OBJECTHOOD IN MODERN LOGIC

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Abstract. Equating objects with logical individuals is a way to get around the problem of objecthood as it has been stressed by Cassirer in 1907. Using the rudimentary Category theory and some hints from the history of 19th century geometry I propose a tentative solution of this problem in the context of today's Categorical mathematics.

1. Historical background

Following Frege many today's philosophers (particularly from the Analytic tradition) equate objects with logical individuals. For example, Charles Parsons describes his most general notion of object as follows:

> I will fix the way I wish to use the term "object" and simultaneously say what I think useful in such abstract discussions [about objects in general - A.R.] by saying that the usable general characterization of the notion of object comes from logic. We speak of particular objects by referring to them by singular terms [..]. [5], p. 3

This familiar promiscuous use of the term "object" (not only in the common talk but also in the current philosophical literature) stands in sharp contrast with the way in which this term has been used by Kant and his Neo-Kantian followers. For this latter category of thinkers being an object always implies being represented in space and time. Equating objects with individuals (as does Frege) destroys the fundamental Kantian distinction between (i) the *transcendental* logic, i.e., logic dealing with objects of possible experience, and (ii) *formal* (or as Kant call it *general*) logic, i.e., logic dealing with all things indiscriminately. I take it here for granted that any theory of objects, which does not involve in some form this distinction between the two sorts of logic, cannot qualify as Kantian. Thus the above terminological difference concerning the term "object" as it is used in the Analytic and in the Kantian philosophical tradition reflects a fundamental difference in how these two influential schools think about logic.

A reason why Kantian philosophy ceased to be a mainstream lays outside the domain of speculative philosophy in the history of mathematics. The rise of Non-Euclidean geometry in the 19th century made doubtful Kant's original arguments which depended on the (implicit) assumption according to which mathematics offers a unique notion of space, namely Euclidean space. In this new context Kant's original analysis of his contemporary mathematics and mathematically-laden physics seemed to be no longer adequate to the changed scientific landscape. At the same time the alternative approach offered by Frege, Russell and other early figures of the rising Analytic philosophy seemed more adequate and more promising. In his important paper *Kant and New Mathematics* published in 1907 Cassirer criticizes Russell's approach (and the name of *logistics*) as follows:

> Here rises a problem that lies wholly outside the scope of "logistics" [..] All empirical judgements [..] must respect the limits of experience. What logistics develops is a system of hypothetical assumptions about which we cannot know, whether they are actually established in experience or whether they allow for some immediate or non-immediate concrete application. According to Russell even the general notion of magnitude does not belong to the domain of pure mathematics and logic but has an empirical element, which can be grasped only through a sensual perception. From the standpoint of logistics the task of thought ends when it manages to establish a strict deductive link between all its constructions and productions. Thus the worry about laws governing the world of objects is left wholly to the direct observation, which alone, within its proper very narrow limits, is supposed to tell us whether we find here certain rules or a pure chaos. [According to Russell] logic and mathematics deal only with the order of concepts and should not care about the order or disorder of objects. As long as one follows this line of conceptual analysis the empirical entity always escapes one's rational understanding. The more mathematical deduction demonstrates us its virtue and its power, the less we can understand the crucial role of deduction in the theoretical natural sciences.([1], p. 43)

As I argue elsewhere the problem stressed by Cassirer in the above quote still remains wide open and becomes particularly pertinent in the context of the alleged "unreasonable" character of the effectiveness of mathematics in today's natural sciences [6], [7]. Leaving further general discussion on this important issue aside I would like to propose in this paper a sketch of object concept, which is of geometrical character (and thus doesn't reduce to the concept of mere logical individual) but unlike the traditional Kant's object concept doesn't require the assumption about the special role of Euclidean geometry. This object concepts issues from an application of mathematical Category theory to the history of 19th century geometry. I shall show how this modern apparatus allows one to reveal in this history some interesting features, which have been missed by Russell and his modern followers. The rudimentary Category theory used in this construction can be found at first several pages of [3]

2. Objecthood and Categorical geometry

Given a surface one can think of it (i) in the usual way as a two-dimensional *object* living in the Euclidean 3-space and (ii) as a 2-space on its own rights (characterized by the intrinsic properties of the given surface), which is a home for its points, lines, triangles, etc.. Generally, a geometrical *object* can be described as a *map* of the form $s : B \to C$ where B is a *type* of the given object and C is a *space* where the given object lives and *instantiates* (or *represents*, which in the given context is the same) its type. This way of thinking about spaces and objects in spaces can be represented by this diagram:

$$TYPE \xrightarrow{object} SPACE$$

It is suggestive also to think about a general categorical morphism in this way. Since we interpret all domains as spaces and all codomains as types these notions are relational in the given context (each type serves as a space for incoming morphisms and each space serves as a type for outgoing morphisms). I shall illustrate this way of thinking about objects, spaces and types at some elementary geometrical examples.

