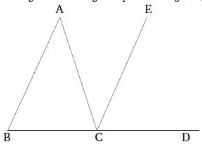
Proposition 32

For any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.

proposition



Let *ABC* be a triangle, and let one of its sides *BC* have been produced to *D*. I say that the external angle *ACD* is equal to the (sum of the) two internal and opposite angles *CAB* and *ABC*, and the (sum of the) three internal angles of the triangle—*ABC*, *BCA*, and *CAB*—is equal to two right-angles.

exposition
specification

For let *CE* have been drawn through point *C* parallel to the straight-line *AB* [Prop. 1.31].

construction

And since *AB* is parallel to *CE*, and *AC* has fallen across them, the alternate angles *BAC* and *ACE* are equal to one another [Prop. 1.29]. Again, since *AB* is parallel to *CE*, and the straight-line *BD* has fallen across them, the external angle *ECD* is equal to the internal and opposite (angle) *ABC* [Prop. 1.29]. But *ACE* was also shown (to be) equal to *BAC*. Thus, the whole angle *ACD* is equal to the (sum of the) two internal and opposite (angles) *BAC* and *ABC*.

proof

Let *ACB* have been added to both. Thus, (the sum of) *ACD* and *ACB* is equal to the (sum of the) three (angles) *ABC*, *BCA*, and *CAB*. But, (the sum of) *ACD* and *ACB* is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) *ACB*, *CBA*, and *CAB* is also equal to two right-angles.

(conclusion of the proof)

Thus, for any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of

conclusion(of the theorem)

Kant on geometrical intuition:

"Give a philosopher the concept of triangle and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on his concept as long as he wants, yet he will never produce anything new. ... But now let the geometer take up this question. He begins at once to construct a triangle. .... In such a way through a chain of inferences that is always guided by intuition, he arrives at a fully illuminated and at the same time general solution of the question." (Kant, *Critique of Pure Reason*)

A chemistry of intuitive awareness? Not quite:

Kant on symbolic intuition:

"But mathematics does not merely construct magnitudes (quanta), as in geometry but also mere magnitudes (quantitatem), as in algebra, where it entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitude. In this case it chooses a certain notation for all construction of magnitudes in general .... and thereby achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction (of objects themselves), which discursive cognition could never achive by means of mere concepts." (Kant, *ib.*)

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- (K2) Intuition underlies human sensual perception. Mathematical objects are, generally, objects of *possible* physical experience.
- (K3) *Objectivity* of mathematics and natural science amounts to the fact that object-construction is a subject to certain universal Principles (*Principles of the Pure Understanding*).

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- K1 and K3 belong to the subject-matter of *Transcendental Logic* . Transcendental Logic is distinguished by Kant from Formal Logic as the logic of "objectual" reasoning from the logic of speculative reasoning with bare concepts.
- K2 is a subject-matter of *Transcendental Aesthetics*

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- "I will not count among my principles those of mathematics, but I will include those on which the possibility and objective a priori validity of the latter are grounded, and which are thus to be regarded as the principle of these principles, and that proceed from concepts to the intuition and not from the intuition to concepts." (Kant, *ib*)

Kant on objecthood and objectivity

"An *object* is that in the concept of which the manifold of intuition is united. .... Transcendental unity of apperception is that unity through which all the manifold given in an intuition is united in a concept of the object. It is called *objective* on that account." (Kant, *ib*)

Kant's philosophy is inadequate to the modern mathematics

"One can only represent a single space, and if one speaks of many spaces, one understands by that only parts of one and the same unique space." (Kant, *ib.*)
Note: The space in question is *both* mathematical and physical

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This couldn't be justified after the discovery of non-Euclidean geometries. Geometrical spaces are many! (What makes us still think that the physical space or space-time is unique?)

Ways of modernization of the Kantian doctrine vis-à-vis the new mathematics

A: Frege (followed by Russell and the whole of Analytic philosophy):
Forget about the transcendental philosophy altogether and do something else! (without providing systematic arguments in favor of this decision)
On this account K1 is no longer granted; the term "object" becomes interchangeable with the term "individual" (as this latter term is used in logic).

A: Frege

"I must also protest against the generality of Kants dictum: without sensibility no object would be given to us. Null and one are objects wich cannot be given to us in sensation. ..... Perhaps Kant used the word "object" in a rather different sense." (*Foundations of Arithmetic*)

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Arguably such challenges can be met with Kant's theory of symbolic intuition.

A: Frege on geometry

"The truths of geometry govern all that is spatially intuitable, whether actual or product of our fancy. The wildest visions of delirium, the boldest inventions of legend and poetry, ... all these remain, so long as they remain intuitable, still subject to the axioms of geometry." (*ib.*)

A: Frege on geometry

Kant would disagree. According to Kant the whole difference between the "wildest visions of delirium" and the geometrical intuition is that the latter unlike the former is guided by certain principles, which Kant calls *Principles of Understanding*.

A: Frege on geometry

"Conceptual thought alone can after a fashion shake off this yoke, when it assumes, say, a space of four dimensions or positive curvature. To study such conceptions ... is to leave the ground of intuition enirely behind. If we do make use of intuition even here as an aid, it is still the same old intuition of Euclidean space, the only space of which we have any picture. Only then the intuition is not taken at its face value, but as symbolic of something else." (*ib.*)

A: Frege

"It is in this way that I understand *objective* to mean what is independent of our sensation, intuition and imagination, and of all construction of mental pictures out of memories or earlier sensations, but not what is independent of the reason." (*ib.*)

A: Frege: Remarks

A straightforward Platonism? By Platonism I mean here: making a self-explanotary distinction between "lower" cognitive capacities like sensation, intuition and imagination, on the one hand, and the reason as the "higher " cognitive capacity, on the other hand.

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A straightforward Platonism? By Platonism I mean here: making a self-explanotary distinction between "lower" cognitive capacities like sensation, intuition and imagination, on the one hand, and the reason as the "higher " cognitive capacity, on the other hand. Return to the pre-critical thinking? A critical approach amounts to a study of cognitive capacities involved in doing science and mathematics rather than simply to separating "higher" capacities from "lower" ones.

A: Parsons:

"There is something absurd about inquiring, with complete generality, what an object is. .... The usable general characterization of the notion of object comes from logic. We speak of particular objects by referring to them by singular terms: names, demonstrative and descriptions." (2008)

B: Cassirer and Meinong

To develop a transcendental theory of *objects* independently of mathematics.

B: Cassirer (reacting on Russell-Couturat)

"With this arises a problem, which lies completely outside the scope of logistics ...Worrying about the rules that govern the world of objects is completely left to direct observation, which is the only one that can teach us ... whether we can find here certain regularities or a pure chaos. Logic and mathematics deal only with the order of concepts; they don't contest the order or the disorder of objects and they don't need to confuse themselves with this issue." (*Kant and Modern Mathematics*)

B: Cassirer

"Thus, a new task arises at the point where logistics ends. What critical philosophy is looking for and what it should advance is a logic of the knowledge of objects." (*ib.*)

B: Cassirer

Remarkably Cassirer shares with Frege the view according to which in the end of 19th century mathematics has profoundly changed its nature and transformed itself into a pure conceptual speculation. This is a reason why Cassirer unlike Kant doesn't see any specific link between mathematics and the transcendental study of objecthood.

B: an observation:

As a matter of fact the modern mathematics still uses the Euclidean Scheme. Instantiation of general concepts by individuals (Euclid's *exposition*) remains an essential feature of mathematical thinking as such!

Proposition 1. *A category* $\mathbf{C}$ *with finite limits and small Hom-sets has a subobject classifier if and only if there is an object* Ω *and an isomorphism*

$$\theta_X : \mathrm{Sub}_{\mathbf{C}}(X) \cong \mathrm{Hom}_{\mathbf{C}}(X, \Omega), \qquad (4)$$

natural for $X \in \mathbf{C}$. *When this holds,* $\mathbf{C}$ *is well-powered.*