I shall write $EPLANE$ for Euclidean plane construed as a 2-space, and write *eplane* for Euclidean plane construed as an object living in the Euclidean 3-space ($ESPACE$). Then an *eplane* can be presented as a map:

$$EPLANE \xrightarrow{eplane} ESPACE$$

Such maps are many (there are many planes in the space) but they all "are of" the same type; this type in its turn is inhabited (as a space) by objects of different types:

$$CIRCLE \xrightarrow{circle} EPLANE$$

A more interesting example I borrow from Lobachevsky [4]. Although Lobachevsky reasoned about the hyperbolic space intuitively without using Euclidean models he actually used what in modern term can be described as a non-standard hyperbolic model of Euclidean plane. Namely, he found in the hyperbolic space a special surface that he called the *horisphere* and showed that intrinsically the geometry of this surface is the plane Euclidean geometry. (This helped Lobachevsky to develop the hyperbolic trigonometry and on this basis build an analytic model for his geometry.) Thus we have got an object of type $EPLANE$ that does not look like *eplane*:

$$EPLANE \xrightarrow{horisphere} HSPACE$$

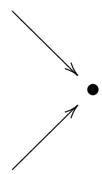
($HSPACE$ stands for hyperbolic space). We can see that the idea to classify geometrical objects into types by their shapes and forms is misleading because it works only when the background space is fixed. However one can learn about any geometrical type by studying it intrinsically as a space, i.e., by studying objects of all types living in it. My suggested approach (unlike Riemann's approach) does not in any way privilege the intrinsic description against the extrinsic one: the fact that a horisphere is intrinsically an Euclidean plane (in the sense of being of type $EPLANE$) is just as significant as the fact that this horisphere is an object in the hyperbolic 3-space ($HSPACE$): when one studies geometrical objects there is, generally, no epistemic reason for privileging their types over their spaces or privileging their spaces over their types.

The geometrical objects so construed are composeable in the obvious way. Here is an example of composite object:

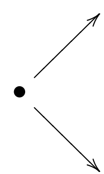
$$\begin{array}{ccc} CIRCLE & \xrightarrow{circle_1} & EPLANE \\ {\scriptstyle circle_2}\downarrow & \swarrow{\scriptstyle eplane} & \\ ESPACE & & \end{array}$$

In the given situation we tend to identify $circle_1$ living on $EPLANE$ with $circle_2$ living in $ESPACE$. However if $ESPACE$ is projected back onto $EPLANE$ and this projection turns $circle_2$ into an oval the difference becomes obvious.

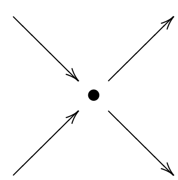
Developing this toy categorical geometry it is suggestive to think about spaces as places where objects meet:



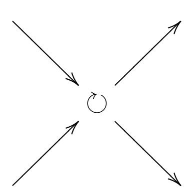
think about types as places where objects split:



and think about the composition of objects as an operation that glues spaces and types together:



and thus form new objects capable for "self-representation":



Among such self-representing objects there is one that we call *identity object* and denote 1; the identity object is distinguished by the usual conditions ($f1 = f$ for each object f represented in the same space, and $1g = g$ for each object g of the same type), which in the given context are read as the conditions of being *neutral* with respect to the composition of objects. Think about $EPLANE$ for example. We know how $EPLANE$ represents objects of various types (circles, triangles and the like) and we also know how $EPLANE$ is represented in its turn in various other spaces. Now in order to make sense of saying that all these representations are representations in and of the same thing one should think of this thing itself as an object that represents itself in a way, which stabilizes the dynamics of all inner (incoming) and outer (outgoing) representations.

The associative composition of objects and the above assumptions about the identity objects makes these objects into a category, which I denote *Geo* for further references. Although *Geo* does not qualify as a well-defined geometrical category [2] it provides a suggestive way of thinking about categories geometrically and thinking about geometry categorically. As a bonus we get here a reasonable general notion of object, which does not involve any fixed representation space (like Euclidean space of Kant's theory of objects).

As the reader may have noticed what in category theory is usually called *morphism* I call *object* and what in category theory is usually called *object* I call *identity object.* This suggested terminological change is not without a reason. The distinction between objects and morphisms is useful in the structural mathematics because it helps to construct categories from structures of certain types (like groups) and appropriate morphisms of these structures (like group homomorphisms). *Geo* can be also construed in this way as the category of differentiable manifolds and differentiable maps (assuming that differentiable manifolds are construed as structured sets) or as some other similar category. However I suggest a different way of thinking about *Geo* and about categories in general. Before I shall try to clarify this different way of thinking let me remind that the usual distinction between objects and morphisms of categories is formally dispensable: since with each object A of given category C is associated a unique identity morphism 1_A, one may formally identify objects of C with their corresponding identity morphisms and thus consider objects as morphisms of special sort. Thus, formally, a general category can be described as a class of things called morphisms provided with a (partial) binary associative operation called composition. I claim that the name of *objects* is more appropriate for these things than the name of morphisms. Saying this I do not mean that any such thing can be called object in the most general sense of the word used by Parsons [5]). Instead I have in mind a particular notion of object, which implies that objects form categories. This way of thinking about objects can be expressed by the slogan *objects are maps.*

One may wonder why I am not happy with the established mathematical terminology and don't want to get rid of the term "object" altogether and talk about maps or morphisms. The reason is that the term "object" does not belong exclusively to mathematics but has also a philosophical meaning. Although the proposed notion of object is not standard for the 20th century philosophy and for the 20th century mathematics, as I have explained in the last section, it is rooted both in an earlier philosophy and in an earlier mathematics.

3. Conclusion

The principal modification of Kant's original viewpoint, which I suggest, is the following: while Kant assumed that all objects are represented in the same space I allow for representations in different spaces (making at the same time the very notion of space relational). We have seen that the 19th century geometry provides us with relevant examples. The example of Lobachevsky's *horisphere* is particularly useful in this respect because it shows that geometrical objects, generally, are determined not only by their types (i.e., by their intrinsic properties) but also by spaces in which they are represented. We have also seen that the modern mathematical notion of category provides a suitable framework for such objects if one identifies these objects with morphisms of some category. Since the language of Category theory is very general and covers most (if not all) of today's mathematics, the proposed categorical notion of object rooted in the 19th century mathematics and philosophy can be considered as a tentative solution of Cassirer's problem in the today's mathematical context. (More details can be found in my recent monograph [6], ch. 8.)

References

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