Proof: Given a subobject classifier as in (3), the correspondence θ_X sending the equivalence class of each monic $S \rightarrowtail X$ to its (unique) "characteristic function" $\phi: X \to \Omega$ is a bijection for each X, as required for (4). Now $\mathrm{Sub}_{\mathbf{C}}(X)$ is a (contravariant) functor of X by pullback (= inverse image); so to prove this bijection natural, we must show that pullback along $f: Y \to X$ in $\mathrm{Sub}_{\mathbf{C}}(-)$ corresponds to composition with f in $\mathrm{Hom}_{\mathbf{C}}(-, \Omega)$. This is immediate by the elementary fact that two pullback squares placed side by side, as in

$$\begin{tikzcd} S' \arrow[r] \arrow[d, tail] & S \arrow[r] \arrow[d, tail] & 1 \arrow[d, tail, "\mathrm{true}"] \\ {} & {} & {} \end{tikzcd}$$

C: Hilbert and Bernays, *Foundations of Mathematics* of 1934

To provide the modern mathematics with a simple intuitive basis (which would be unproblematic for the classical Kantian approach) through the (*formalization*) of mathematics. This project aims at changing mathematics itself rather than its philosophy. The philosophy remains Kantian.

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- "Following the presentation of the elementary arithmetic, we will now also briefly characterize the elementary conceptual viewpoint in algebra. ... The <u>objects</u> of the theory are again certain <u>figures</u>, the polynomials, constructed with the help of the symbols $+, -, \bullet$ and parentheses."

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- "Following the presentation of the elementary arithmetic, we will now also briefly characterize the elementary conceptual viewpoint in algebra. ... The objects of the theory are again certain figures, the polynomials, constructed with the help of the symbols $+, -, \bullet$ and parentheses."
- "When the usual Calculus is formalized (i.e. when its presuppositions and inferences are translated into initial formulas and rules of deduction), then a proof in Calculus becomes a succession of intuitively comprehensible processes. ... Then, in principle, we have the same situation as in our treatment of the elementary arithmetic."

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- Symbolic Logic serves here as a *means* supposed to provide the whole of mathematics with a simple and uniform "figural" intuitive support (I leave now aside the well-known difficulties of Hilbert's Program).
- These foundations are "liberal" in the sense that they don't rule out a possibility of providing mathematics (or a part of mathematics) with a different intuitive basis.

D: Hintikka

To retain (a modernized version of) Transcendental Logic but get rid of Transcendental Aesthetics. Hintikka's Transcendental Logic is a version of modern first-order logic, namely, IF-logic provided with Game-Theoretic semantics.

D: Hintikka

"Kant thought that the general character of the mathematical method amounts to using particular representatives of general concepts. In modern language this is tantamount to a systematic use of rules of exemplification. Since those rules make part of the modern first-order logic Kants theory applies to the logic of quantification (the first-order logic) rather than mathematics." (*Transcendental argumentation revived*)

D: Hintikka

"Since according to Kant the last word of the mathematical method is the treatment of particular objects, those objects are, for him, what modern logicians call "individual". But where such individuals come from? Kants answer is attractive but false. He thinks that those objects "are given to us through the sensual perception, which alone is capable of providing us with the intuition" and he defines such intuitions as representation of particulars." (*ib.*)

Transcendental Aesthetics also matters!

We can think no line without drawing it in thought, no circle without describing it. .... (Kant, *Critique of Pure Reason*)

Euclid's Postulates

Not to be confused with axioms! Postulates 1-3 are NOT propositions.

Αἰτήματα.

α΄. Ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

β΄. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ᾽ εὐθείας ἐκβαλεῖν.

γ΄. Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

δ΄. Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

ε΄. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ᾽ ἄπειρον συμπίπτειν, ἐφ᾽ ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

Postulates

1. Let it have been postulated to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).†

† This postulate effectively specifies that we are dealing with the geometry of *flat*, rather than curved, space.

Hilbert and Bernays on Postulates

"The intuitive meaning of the figures is not ignored in Euclid. Furthermore, its axioms are not in existential form either: Euclid does not presuppose that points or lines constitute any fixed domain of individuals. Therefore, he does not state any existence axioms either, but only construction postulates." (Hilbert and Bernays, *ib.*)

Constructive Postulates

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Constructive Postulates

- In a formal axiomatics the role of non-propositional constructive postulates is played by *formation rules*
- In my view such constructive postulates is an indispensable part of any mathematical theory.

General Remarks on Mathematical Intuition

- New mathematical concepts require new intuitions. Development of new intuitions is an essential part of mathematical progress. The idea to develop mathematics wholly "abstractly" turns mathematics into a sheer speculation. It brakes the (transcendental) mechanism linking the pure mathematics with natural sciences.

General Remarks on Mathematical Intuition

- New mathematical concepts require new intuitions. Development of new intuitions is an essential part of mathematical progress. The idea to develop mathematics wholly "abstractly" turns mathematics into a sheer speculation. It brakes the (transcendental) mechanism linking the pure mathematics with natural sciences.
- The idea of developing mathematics "formally" restricts mathematical thinking with a single preferred syntax. The development of new syntaxes - and, more generally, of new intuitive supports - is as much important as the development of new mathematical concepts. The two things are mutually dependent. Whether or not a given mathematical syntax qualifies as "formal" is a non-issue. The real issue is whether or not it supports a sound mathematical reasoning.

Towards new Transcendental Aesthetics

The issue of *representation*, which is central for Kant's Transcendental philosophy, was a hot issue in mathematics itself already in Kant's times (think of Projective geometry). However Kant didn't take these (as well as many other) mathematical developments into his account.

Representations in mathematics and philosophy

In Geometry representations are *maps* between geometrical spaces. Projective geometry originates from the *study of perspective* (that began during the Renaissance), i.e. the study of maps (projections) between the Euclidean space and the Euclidean plane. This study involved different geometrical spaces from the very outset.

Representations in mathematics and philosophy

Since according to Kant "the" space is essentially unique the study of perspective is in odds with Kant's Transcendental Aesthetics. However the study of perspective *is* supported by the spatial intuition and *is* relevant to the human sensual perception. A representation of a 3D object by its plane picture *is* intuitively conceivable. This shows that Kant's Transcendental Aesthetics doesn't wholly cover its own subject.

Examples

The following two examples will help me to introduce a concept of mathematical object.

Ex.1: Euclidean plane

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- the universe of Euclidean Planimetry (*EPLANE*) and
- an object of Euclidean stereometry (an *eplane* in *ESPACE*)

Than an eplane can be described as an injective map
$eplane : EPLANE \to ESPACE$

Ex.1: Euclidean plane

The EPLANE cannot be (and needs not to be) visualized within the Planimetry. However one gets an intuitive *image* of EPLANE by mapping it into ESPACE (i.e., into a different space). While EPLANE is unique its images in ESPACE (i.e., eplanes) are many.

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Every eplane “carries with it” the whole world of Planimetry. Through this operation plane figures become habitants of a 3D world. The capacity of switching between the plane intuition and the stereometric intuition is required for doing the traditional Euclid-style geometry. Such an intuitive capacity is not accounted for by Kant’s Transcendental Aesthetics.

Ex.2: horosphere

Like a number of his predecessors Lobachevsky first developed Hyperbolic geometry by combining Euclid-style intuitive reasoning with some analytic tools (in the sense of Analytic geometry of 18th century). His main achievement was fixing the *analytic* part of this new theory.

Ex.2: horosphere

Like a number of his predecessors Lobachevsky first developed Hyperbolic geometry by combining Euclid-style intuitive reasoning with some analytic tools (in the sense of Analytic geometry of 18th century). His main achievement was fixing the *analytic* part of this new theory. For this end Lobachevsky considered a special surface in the hyperbolic 3-space (HSPACE), which he called a *horosphere*. Lobachevsky observed that the intrinsic geometry of a horosphere is the same as that of an eplane. Then by applying the usual plane trigonometric identities to triangles on a horosphere he figured out the corresponding identities for triangles on a tangent hyperbolic plane. (Noticeably Lobachevsky didn't use anything like a *model* of Hyperbolic geometry but used what we may call today a non-standard model of Euclidean geometry!)

Ex.2: horosphere

A horosphere can be described as an injective map
$horosphere : EPLANE \rightarrow HSPACE$
it is an object living in the Hyperbolic 3-space.

Objects are maps!

Generalizing upon these and similar examples I shall call a *geometrical object* any map (not necessarily injective) of geometrical spaces. I shall try to think similarly about *all* mathematical objects (through replacing geometrical spaces by representation schemes of other sorts).

A comparison with Kant

A geometrical object (in my sense), generally, involves *two* spaces (but not one fixed space as Kant assumed), which may only eventually coincide. The distinction between these two spaces is made through the usual (after Gauss and Riemann) distinction between *intrinsic* and *extrinsic* properties of a given geometrical object.

A suggestive terminology

It is appropriate to call the source-space of a given object by the term "type" reserving the word "space" for its target-space. Think again about $eplane : EPLANE \rightarrow ESPACE$. Actually we think about the EPLANE as a *space* only when we study objects of the form $x : X \rightarrow EPLANE$ living *in* the EPLANE (i.e., plane figures). Given an eplane we rather think about the EPLANE as a *type* shared by all eplanes.

A suggestive terminology

It is also suggestive to think of an object (in my sense) as an *instantiation* of given type in a given space. The latter point is crucial for my Kantian approach: types are not instantiated "nowhere". Unlike Kant I assume that spaces allowing for the instantiation of types are many (just like types themselves are many). I do NOT assume that any type can be instantiated in any space.

WARNING

Using the above terminology one should not confuse the distinction between objects and their *types* with the usual type/token distinction. According to my suggested view an eplane and a horosphere are of the same type (= they are *intrinsically* the same). A given objects is determined both by its type and by its space. Yet these data do NOT determine an object uniquely.

A duality between types and spaces

Being a *type* and being a *space* (in my sense) are relational properties. Studying objects of the form $X \rightarrow Y$ with fixed Y and variable X one describes Y as a *space*. This provides an *intrinsic* view on Y.
Studying objects of the form $Y \rightarrow X$ with fixed Y and variable X (once again) one describes the same Y as a *type*. This provides an *extrinsic* view on Y.

A comparison with Hilbert's (nowadays standard) view

According to the standard view an eplane and a horosphere are two different intuitive images (representations, realizations, models) of the same plane Euclidean *structure*; this structure is supposed to catch the notion of EPLANE in purely conceptual terms. Such an approach privileges the *intrinsic view* on a given object against the *extrinsic view*. I can see no epistemic reason for making this preference.

A comparison with the standard view

The standard thinking is this: the type EPLANE determines what is a horosphere "essentially" while the fact that this thing lives in the HSPACE is somewhat "accidental". Similarly for an eplane. However this thinking is erroneous. There is no eplane outside the ESPACE just like there is no horosphere outside the HSPACE.

intrinsic and extrinsic features

The habit to privilege epistemically "intrinsic features" over "extrinsic features", which one finds in the *structural* mathematics of the first half of the 20th century, is a remnant of the traditional *essentialism* (that can be traced back to Plato), which stands against the general tendency towards a *relational* thinking promoted by partisans of the structural trend in mathematics.

beer mugs

Indeed, as Hilbert repeatedly stresses, primitive geometrical objects like points and straight lines have no “intrinsic” (i.e. non-relational) properties; they are pure logical individuals holding certain relations (they can thought of as beer mugs!). Any system of such objects holding required relations qualifies as a geometrical space. BUT....

Structuralism and Essentialism

(i) In spite of the fact that Hilbert in 1899 avoids to describe *primitive objects* intrinsically he DOES describe everything else (including geometrical *spaces*) intrinsically. Such an intrinsic description is given in terms of relations between primitive objects, which serve as buildings blocks for complex objects and spaces. An "essence" of a complex mathematical construction like a geometrical space is its internal *structure*. This is a traditional *essentialist* aspect of the Hilbertian Axiomatic method.

Structuralism and Substantialism

(ii) The concept of logical *individual* (identified here with that of primitive *object*) is non-relational, i.e., absolute. Identities of primitive objects are supposed to be fixable independently of relations held by these objects. This is a traditional *substantialist* aspect of Hilbert's Axiomatic method.

Structuralism and Substantialism

Structuralism and (Set-theoretic) Substantialism are different philosophical interpretations of the same theoretical method. Structuralism stresses the essentialist aspect of this method (as described above) and downplays its substantialist aspect while Substantialism does it the other way round. The controversy between the two interpretations hardly leads anywhere. A further progress is hardly possible without changing the theoretical method itself.

Aim

My goal is to dispel with the traditional features (i)-(ii) of the standard Axiomatic method through a further "relativization" of this method. A more limited aim of this talk is to suggest a relational notion of object. My idea is to push further forward Kant's "Copernican revolution" by replacing Kant's "absolute" notion space (which reflects the absolute space of the Newtonian physics) by a relational framework, which allows for multiple spaces. (An anticipation of what follows: actually I have a *category* of spaces in mind.)

A Transcendental Deduction of Category Theory

Consider after Kant a "manifold of intuition", which is NOT yet "united" and hence does not yet provide us with well-distinguishable *objects*. Elements of this manifold are "proto-objects" that turn into full-fledged objects through a unification of this manifold. Let me now describe a specific *way* of unification of such a manifold, which I shall call *categorification* (for reasons that will become very soon clear).

Transcendental Unity through Categorification

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Categorification comprises three “transcendental” operations (which are not independent):

- typing : specification of elements sharing a type
- spacing : specification of elements sharing a space
- combining : specification of pairs of elements such that the type of one element is the space of the other element.

Typing

Typing is understood here as usual. (I leave aside the old controversy between nominalistic and realistic interpretations of typing.)

Spacing

Here is an example. One may equally well imagine Euclidean and Hyperbolic geometrical objects of the same type (Frege's philosophical prejudges notwithstanding). Think of Euclidean and hyperbolic triangles. But one cannot place all these things side-by side into the same space! Some of those triangles do share a space while some other don't.

Combining

I assume - as usual - that *combining* of two objects always brings about a (unique) new object. However in the given context this notion of combining has a specific sense. Here is an example. By combining a plane circle $c : C \rightarrow EPLANE$ with an eplane $e : EPLANE \rightarrow ESPACE$ one gets a disk $d : C \rightarrow ESPACE$ living in the 3D space. This combination can be shown with a diagram:

$$
\begin{tikzcd}
& EPLANE \arrow[dr, "e"] & \\
C \arrow[ur, "c"] \arrow[rr, "d"'] & & ESPACE
\end{tikzcd}
$$

Categorification

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I leave the Transcendental Deduction of these final steps of categorification for a future work.

WARNING

My terminology differs from one established in Category theory. My types/spaces are called in Category theory "objects" and my objects are called here "maps" or "morphisms". The suggested terminological change helps me to revise the usual way of thinking about categories. The usual terminology reflects Frege/Russell's notion of object. My suggested terminology is Kantian.

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- the categorical objecthood remains relevant to sensual perception like in Kant
- the categorical objectivity and objecthood allow for a multiplicity of representation spaces, which is not the case in Kant
- the categorical notion of object is *relational* in the sense that it relates *what* is represented to *where* it is represented (which is not the case in Hilbert et al.).

Principles of Pure Understanding revised

We see that Kant's "Principles of Pure Understanding" do *not need* to ground axioms determining a unique representation space - even if Kant's own version of these Principles indeed does so. A transcendental unity of apperception, and hence reasonable notions of objectivity and objecthood, can be based on rules applicable to any representation space - and even more generally, to any representation *scheme*). One may tentatively think of the usual axioms of Category theory as of such general rules of representation.

From axioms to postulates

In fact the above background suggests a reformulation of these axioms (or a part of these axioms) in the form of non-propositional *constructive postulates*:
Given object $A \rightarrow B$ *and object* $B \rightarrow C$ to produce *object* $A \rightarrow C$, etc.

Syntax

The diagrammatic syntax developed for Category theory serves as an intuitive support for doing mathematics. It does not need to *replace* the more convenient alphabetic syntax but can be used along with it. Generally, I don't think that the idea to reduce the whole of mathematics to a single syntax is justified. In Euclid we find at least three different interacting syntaxes: diagrammatical, notational and the usual phonetic syntax of Greek.

An open problem

Is it possible to make alternative choices of the two spaces for the same object? Here is a hint. By presenting a given object O as a bare set of *points* living in some given space S and assuming that the intrinsic geometry of a point is null one gets a "purely extrinsic" presentation of this object. Alternatively one may think of a given object "purely intrinsically" as a *space* inhabited by some other objects and independent of any "external" space. Such "extreme" presentations in different ways "kill" the given object. A bare set of points in a space is no longer a single object; in this case one reduces the *type* of the given object. A space is not an object either. Considering the given object "purely intrinsically" as a space one forgets about the "ambient" space, in which this given object is represented. It is interesting to study the spectrum of all intermediate cases.